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PROCEEDINGS

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RECORDS OF PROCEEDINGS AT MEETINGS.

Annual General Meeting.

Thursday, November 13th, 1924.

Prof. H. HILTON, Vice-President, and later Prof. A. L. DIXON, President, in the Chair.

There were present thirty-three members and six visitors.

The Minutes of the last meeting were read and confirmed.

Mr. D. D. Chelliah was elected a member.

Messrs. F. Bath, T. M. Cherry, A. E. Daniels, A. C. Heath, and A. E. Ludlam were nominated for election.

Messrs. H. Cansdale and A. Oppenheim were admitted into the Society.

The Treasurer presented his report, and Lt.-Col. A. J. C. Cunningham was appointed Auditor.

The meeting proceeded to the election of Officers and Council for the ensuing session (1924–5).

Mr. J. Brill and Prof. M. J. M. Hill were appointed Scrutineers, and reported that the result of the election was as follows :—President, Prof. A. L. Dixon; Vice-Presidents, Prof. L. N. G. Filon, Prof. H. Hilton, Prof. W. H. Young; Treasurer, Dr. A. E. Western; Secretaries, Prof. G. H. Hardy, Prof. G. N. Watson; other members of the Council, Prof. S. Chapman, Miss H. P. Hudson, Prof. G. B. Jeffery, Prof. A. E. Jolliffe, Mr. J. E. Littlewood, Prof. A. E. H. Love, Prof. L. J. Mordell, Mr. H. W. Richmond, Mr. F. P. White.

Prof. A. L. Dixon then took the Chair.

The address of the retiring President (Prof. W. H. Young), "The Progress of Mathematical Analysis in the Twentieth Century,"* was, owing to Prof. Young's absence, read by Prof. Hilton.

* Printed in this volume.

The following papers were communicated by title from the Chair :—

The General Equation of the n-th Degree : H. Cansdale.

Poncelet's Poristic Polygons (second paper) : T. W. Chaundy.

On Successive Evolutes of a Plane Curve : H. Hilton.

The Real Zeros of Solutions of a Linear Differential Equation with Periodic Coefficients : E. L. Ince.

On some Tauberian Theorems of Hardy and Littlewood : A. E. Ingham.

*On the Riemann Zeta function : J. E. Littlewood.

*Contravariant Envelopes of the Plane Quartic Curve : W. P. Milne.

(i) *The Convergence of Certain Integrals, (ii) An Inversion Formula involving Bessel Functions : E. C. Titchmarsh.

*A Class of Problems in Electrostatics and Hydrodynamics : D. M. Wrinch.

NOTES AND ABSTRACTS.

On Successive Evolutes of a Plane Curve

HAROLD HILTON.

A singularity such as is possessed by the curve $y^2 = x^{2k+1}$ at the origin will be called a "k-point." The singularity is equivalent to $k-1$ nodes, 1 cusp, $2(k-1)^2$ bitangents, and $2(k-1)$ inflexions as far as its effect on Plücker's numbers is concerned, and the number of tangents from a point on $y = 0$, which touch at the origin, is $2k-1$.

Each intersection of a curve with the line at infinity gives rise to a 1-point (cusp) on the evolute at infinity, the line at infinity being the tangent. Each k-point on the curve, having the line at infinity as tangent, gives rise to a $(k+1)$-point on the evolute, having the line at infinity as tangent.

From this follows by induction the result :—

A curve has degree n, class m, δ nodes, κ cusps, τ bitangents, ι inflexions, deficiency D; and has no special relationship with the line at infinity or the circular points.

The corresponding quantities for the k-th evolute (the evolute of the $(k-1)$-th evolute) are

$$n_k, \quad m_k, \quad \delta_k, \quad \kappa_k, \quad \tau_k, \quad \iota_k, \quad D_k.$$

Then $n_1 = 3n+\iota, \quad m_1 = m+n, \quad \iota_1 = 0, \quad D_1 = D;$

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* Printed in this volume.

and, if $k > 1$,

$$n_k = (2k+1)n+(2k-1)\iota, \quad m_k = m+(2k-1)n+(2k-3)\iota,$$
$$\iota_k = 2(k-1)n+2(k-2)\iota, \quad D_k = D.$$

The first evolute has n cusps at which the line at infinity is tangent, and ι ordinary points at infinity. The k-th evolute ($k > 1$) has n k-points and ι $(k-1)$-points, at all of which the line at infinity is tangent.

We deduce δ_k, κ_k, τ_k from Plücker's equations. For instance, $\kappa_k = (2k+1)(n+\iota)+\kappa$. All the inflexions and $n+\iota$ of the cusps of the k-th evolute ($k > 1$) are involved in the singularities at infinity.

An Inversion Formula involving Bessel Functions

E. C. TITCHMARSH.

I have obtained the following inversion formula :—

$$(1.1) \quad \phi(x) = \int_0^\infty [\sin a\pi \{J_\nu^2(xt) - Y_\nu^2(xt)\} - 2\cos a\pi J_\nu(xt) Y_\nu(xt)] t f(t) dt,$$

$$(1.2) \quad f(x) = \pi \int_0^\infty \frac{d}{dx} \{xG(xt)\} t \phi(t) dt,$$

where

$$G(x) = \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\nu+n+a+\frac{1}{2}) x^{2\nu+2n+2a}}{\Gamma(\frac{1}{2}) \Gamma(a+n+1) \Gamma(\nu+a+n+1) \Gamma(2\nu+a+n+1)}.$$

When $a = 0$, $G(x) = J_\nu^2(x)$, and the formula reduces to one given previously by me as a generalization of a formula of Bateman.*

The result is deduced from Hardy's inversion formula†

$$(2.1) \quad \phi(x) = \int_0^\infty \{\cos a\pi J_\nu(xt) + \sin a\pi Y_\nu(xt)\} t f(t) dt,$$

$$(2.2) \quad f(x) = \int_0^\infty F_{\nu, a}(xt) t \phi(t) dt,$$

where

$$F_{\nu, a}(x) = \sum_{n=0}^\infty \frac{(-1)^n (\frac{1}{2}x)^{\nu+2a+2n}}{\Gamma(a+1+n) \Gamma(\nu+a+1+n)}.$$

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* E. C. Titchmarsh, "Extensions of Fourier's Integral Formula to Formulae involving Bessel Functions", *Proc. London Math. Soc.*, 17th January, 1924 (formula 5, p. xxiii).

† G. H. Hardy, "Some formulae in the theory of Bessel functions", *Proc. London Math. Soc.*, Records for 12th June, 1924 (formulae 6.1, 6.2, p. lxii).

We also use Nicholson's integral*

$$\int_1^\infty K_{2\nu}(xt)\frac{dt}{\sqrt{(t^2-1)}} = \tfrac{1}{2}K_\nu^2(\tfrac{1}{2}x). \tag{3}$$

Replacing x by a purely imaginary number (as we may by the theory of analytic continuation), and then separating real and imaginary parts, we obtain

$$\int_1^\infty J_{2\nu}(xt)\frac{dt}{\sqrt{(t^2-1)}} = -\tfrac{1}{2}\pi J_\nu(\tfrac{1}{2}x)\,Y_\nu(\tfrac{1}{2}x), \tag{3.1}$$

$$\int_1^\infty Y_\nu(xt)\frac{dt}{\sqrt{(t^2-1)}} = \tfrac{1}{4}\pi\{J_\nu^2(\tfrac{1}{2}x)-Y_\nu^2(\tfrac{1}{2}x)\}. \tag{3.2}$$

In (2.1) put $f(t) = t^{-1}(t^2-1)^{-\frac{1}{2}}$ $(t>1)$, 0 $(t<1)$. Then it follows from (2.2) that

$$\int_0^\infty F_{\nu,a}(2\mu t)\left[\sin a\pi\{J_{\frac{1}{2}\nu}^2(\lambda t)-Y_{\frac{1}{2}\nu}^2(\lambda t)\}-2\cos a\pi J_{\frac{1}{2}\nu}(\lambda t)\,Y_{\frac{1}{2}\nu}(\lambda t)\right]t\,dt = \frac{1}{\pi\mu\sqrt{(\mu^2-\lambda^2)}} \quad (\lambda<\mu), \quad 0 \quad (\lambda>\mu). \tag{4}$$

To prove the inversion formula, replace $\phi(t)$ by $\psi(t)$ in (1.2). Then we have to verify that $\psi(t)=\phi(t)$. Multiply both sides of (1.1) by $x\,F_{2\nu,a}(2\xi x)$ and integrate from 0 to ∞, inverting the order of integration on the right. Using (4), we obtain

$$\begin{aligned}\int_0^\infty F_{2\nu,a}(2\xi x)\,x\,\phi(x)\,dx &= \frac{1}{\pi\xi}\int_0^\xi \frac{t}{\sqrt{(\xi^2-t^2)}}f(t)\,dt\\ &= \frac{1}{\xi}\int_0^\xi \frac{t\,dt}{\sqrt{(\xi^2-t^2)}}\int_0^\infty \frac{d}{dt}\{tG(tu)\}\,u\psi(u)\,du\\ &= \int_0^1 \frac{s\,ds}{\sqrt{(1-s^2)}}\int_0^\infty \frac{d}{du}\{uG(s\xi u)\}\,u\psi(u)\,du\\ &= \int_0^\infty F_{2\nu,a}(2\xi u)\,u\psi(u)\,du,\end{aligned}$$

since

$$\int_0^1 \frac{d}{du}\{uG(s\xi u)\}\frac{s\,ds}{\sqrt{(1-s^2)}} = F_{2\nu,a}(2\xi u),$$

as is easily verified by integrating term by term. Using Hardy's inversion formula again, it follows that $\phi(x)=\psi(x)$.

The work is, of course, not rigorous, and conditions under which it may be made so remain to be investigated.

* J. W. Nicholson, "Notes on Bessel Functions", *Quart. J. of Math.*, 42 (1911), 216-224 (221).

*On Borel Intervals**

M. J. M. HILL, R. G. COOKE, *and* Miss F. M. WOOD.

This investigation had its origin in an attempt to determine what part of the unit length was covered by the intervals

$$\left(\frac{p}{q}-\frac{1}{q^3},\quad \frac{p}{q}+\frac{1}{q^3}\right),$$

to which Borel had drawn attention in his *Leçons sur la théorie des fonctions*, 39.

Borel showed that the total length did not exceed the value

$$2\sum_{q=2}^{\infty}(q^{-2}-q^{-3}) = 0{\cdot}885754327\ldots,$$

but this includes an infinite number of lengths counted many times over. Thus corresponding to the fraction $\frac{1}{2}$, which appears in the above estimate as $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, ..., there are intervals whose total length is

$$2(2^{-3}+4^{-3}+6^{-3}+\ldots),$$

but all these are included in the first. Again, the interval $(\frac{49}{125}, \frac{51}{125})$ corresponding to $\frac{2}{5}$ is completely included in the interval $(\frac{3}{8}, \frac{5}{8})$ corresponding to $\frac{1}{2}$.

It was therefore desirable to find some method of rejecting intervals already overlapped, without examining each one separately. In doing this it was found that the overlapping of intervals was subject to definite laws, the interest of which overshadowed that of the original quest. These laws are as follows :—

(i) If the centre of an interval be inside another interval (at any point but its centre), then the whole of the first interval lies inside that half of the second interval in which the centre of the first interval is situated, *i.e.* it does not extend as far as the centre nor as far as an end-point of the second interval.

(ii) If the centre of an interval is at an end-point of another, then the half of the first interval which lies inside the second interval does not extend as far as its centre.

(iii) If the centre of one interval is outside a second interval, then either the first interval is wholly outside the second interval, or else the first interval contains the second interval in one of its halves.

(iv) Two intervals cannot have a common end-point.

* Read at the meeting of April 24th, 1924.

When the paper was read before the Society, Professor Hardy suggested that more fruitful results than the object originally sought might flow from the consideration of the intervals

$$\left(\frac{p}{q}-\frac{\epsilon}{q^3},\quad \frac{p}{q}+\frac{\epsilon}{q^3}\right)$$

mentioned by Borel (*loc. cit.*, 44). In particular, he suggested that the above mentioned laws might be valid for *some* types of ϵ, although a complete investigation of *all* the values of ϵ for which the laws were valid might be attended by great arithmetical difficulties. Accordingly it is here shown that the four laws above mentioned are valid for the intervals

$$\left(\frac{p}{q}-\frac{\alpha}{\beta}\frac{1}{q^n},\quad \frac{p}{q}+\frac{\alpha}{\beta}\frac{1}{q^n}\right),$$

where n is any positive integer not less than 3, and α, β are integers prime to one another, such that $\beta^{n-1} > \alpha^n$, which involves $\beta > \alpha$. In addition the case $\alpha = \beta = 1$ is included.

The paper concludes with a brief outline of the argument employed to determine approximately the portion of the unit length covered by the intervals

$$\left(\frac{p}{q}-\frac{1}{q^3},\quad \frac{p}{q}+\frac{1}{q^3}\right).$$

The result is that the covered portion exceeds 0·620705785..., but does not exceed 0·623475738.

The Inflexional Tangents of Plane Quartic Curves

A. E. JOLLIFFE.

I should like to put on record some additional properties of the non-singular plane quartic which came to my notice when reading the proofs of my paper with the title above. The main theorem of that paper is that the 12 bitangents of a Steiner's complex of a quartic, and the 24 inflexional tangents of the quartic, touch a class-sextic. The 12 bitangents touch a class-cubic, called in the paper $\Psi = 0$. It is well known that the 12 bitangents of a Steiner's complex can be divided into two groups of 6, such that the bitangents of each group touch a conic. It follows from the principles of residuation that the remaining 6 common tangents of the class-sextic and $\Psi = 0$ touch a conic. Now it can be verified that the equation of the class-sextic may be written in

the form $KU_4+\Psi^2=0$, where U_4 is a tangential form of the 4th degree, and $K=0$ is the tangential equation of a certain conic associated with the complex. The equation $J=0$, of the envelope of straight lines divided harmonically by the quartic, is of a similar form, and it follows that the class-sextic and the curve $J=0$ touch each other and $K=0$ at the same 6 points. Since there is a conic $K=0$ associated with each Steiner's complex, we obtain the remarkable result that $J=0$ touches each of 63 conics at 6 points.

These results have been obtained also by Professor W. P. Milne, and follow quite obviously from a singularly elegant proof which he has discovered of the main theorem. This proof also gives geometrical interpretations of the class-sextic and of the conic $K=0$.

On the Deformation and Applicability of Surfaces

B. M. SEN.

Beltrami* has discovered that to every ruled surface corresponds another with corresponding generators parallel, the two surfaces being applicable, but not mutually deformable, as the respective tangent planes rotate in opposite directions when the points of contact proceed along corresponding generators on the surfaces. A distinction must therefore be made between the applicability and the mutual deformability of two surfaces. It is easily verified that some well known cases, *e.g.* the so-called deformation of a surface with a particular curve deformed into a given curve, are cases of applicability only. A surface and its image on any plane afford another example.

The general problem of determination of all surfaces applicable on a given surface can be made to depend on the solution of the equations of Gauss and Codazzi. These determine the three coefficients D, D', D'', and by Bonnet's theorem† each set of solutions gives a distinct surface having the same arc-element as the original surface.

All the three equations are unchanged if the signs of D, D', D'' are reversed. Corresponding to any set of solutions, there is another with the sign reversed. These two sets give rise to two distinct surfaces. The case of a ruled surface with corresponding generators parallel may be immediately verified.

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* Beltrami, *Annali di Mat.*, 7 (1865), 139.

† Bonnet, *Journ. Éc. Polytechnique*, 42 (1867), 31.

Also if p represents the perpendicular on the tangent plane at any point from an adjacent point on the surface,

$$p = \tfrac{1}{2}(D\,du^2 + 2D'\,du\,dv + D''\,dv^2).$$

The two sets of solutions, therefore, imply two equal and opposite values for p. The rotation of the respective tangent planes in opposite directions, as the points of contact move along two corresponding curves on the respective surfaces, is therefore verified.

Thursday, December 11th, 1924.

Prof. H. HILTON, Vice-President, in the Chair.

There were present fifteen members.

Messrs. F. Bath, T. M. Cherry, A. E. Daniels, A. C. Heath, and A. E. Ludlam were elected members of the Society.

Dr. G. Pólya, Prof. J. L. Synge, and Dr. W. H. Thürow were nominated for election.

Mr. F. Bath was admitted into the Society.

The Auditor (Lt.-Col. A. Cunningham) presented his Report, which was accepted.

Prof. G. N. Watson, as Secretary, reported that the number of members of the Society on November 1st, 1923, was 382. Fifteen new members had been elected during the session 1923-4, nine had died, and five had resigned. The number of members on November 1st, 1924, was 383.

Mr. C. G. F. James read a paper *"Formulae for the Intersection of Line Systems in Higher Space."

Mr. H. W. Richmond read a paper "Surfaces on which lie Five Systems of Conics."

Mr. R. F. Budden read a paper *"The Summation of certain Kapteyn Series of Bessel Functions, with an Application to Electromagnetic Radiation."

Mr. E. C. Titchmarsh made an informal communication on "Conjugate Functions."

The following papers were communicated by title from the Chair :—

On the Parametric Representation of Plane Cubic Curves : H. Hilton.

Analytic Completeness of Sets of Postulates : C. H. Langford.

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* Printed in this volume.

A Contribution to the Theory of Invariant Integrals: F. H. Murray.

Pseudo-orthogonal Functions: B. M. Wilson.

Some Applications of a Transformation of Series due to Schlömilch: J. R. Wilton.

NOTES AND ABSTRACTS.

Some Applications of a Transformation of Series due to Schlömilch

J. R. WILTON.

A slight extension of an equation due to Schlömilch, namely,

$$\sum_{a}^{b}{}' z^n f(n) = \operatorname*{Lt}_{m\to\infty} \sum_{-m}^{m} F(\log z + 2n\pi i),$$

where the accent denotes that the first term is to be halved if a is an integer, and the last term is to be halved if b is an integer, and

$$F(x) = \int_a^b e^{xt} f(t)\, dt,$$

may be employed to obtain a large number of transformations of series, many of which exhibit the behaviour of the sums of power series as z tends to a point on the circle of divergence; a typical example is

$$\operatorname*{Lt}_{z\to 1} \left\{ \sum_1^\infty \frac{z^n}{\sqrt{n}} \cos 2x\sqrt{n} - \sqrt{\pi} \left(\log \frac{1}{z}\right)^{-\frac{1}{2}} \exp\left(\frac{x^2}{\log z}\right) \right\}$$

$$= \zeta(\tfrac{1}{2}) - \sum_1^\infty \frac{1}{\sqrt{n}} \left(1 - \cos\frac{x^2}{2n\pi} - \sin\frac{x^2}{2n\pi}\right).$$

In many cases the transformations lead to expressions in finite terms for the sums of Fourier series, several of which are also extensions of Schlömilch series; a number of null-functions, some of them of very general form, may also be obtained in this way.

For instance, if $y \neq 0$,

$$\tfrac{1}{2}\pi y + \sum_1^\infty \frac{J_1(2n\pi y)}{n} \cos 2n\pi x = \frac{1}{y} \sum_{-y-x}^{y-x} \{y^2 - (n+x)^2\}^{\frac{1}{2}};$$

while, if $R(\mu+\nu) > 0$, and ξ and η are arbitrary complex numbers,

$$\sum_{-\infty}^{\infty} (-)^n \frac{J_{\mu+ny}(\xi)}{\xi^{\mu+ny}} \frac{J_{\nu-ny}(\eta)}{\eta^{\nu-ny}} = 0 \quad (0 < y \leqslant 1),$$

except that, if $y = 1$, $\xi^2 \neq \eta^2$.

Finally the same transformation may be employed to obtain an extension of the Euler-Maclaurin sum-formula in a form which makes it possible, for instance, to write down an asymptotic expression for the remainder after n terms of a power series (or Fourier series) when n is large. The general formula of this type is

$$\sum_{N+1}^{\infty} z^n f(n) = z^N \sum_{m=0}^{r-1} \frac{f^{(m)}(N)}{m!} \left(z \frac{d}{dz}\right)^m \left(\frac{z}{1-z}\right) + R_r \quad (z \neq 1),$$

where

$$|R_r| \leqslant \sum_{-\infty}^{\infty} \{(\log|z|)^2 + (2n\pi + \arg z)^2\}^{-\frac{1}{2}r} \int_N^{\infty} e^{t\log|z|} |f^{(r)}(t)| \, dt.$$

On the Parametric Representation of Plane Cubic Curves

HAROLD HILTON.

It is well known that the coordinates of a point on a non-singular plane cubic curve can be expressed rationally in terms of elliptic functions. For this purpose it is usual to throw the equation of the cubic into the form

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 \qquad \text{(i)}$$

or

$$z^2 x = y(y-x)(y-k^2 x) \quad (0 < k^2 < 1). \qquad \text{(ii)}$$

The form

$$z^2(y-x) = (y+x)(k^2 x^2 + k'^2 y^2) \quad (k^2 + k'^2 = 1) \qquad \text{(iii)}$$

has been also suggested,* since (ii) is not available for the one-circuited cubic.

By means of (i) we can express the coordinates of any point on the curve rationally in terms of $\wp(u)$ and $\wp'(u)$, while by means of (ii) and (iii) we can express the coordinates rationally in terms of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

Now the forms (i), (ii), (iii) are equations of the cubic which are not useful for any other purpose, and the question arises whether the coordinates can be expressed rationally in terms of $\wp(u)$, $\operatorname{sn} u$, &c., in any simple manner when the cubic is put into either of its canonical forms

$$(x+y+z)^3 + 6nxyz = 0, \qquad \text{(iv)}$$

$$x^3 + y^3 + z^3 + 6mxyz = 0. \qquad \text{(v)}$$

* H. Hilton, *Plane Algebraic Curves*, 255.

It will suffice to take equation (iv), since these two forms can be at once transformed into each other.* It will be found that any point on this curve may be taken as

$$x : y : z = \{\wp(u)-\wp(\tfrac{2}{3}\omega)\}^{-1} : \{\wp(u+\tfrac{2}{3}\omega)-\wp(\tfrac{2}{3}\omega)\}^{-1} : \{\wp(u+\tfrac{4}{3}\omega)-\wp(\tfrac{2}{3}\omega)\}^{-1}, \tag{vi}$$

where
$$n = -24\{\wp(\tfrac{2}{3}\omega)\}^2/\wp''(\tfrac{2}{3}\omega). \tag{vii}$$

As an alternative we have

$$x : y : z = f(u) : f(u+\tfrac{4}{3}K) : f(u+\tfrac{8}{3}K), \tag{viii}$$

in which for the two circuited cubic

$$f(u) = \operatorname{sn}^2 u/(\operatorname{sn}^2 u-\operatorname{sn}^2 \tfrac{4}{3}K), \tag{ix}$$

or as alternatives,

$$f(u) = \operatorname{sn} u \operatorname{sn}(u+\tfrac{4}{3}K) \operatorname{sn}(u+\tfrac{8}{3}K)+\operatorname{sn}^2 \tfrac{4}{3}K \operatorname{sn} u, \tag{x}$$

$$f(u) = (1-k^2 \operatorname{sn}^4 \tfrac{4}{3}K) \operatorname{sn} u+\operatorname{sn}(u+\tfrac{4}{3}K)+\operatorname{sn}(u+\tfrac{8}{3}K), \tag{xi}$$

where
$$n = -(3-k^2 \operatorname{sn}^4 \tfrac{4}{3}K)^3/6(1-k^2 \operatorname{sn}^4 \tfrac{4}{3}K); \tag{xii}$$

while for the one-circuited cubic

$$f(u) = (1-\operatorname{cn} u)/(\operatorname{cn} \tfrac{4}{3}K-\operatorname{cn} u) \tag{xiii}$$

or
$$f(u) = \operatorname{sn}^2 \tfrac{4}{3}K \operatorname{sn} u\{\operatorname{cn}(u+\tfrac{4}{3}K)+\operatorname{cn} \tfrac{4}{3}K\}\{\operatorname{cn}(u+\tfrac{8}{3}K)+\operatorname{cn} \tfrac{4}{3}K\}$$
$$+4 \operatorname{cn}^2 \tfrac{4}{3}K \operatorname{sn} u \operatorname{sn}(u+\tfrac{4}{3}K) \operatorname{sn}(u+\tfrac{8}{3}K), \tag{xiv}$$

where
$$n = -(1+4 \operatorname{cn} \tfrac{4}{3}K+\operatorname{cn}^2 \tfrac{4}{3}K)^3/3 \operatorname{cn} \tfrac{4}{3}K(1+\operatorname{cn} \tfrac{4}{3}K)^4. \tag{xv}$$

We may, of course, at once verify these results; the only difficulty is to obtain them in the first instance. Equations (vi) were obtained by writing down the real inflexional tangents of (i), namely the tangents at $u = 0$, $\frac{2}{3}\omega$, $\frac{4}{3}\omega$. They are $X = 0$, $Y = 0$, $Z = 0$, where

$$X = 2\wp''(\tfrac{2}{3}\omega)\, z,$$

$$Y, Z = -12\wp(\tfrac{2}{3}\omega)\, x \mp \{12\wp(\tfrac{2}{3}\omega)\}^{\frac{1}{2}} y + \{12\wp^2(\tfrac{2}{3}\omega)-\wp''(\tfrac{2}{3}\omega)\}\, z.$$

If we put
$$x = \wp(u), \quad y = \wp'(u), \quad z = 1$$

in these, we have values of $X : Y : Z$ which are equivalent to the values of $x : y : z$ given by (vi).

---

* *Plane Algebraic Curves*, 233, where, however, read $(2m-1)x+y+z$ in line 11 for $(2m+1)x+y+z$, and $8(2m+1)^2(1-m)^3$ in line 15 for $(2m+1)^2$.

Equations (ix), (xiii) were obtained in the same way, and were verified by deducing them from (vi), using

$$\wp(u) = a^2\{1/\mathrm{sn}^2 au - \tfrac{1}{3}(1+k^2)\}, \quad (\wp)u = \tfrac{1}{2}a^2\{1/(1-\mathrm{cn}\, au) - \tfrac{1}{6}(1+4k^2)\},$$

according as $g_2^3 - 27g_3^2$ is positive or negative.

For the acnodal cubic (iv), with $n = -\frac{9}{2}$, we may take

$$x : y : z = \sin^3\phi : \sin^3(\phi+\tfrac{2}{3}\pi) : \sin^3(\phi+\tfrac{4}{3}\pi).$$

Formulae for the Intersection of Line Systems in Higher Space

C. G. F. JAMES.

The systems of a space $[r]$ of dimension r are divided into those of *unrestricted type*, for which the dimension i exceeds $r-2$, and the *restricted type*, for which it is less than $r-1$. The following definitions are given and justified :—

The *order*, m, of an unrestricted system ∞^i equals the order of the cone of its lines through a fixed point; and the *s-th class*, n_s, equals the order of the construct of its lines in a space $[2r-i-s-1]$ of $[r]$.

The *order*, m, of a restricted system equals the order of the construct generated by its lines; and the *s-th class*, n_s, equals the order of the construct of its lines in a space $[r-s]$ of $[r]$.

The principal theorems of the paper are :—

I. The number of lines common to two generalized congruences, or systems ∞^{r-1}, equals

$$mm' + \sum_\rho n_\rho n'_\rho,$$

where the characteristics of the two systems are distinguished by unaccented and accented symbols.

II. The same formula also expresses the number of lines common to a restricted system of dimension i, and an unrestricted system of dimension j, where

$$i+j = 2r-2.$$

It is seen that the characteristics of the system common to two given systems can be read off, using these theorems, from certain systematic preliminary results.

Surfaces on which lie Five Systems of Conics

H. W. RICHMOND.

1. As long ago as 1871 Darboux* discovered that, if tangent planes are drawn from a fixed point to a family of confocal quadrics, the locus of the points of contact is a surface of order five, upon which lie five distinct systems of conics. Through any ordinary point of the surface pass five conics, one from each system; any two conics of different systems have in general one common point. The surface can be mapped, point for point, upon a plane; the coordinates of points on the surface are, in fact, proportional to cubic functions of coordinates of points in a plane, which represent cubic curves through four common points. By considering parametric equations of a certain type, we find that the surface of Darboux is one of many surfaces possessing five such systems of conics, which exist in ordinary space and in space of four dimensions,† all derivable by projection from a fundamental surface in space of five dimensions.‡ From the fundamental surface in S_5 projection derives other surfaces of order five or less than five in S_4, or in ordinary space S_3; from the lines, conics, etc. that lie on the fundamental surface projection derives similar curves on the projected surfaces.

2. *Rational Parameter-equations. Invariantive functions.* If the points of a locus have coordinates proportional to polynomials (rational homogeneous algebraic functions of the same degree) of r parameters, the locus is generally of $r-1$ dimensions. The number of dimensions is reduced if the polynomials are such that an infinite number of sets of values of the parameters give the same point; for instance, if the polynomials are functions of differences of the parameters, the locus is of $r-2$ dimensions, since the same coordinates are obtained by giving to the parameters the values $\theta_1+\gamma$, $\theta_2+\gamma$, ..., $\theta_r+\gamma$, as by giving them the values θ_1, θ_2, ..., θ_r.

The number of dimensions of the locus is reduced to $r-3$ *if the polynomials are invariantive functions of the parameters* θ; *i.e.* if the mutual ratios of the polynomials are unchanged when every parameter θ_s is replaced by

$$(p\theta_s+q)/(m\theta_s+n)$$

whatever p, q, m, n may be. Invariantive functions§ are products of

* "Sur une surface du cinquième ordre et sa représentation sur le plan", *Bull. des sciences math. et astron.* (1), 2 (1871), 40–64.

† Some of them already known.

‡ "Surface" here denotes a locus or manifold of two dimensions, *i.e.* a *two-fold*. A space of n dimensions is denoted by S_n.

§ E. B. Elliott, *Algebra of Quantics*, 96 (§75).

differences of the parameters in which each parameter appears the same number of times, or sums of such functions. Thus with six parameters, $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$, the fifteen functions of the type

$$(\theta_1-\theta_5)(\theta_3-\theta_6)(\theta_2-\theta_4)$$

are invariantive, and a locus defined by the property that the coordinates of its points are proportional to such functions is not a five-fold, but a three-fold. Moreover, if we take a set of parameters defining a point on the locus and allow one parameter to vary, we obtain a straight line lying wholly on the locus; through every point of the locus pass six such lines, belonging to six separate systems. But these functions were noticed long ago by Joubert* in connexion with a proposed resolvent of the sextic equation; and the locus was shown by Castelnuovo† to be Segre's cubic variety in S_4 with ten double points. It does not need discussion here.

With more than six parameters the matter grows very complicated; little or nothing is known of the invariantive functions.‡ But loci of points whose coordinates are proportional to invariantive functions of five parameters,§ though less simple than those of six, are not difficult to discuss and have some interest. The loci are two-folds, *i.e.* "surfaces"; some of them, like the surface of Darboux in § 1, are already known; but, even so, it appears to be worth while to bring them together and derive them and their properties from one fundamental source, the surface in five dimensions. The present paper is no more than a sketch of what may be done with this object in view.

3. *The simplest invariantive functions of five parameters.* There are no invariantive functions of five parameters $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ in which each parameter occurs only once, but there are twelve like

$$(\theta_1-\theta_4)(\theta_4-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_5)(\theta_5-\theta_1),$$

and ten like $$(\theta_1-\theta_3)^2(\theta_2-\theta_4)(\theta_4-\theta_5)(\theta_5-\theta_2),$$

in which each parameter occurs in the second degree. Any surface defined by the fact that the coordinates of its points are proportional to certain of these functions, or to sums of these functions, each multiplied by a numerical constant, must have five conics, which belong to five

* *Comptes Rendus* (1st sem. 1867), 1025, 1081.

† *Atti di Reale Istituto Veneto* (7), 2 (1891), 855–901.

‡ From a short notice in the *Bull. Amer. Math. Soc.*, 29 (1923), 147, of a paper on "The Theory of the Octavic", by Miss A. M. Whelan, it seems that ground has been broken for the case of eight parameters; but the note is very brief.

§ Invariantive functions of four parameters may be passed over.

separate systems and lie wholly on the surface, passing through each of its points. For let each parameter in turn vary, the other four remaining constant; since each parameter occurs in each coordinate in the second power, the point describes a conic which belongs to the first, second, third, ... system according as the parameter that varies is $\theta_1, \theta_2, \theta_3, \ldots$. Again, if we take any two points on the surface, excluding those for which two of the parameters are equal, we may transform the parameters of one by substituting $(p\theta+q)/(m\theta+n)$ for θ without altering its coordinates, in such a way as to make three of the parameters (say $\theta_3, \theta_4, \theta_5$) the same for both points; the two points now have parameters $(\theta_1\theta_2\theta_3\theta_4\theta_5)$ and $(\phi_1\phi_2\theta_3\theta_4\theta_5)$. Clearly the point $(\phi_1\theta_2\theta_3\theta_4\theta_5)$ is common to the conic of the first system through the former point, and to the conic of the second system through the latter. Any two conics of different systems have one common point.

This reasoning holds good for surfaces in space of any number of dimensions, but we shall find that only six of the twenty-two functions given above are linearly independent. Every one of the surfaces just considered therefore lies wholly in a space of five dimensions; some of them lie wholly in space of four or three dimensions. For, since we may select six of the functions quoted above and express all the rest in terms of the six by linear equations, it follows that six homogeneous coordinates suffice to define all the surfaces under discussion, and a space of five dimensions must contain them.

4. *Ordinary parametric equations of these surfaces.* By substituting $(p\theta+q)/(m\theta+n)$ for θ, which does not affect the coordinates, we may make certain of the parameters (three at most) have assigned values. Except for those points where θ_4 and θ_5 are equal, we may take θ_5 to be infinite and θ_4 to be zero. The symmetry of the surface and its five systems of conics is lost, or rather is obscured, but the coordinates of its points are now seen to be proportional to homogeneous polynomials in three parameters, $\theta_1, \theta_2, \theta_3$, as is usual with a rational algebraic surface. The two invariantive functions quoted at the beginning of § 3 become proportional to

$$\theta_1\theta_2(\theta_2-\theta_3), \quad (\theta_1-\theta_3)^2\theta_2$$

respectively, and on examination it appears that all the twenty-two such functions are reduced to cubic polynomials in $\theta_1, \theta_2, \theta_3$, which vanish when $\theta_1, \theta_2, \theta_3$ receive the values

$$(1, 0, 0), \ (0, 1, 0), \ (0, 0, 1), \ (1, 1, 1);$$

if we regard θ_1, θ_2, θ_3 as coordinates of points in a plane, the polynomials equated to zero represent cubic curves passing through four fixed points in the plane.* Only six of these curves are linearly independent; this justifies the similar statement concerning the invariantive functions. Two cubics through four given points intersect in five other points, of which four may be chosen arbitrarily, and determine the fifth; the surfaces under consideration are therefore generally of order five, but may be lower. The surfaces that belong to S_5, *i.e.* are not contained in a space of fewer than five dimensions, *must* be of order five. They are cut by an arbitrary S_3 in five points, four of which suffice to define the S_3, and so determine the fifth point. The section by an arbitrary S_4 is a curve of order five, which, being in $(1, 1)$-correspondence with a plane cubic, is of genus $p = 1$.

5. The surfaces in space of five dimensions seem to be of fundamental importance and likely to repay further study, both as illustrating invariantive functions of five parameters and because of the derived loci in ordinary space and in S_4. The surfaces in S_5 are all projectively identical —examples of a unique surface represented by equations which involve no arbitrary constant. On it lie ten straight lines, loci of points at which two parameters are equal. Every conic on the surface meets four of the ten lines, and the planes of the conics are defined by the fact that they meet the four fixed straight lines. They form interesting systems of planes, for two planes of different systems (like the conics they contain) have one common point; and therefore any five planes, one from each system, lie on a quadric. There are also five systems of twisted cubics on the surface, residual to the conics, a double infinity in each system.

I have to thank Mr. C. G. F. James for two references :—

1. F. Severi, "Sulle intersezioni delle varieta algebriche, e sopra i loro caratteri e singolarita proiettivi," *Mem. Torino* (2), 52 (1903), 61–188, where this surface is noted on page 102.

2. P. del Pezzo, "Sulle superficie dell' n^{mo} ordine immerso nello spazio di n dimenzioni," *Rend. di Palermo*, 1 (1887), 241–271. *All* surfaces whose points are proportional to cubic functions of three parameters are considered in this well known paper. The representation by invariantive functions, an almost unique property of this surface and Segre's variety, is not mentioned; and to the writer this appears the chief reason for further investigation of this particular surface.

* Cf. § 1.

Thursday, January 15th, 1925.

Prof. A. L. DIXON, President, in the Chair.

There were present twenty-one members.

Dr. G. Pólya, Prof. J. L. Synge, and Dr. W. H. Thürow were elected members of the Society.

Mr. E. C. Titchmarsh made an informal communication *"A Theorem on Trigonometrical Series."

Mr. R. G. Cooke read a paper *"The Inversion Formulae of Hardy and Titchmarsh."

Mr. J. E. Littlewood made an informal communication on "A Point in the Theory of Conformal Representation."

The following papers were communicated by title from the Chair :—

On the Expression of Numbers in the Form $ax^2+by^2+cz^2+dt^2$: H. D. Kloosterman.

Two theorems on Symmetric Sums of Determinants : L. L. Dines.

A further note on the converse of Abel's theorem : G. H. Hardy and J. E. Littlewood.

Some Identities in the Theory of Numbers : A. Oppenheim.

NOTES AND ABSTRACTS.

Two Theorems on Symmetric Sums of Determinants

L. L. DINES

The first theorem is relative to a certain transformation

$$T_{\substack{s_1 s_2 \dots s_n \\ \alpha_1 \alpha_2 \dots \alpha_n}} D \tag{1}$$

of an n-th order determinant D which may be described as follows. The subscripts $\alpha_1, \alpha_2, \dots, \alpha_n$ denote a sequence of numbers (not necessarily distinct), each chosen from the set $0, 1, 2, \dots, n$, but not all zero. For each κ the symbol s_κ denotes a sequence of $n-\alpha_\kappa$ numbers

$$s_\kappa \equiv (s_{1\kappa}, s_{2\kappa}, \dots, s_{n-\alpha_\kappa, \kappa})$$

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* Printed in this volume.

subject only to the restriction that if $a_i = a_j$ then $s_i \equiv s_j$. The notation (1) then signifies the determinant obtained from D by shifting the elements of the κ-th column upward a_κ places (thus eliminating the upper a_κ of them from the determinant), filling in the vacant places at the bottom with zeros, and multiplying the non-zero elements of the resulting column respectively by the elements of the sequence s_κ; this procedure to be followed for each column of the determinant.

By the notation

$$ST_{\substack{s_1 s_2 \dots s_n \\ a_1 a_2 \dots a_n}} D$$

we shall mean the sum of all the determinants

$$T_{\substack{s_{i_1} s_{i_2} \dots s_{i_n} \\ a_{i_1} a_{i_2} \dots a_{i_n}}} D,$$

where $(i_1, i_2, \dots, i_n)$ is a permutation of the set of subscripts $(1, 2, \dots, n)$, with the understanding that if (owing to the possible equality of two or more a's) different permutations give rise to identical sequences of a's, only one such permutation shall be represented in the sum. *The theorem is then to the effect that*

$$ST_{\substack{s_1 s_2 \dots s_n \\ a_1 a_2 \dots a_n}} D \equiv 0.$$

The second theorem has to do with sums of determinants of which the elements are elements of a persymmetric array

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & c_4 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ c_n & c_{n+1} & c_{n+2} & \dots, \end{array}$$

each multiplied by a binomial coefficient according to a well-defined rule.

Denoting the binomial coefficients by the notation

$$(m)_\kappa = \frac{m(m-1)\dots(m-\kappa+1)}{1 \cdot 2 \dots \kappa},$$

we state the theorem:

For any non-negative integer p, and any sequence of non-negative

integers $r_1, r_2, \ldots, r_n$,

$$\sum_{j_1+j_2+\ldots+j_n=p}$$

$$\begin{vmatrix} (j_1+r_1+1)_{r_1+1}\, c_{j_1+r_1+1} & (j_2+r_2+1)_{r_2+1}\, c_{j_2+r_2+1} & \cdots & (j_n+r_n+1)_{r_n+1}\, c_{j_n+r_n+1} \\ (j_1+r_1+2)_{r_1+2}\, c_{j_1+r_1+2} & (j_2+r_2+2)_{r_2+2}\, c_{j_2+r_2+2} & \cdots & (j_n+r_n+2)_{r_n+2}\, c_{j_n+r_n+2} \\ \vdots & \vdots & \vdots & \vdots \\ (j_1+r_1+n)_{r_1+n}\, c_{j_1+r_1+n} & (j_2+r_2+n)_{r_2+n}\, c_{j_2+r_2+n} & \cdots & (j_n+r_n+n)_{r_n+n}\, c_{j_n+r_n+n} \end{vmatrix}$$

$$= \sum_{j_1+j_2+\ldots+j_n=p} (j_1+r_1+n)_{r_1+n}\ (j_2+r_2+n)_{r_2+n}\ \cdots\ (j_n+r_n+n)_{r_n+n}$$

$$\begin{vmatrix} c_{j_1+r_1+1} & c_{j_2+r_2+1} & \cdots & c_{j_n+r_n+1} \\ c_{j_1+r_1+2} & c_{j_2+r_2+2} & \cdots & c_{j_n+r_n+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{j_1+r_1+n} & c_{j_2+r_2+n} & \cdots & c_{j_n+r_n+n} \end{vmatrix}.$$

It will be noted that the determinants on the right side of the identity are determinants from the persymmetric array, unaffected by binomial coefficients.

A Theorem on Trigonometrical Series

E. C. TITCHMARSH

It is known from the work of F. Riesz, Fischer, Young, and Hausdorff that, if the series

$$\sum_{n=1}^{\infty} |a_n|^{p/(p-1)} \quad (p \geqslant 2)$$

is convergent, then the series

$$\Sigma a_n e^{inx} \tag{1}$$

converges "in mean" with exponent p; *i.e.* there is a unique function

$f(x)$, whose p-th power is integrable, such that

$$\lim_{m\to\infty}\int_a^b |f(x)-\sum_1^m a_n e^{inx}|^p\,dx = 0.$$

It is natural to inquire in what circumstances the same is true of the more general series

$$\Sigma a_n e^{i\lambda_n x}, \tag{2}$$

where λ_n denotes any increasing sequence of real numbers. I have observed a set of conditions under which the result holds.

THEOREM.—*Suppose there are constants* K, q $(0<q\leqslant 1)$, *such that*

$$|a_n|^q \leqslant K(\lambda_{n+1}-\lambda_m)$$

for all values of n, *and the series*

$$\Sigma |a_n|^{(p-q)/(p-1)} \quad (p\geqslant 2)$$

is convergent. Then the series (2) *converges in mean with exponent* p.

The theorem is trivial if $q=1$. I have proved it in the general case by approximating to the series by means of integrals, and using the theory of Fourier transforms.* I have not been able to construct a more direct proof. I hope to give more details later, if the conditions found turn out to be of importance. They are, of course, by no means necessary.

Some Identities in the Theory of Numbers

A. OPPENHEIM

1. *Notation.*—If x is a positive integer, let

$$\sigma_k(x) = \sum_{d|x} d^k,$$

and, if x is not a positive integer, let

$$\sigma_k(x) = 0.$$

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* E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 23 (1924), 279–289. For references to the other authors, see p. 281.

By $r_p(x)$ I denote the number of integral solutions of the equation

$$x_1^2+\ldots+x_p^2 = x \quad (p \geqslant 2),$$

so that $r_p(x)$ is the number of lattice points on the hyper-sphere $\Sigma x_i^2 = x$. Finally I write

$$F_\nu(z) \equiv \cos \tfrac{1}{2}\nu\pi\, J_\nu(z) - \sin \tfrac{1}{2}\nu\pi \left\{ Y_\nu(z) + \frac{2}{\pi} K_\nu(z) \right\},$$

where J_ν, Y_ν, K_ν are the Bessel functions of order ν usually so designated*; and $\Phi_k(x)$ for the sum of the residues of the function†

$$\frac{\zeta(z)\,\zeta(z-k)}{z}\, x^z.$$

2. It is the object of this paper to prove the following theorems.

THEOREM 1.‡—*If k is real and x is positive, then*

$$\sum_{n \leqslant x} \sigma_k(n) - \tfrac{1}{2}\sigma_k(x) = \Phi_k(x) + x^{\frac{1}{2}+\frac{1}{2}k} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}} F_{1+k}(4\pi\sqrt{nx}),$$

the series being convergent if $|k| < \frac{1}{2}$, and summable $(R, n, |k| - \frac{1}{2} + \epsilon)$ for any $\epsilon > 0$ if $|k| \geqslant \frac{1}{2}$; and the convergence and summability being uniform in any closed interval free from integral x.

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* G. N. Watson, *Theory of Bessel Functions*, Cambridge (1922), §§ 3.1 (8), 3.54 (1), 3.7 (6).

† In particular

$$\Phi_0(x) = x\log x + (2\gamma-1)x + \tfrac{1}{4}; \quad \Phi_{-1}(x) = \tfrac{1}{6}\pi^2 x - \tfrac{1}{2}(\gamma + \log 2\pi + \log x);$$

and, if $k \neq 0, -1$, $\quad \Phi_k(x) = x\zeta(1-k) + x^{1+k}\zeta(1+k)/(1+k) - \tfrac{1}{2}\zeta(-k)$.

‡ The case $k = 0$ was proved initially by G. Voronoï, "Sur une fonction transcendante et ses applications à la sommation de quelques séries", *Ann. de l'Ecole Norm. Sup.* (3), 21 (1904), 207–267 and 459–533 (513); later by G. H. Hardy, "On Dirichlet's Divisor Problem", *Proc. London Math. Soc.* (2), 15 (1915), 1–25 (16–22); and quite recently by W. Rogosinski, "Neue Anwendung der Pfeifferschen Methode bei Dirichlets Teilerproblem", *Inauguraldissertation*, Göttingen (1922), 1–32. The proofs by Voronoï and Hardy involve complex analysis; that by Rogosinski uses real analysis.

The cases $k = \pm 1$ are due to A. Walfisz, "Über die summatorischen Funktionen einiger Dirichletscher Reihen", *Inauguraldissertation*, Göttingen (1922), 52–53; he, however, only gives summability $(R, n, 1)$ whereas, actually, when $k = \pm 1$, the series are summable $(R, n, \frac{1}{2}+\epsilon)$.

THEOREM 2.*—*If $x > 0$ and p is an integer $\geqslant 2$, then*

$$\sum_{0 \leqslant n \leqslant x} r_p(n) - \tfrac{1}{2} r_p(x) = \frac{\pi^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)} x^{\frac{1}{2}p} + x^{\frac{1}{2}p} \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{\frac{1}{2}p}} J_{\frac{1}{2}p}(2\pi\sqrt{nx}),$$

the series being summable $[R, n, \{\frac{1}{2}(p-3)\}+\epsilon]$ if $p \geqslant 3$, and convergent if $p = 2$, and the summability and convergence being uniform in any closed interval free from integral x.

The next two theorems show what happens when we formally differentiate the identities of Theorems 1 and 2.

THEOREM 3.—*If x is positive and not integral, then the series*

$$2\pi x^{\frac{1}{2}k} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \left\{ -\sin \tfrac{1}{2}k\pi J_k(4\pi\sqrt{nx}) - \cos \tfrac{1}{2}k\pi \left(Y_k(4\pi\sqrt{nx}) - \frac{2}{\pi} K_k(4\pi\sqrt{nx}) \right) \right\},$$

obtained by formal differentiation of the series of Theorem 1, is summable $(R, n, |k|+\frac{1}{2}+\epsilon)$ to the sum

$$-\Phi_k'(x).\dagger$$

THEOREM 4.—*If x is positive and not integral, then the series*

$$\pi x^{\frac{1}{2}p-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{\frac{1}{2}p-\frac{1}{2}}} J_{\frac{1}{2}p-1}(2\pi\sqrt{nx})$$

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* The case $p = 2$ was first stated but not proved by Voronoï, "Sur la developpement, à l'aide des fonctions cylindriques, des sommes doubles $\Sigma f(pm^2+2qmn+rn^2)$, où $pm^2+2qmn+rn^2$ est une forme positive à coefficients entiers", *Verhandlungen des dritten internationalen Math.-Kongresses in Heidelberg* (1904), 241–245. The first rigorous proof was given by Hardy, "On the Representation of a Number as the Sum of Two Squares", *Quart. J. of Math.*, 46 (1915), 268–283; this proof uses complex analysis. The next (by real analysis) was given by E. Landau, "Über die Gitterpunkte in einem Kreise (3)", *Göttinger Nachrichten* (1920), 109–134. Simplified forms of these proofs have recently been published in a joint paper by Hardy and Landau, "The Lattice Points of a Circle", *Proc. Royal Soc.* (A), 105 (1924), 244–258. By developing a method used in the latter paper, Hardy has obtained yet another proof by real analysis: see "The Lattice Points of a Circle", *Proc. Royal Soc.* (A), 107 (1925), 623–635. Hardy and Laudau give also the corresponding identity for the ellipse.

The general theorem, but with summability $[R, n, \{\frac{1}{2}(p-1)\}]$, is due to Walfisz, *l.c.*, 54. Walfisz, indeed, has given the more general identity corresponding to the sum

$$\Sigma \exp\left(2\pi i \sum_{1}^{p} v_j h_j\right),$$

where $Q(w) = Q(w_1 \ldots w_p)$ is a positive quadratic form in p variables, the numbers h and z are given real numbers, and the summation extends over all integers v such that $0 < Q(v+z) \leqslant x$.

It will be observed that Theorem 2 improves the order of summability required by Walfisz, by nearly $\frac{1}{2}$ if p is even and $\geqslant 4$, and by nearly 1 if p is odd and $\geqslant 3$.

† $\Phi_0'(x) = \log x + 2\gamma$; $\Phi_k'(x) = \zeta(1-k) + x^k \zeta(1+k)$, if $k \neq 0$.

is summable $[R, n, \{\frac{1}{2}(p-1)\}+\epsilon]$ *to the sum*

$$-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\, x^{\frac{1}{2}p-1}.$$

The idea of investigating these differentiated identities was suggested to me by Prof. Hardy, to whom I am deeply indebted for much assistance and advice.

3. The method of proof is based on the first proof, in the joint paper of Hardy and Landau,* of the "circle-identity." It depends essentially on the discontinuous factor

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H_\mu^{(1)}(isa)\, H_{\mu+1}^{(2)}(isb)\, ds = \begin{cases} 0 & (a > b) \\ (1/\pi b) & (a = b) \\ (2a^\mu/\pi b^{\mu+1}) & (a < b) \end{cases},$$

where $c > 0$, $a > 0$, $b > 0$ and $H_\mu^{(1)}$, $H_\mu^{(2)}$ are Hankel's cylinder-functions† ; secondly, it depends on the analytic continuations of the functions $\Phi(s)$ and $\Psi(s)$ defined, when $\sigma > 0$, by what are practically Dirichlet's series,

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}}\, H_k^{(1)}(4\pi is\sqrt{n}), \qquad \Psi(s) = \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{\frac{1}{4}p-\frac{1}{2}}}\, H_{\frac{1}{2}p-1}^{(1)}(2\pi is\sqrt{n})\,;$$

and thirdly on an elaborate use of Cauchy's theorem.

GENERAL MEETING.

Thursday, February 12th, 1925.

Prof. A. L. DIXON, President, in the Chair.

There were present twenty-five members.

The President announced that Dr. H. Jeffreys would give a lecture on "The Interior of the Earth" at the meeting of April 23rd, and that Mr. Littlewood would give a lecture at the June meeting.

Prof. H. Levy read a paper on ‡"A Numerical Solution of a Certain Differential Equation."

Dr. G. Pólya read two papers : (1) § "On an Integral Function of an Integral Function"; (2) "On the Minimum Modulus of an Integral Function of Order less than Unity."

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* *l.c.*, § 2.

† Watson, *l.c.*, 73.

‡ Printed in this volume.

§ Printed in *Journal London Math. Soc.*, 1 (1926).

Mr. T. L. Wren read a paper* (written in collaboration with Miss H. P. Hudson) on "Involutory Point-pairs in the Quadro-quadric Cremona Space-transformation."

Lt.-Col. Allan Cunningham called the attention of the meeting to certain tables of squares and cubes.

The following papers were communicated by title from the Chair :—

On Some Cases of Two-dimensional Fluid Motions : D. Riabouchinsky.

On a Hydrodynamical Problem considered by Mr. Riabouchinsky : E. G. C. Poole.

SPECIAL GENERAL MEETING.

Thursday, February 12th, 1925.

Prof. A. L. DIXON, President, in the Chair.

There were present twenty-five members.

Resolutions 1 and 2—

1. That By-law V (1) be altered by substituting "five o'clock" for "half-past five o'clock";

2. That By-law X (1) be cancelled and that the following new By-law be adopted in its place :—

"X (1). The Council shall provide as they shall from time to time think expedient for the custody of the Common Seal of the Society and of all deeds and other documents relating to the trust funds of the Society and to the investments which are the property of the Society";

were proposed by the President, seconded by the Treasurer, and carried by a two-thirds majority.

Resolution 3—

3. That the following new By-law be adopted :—

XI. OF THE LIBRARY.

"(1) The Library of the Society shall be kept at such place as shall from time to time be determined by the Council. Every member of the Society shall be entitled to take books out of the Library, and shall return them within one month. Any member who shall not return any book taken out by him on being required to do so shall pay to the Treasuer the cost of another copy, if procurable, or such compensation as the Council may decide";

* Printed in this volume.

"(2) There shall be a Librarian who shall be one of the Officers of the Society referred to in No. 7 of the Articles of the Society. The Librarian may be one of the other Officers of the Society";

was proposed by the President and seconded by Prof. Watson.

The Treasurer then moved, as an amendment, to delete the words "one month" and substitute "such period as the Council shall from time to time determine."

The amendment was seconded by Prof. Neville and agreed to unanimously.

The resolution as amended was then carried by a two-thirds majority.

NOTES AND ABSTRACTS.

Involutory Point-Pairs in the Quadro-Quadric Cremona Space Transformation

H. P. HUDSON *and* T. L. WREN.

1. The only references to this subject known to the writers are :—

(i) Aschieri, *Rend. R. Ist. Lomb.* (2), 14 (1881), 21–26 (25), states that it is obvious that there are no such pairs in the general case, and one if the collinear sheaves have a united plane.

(ii) Doehlemann, *Geometrische Transformationen II* (1908), 189, states that there are no such pairs because a certain pair of twisted curves do not in general intersect; no proof is offered of the last statement. The attempt to prove this has led to the result that there are two such pairs.

The symbol $\sim$ is here used for "corresponds to." Thus, when a given collineation **C** transforms a point P into P', this will be written : in **C**, the point $P \sim P'$.

2. A quadratic Cremona transformation **U** between two superposed spaces can be defined by means of a collineation **C** between the sheaves through two points S, T and a correlation **L**. In **C**, a ray $SP \sim$ a ray TP'; and in **L**, a point $P \sim$ a plane π, and a plane $\pi^{-1} \sim P$. Then in **U**, the point $P \sim$ the point of intersection P' of TP', π.

Let P, P' be involutory for **U**; then (i) in **C** we have $SP \sim TP'$ and $SP' \sim TP$; and (ii) in **L** we have P' lying on both the planes π, π^{-1}.

3. First consider pairs P, P' which satisfy (i) only. Let the ray a, through S, $\sim TS$, and let the ray $ST \sim$ the ray b through T. In general, a, b are skew. Any plane λ through ST defines two planes α, β, such that $\alpha \sim \lambda$ and $\lambda \sim \beta$. Then if P lies in λ, P' must lie on the meet l_1 of α, β. As P' describes l_1, so SP, TP describe projective pencils in λ, and the locus of P is a conic c.

The planes λ, α, β describe projective pencils about ST, a, b; and the locus of l_1 is a regulus R_1 of a quadric κ, which is the locus of P' and also of P, since P, P' satisfy condition (i). Hence the locus of c is κ.

Since $SPP' \sim TP'P$, corresponding rays of the sheaves determine on $l_2 \equiv PP'$ a projectivity which has P, P' as one involutory pair, and therefore is an involution; as P describes l_2, so P' describes the same line. Hence l_2 lies on κ and is a generator of the other regulus R_2.

Thus we have an involution of pairs P, P' on each generator of R_2.

4. Next consider correspondence in **L**. This determines on any generator l_2 of R_2 a homography between P and the point Q in which l_2 is met by π. If l_2 contains an involutory pair of **U**, this pair must correspond doubly in the homography, which must therefore be an involution.

Now as P describes an arbitrary generator l_1 of R_1, the points Q, Q^{-1} in which π, π^{-1} meet the generator l_2 through P describe space cubics on κ. These cubics are in general distinct, and they have in common the two points, variable with l_1, in which l_1 meets the quadric locus of points lying in their corresponding planes in **L**. They therefore have two other common points lying on two different generators of R_2; and these are the only generators of R_2 upon which **L** determines involutions (P, Q).

On each of these generators we have two involutions (P, P') and (P, Q), of which the first only is dependent on S, T. They are therefore in general different, and have one common pair.

Hence the general quadratic space transformation has two involutory pairs.

5. If a, b intersect, then R_2 is a pair of flat pencils in different planes. The relation between P, P' is still an involution on each ray of R_2. Now **L** determines an involution (P, Q) on a single ray of each of the two pencils; and this type of quadratic transformation also has two involutory pairs.

*A definite integral which occurs in physical optics**

G. H. HARDY.

1. The integral in question is

$$J = \int_0^\infty (C^2+S^2)\,dx,$$

where
$$C = \int_x^\infty \cos t^2\,dt, \qquad S = \int_x^\infty \sin t^2\,dt.$$

Sir Arthur Schuster, in a paper published recently in the *Proceedings* of the Royal Society,† suggests that the value of the integral is

$$\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)},$$

and shows what would be the physical significance of this result.

The correctness of Schuster's conjecture may be proved in various ways. The method which is formally the simplest is as follows. It is plain that

$$C^2+S^2 = \int_x^\infty \int_x^\infty \cos(t^2-u^2)\,dt\,du.$$

If we integrate with respect to x, and ignore any difficulties in changing the order of the integrations, we obtain

$$J = \int_0^\infty dx \int_x^\infty \int_x^\infty \cos(t^2-u^2)\,dt\,du = \int_0^\infty \int_0^\infty \cos(t^2-u^2)\,dt\,du \int_0^w dx,$$

where $w = \mathrm{Min}\,(t, u)$. The integral now falls apart into two equal integrals, of which one is

$$\int_0^\infty dt \int_0^t u\cos(t^2-u^2)\,du = \tfrac{1}{2}\int_0^\infty \sin t^2\,dt = \tfrac{1}{4}\sqrt{(\tfrac{1}{2}\pi)}.$$

We thus obtain Schuster's result almost without calculation; but it is plain that a strict analytical proof would be rather troublesome to write out.

An alternative proof may be based on the theory of "Fourier trans-

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* Read at the meeting of March 12th, 1925.

† Sir A. Schuster, "On the total reflexion of light", *Proc. Royal Soc.* (A), 107 (1925), 15–30, especially p. 25, bottom. I have altered the notation slightly.

forms."* Here we can appeal to known general theorems. If

$$\phi(x) = \int_0^\infty \sin xt\, C(t)\, dt, \qquad \psi(x) = \int_0^\infty \sin xt\, S(t)\, dt,$$

then

$$\int_0^\infty (C^2+S^2)\, dx = \frac{2}{\pi}\int_0^\infty (\phi^2+\psi^2)\, dx.$$

The functions ϕ and ψ may be calculated by partial integration; we find

$$\phi+i\psi = \frac{1+i}{2x}\sqrt{(\tfrac{1}{2}\pi)}(1-\cos\tfrac{1}{4}x^2+i\sin\tfrac{1}{4}x^2),$$

$$\phi^2+\psi^2 = \frac{\pi}{2x^2}(1-\cos\tfrac{1}{4}x^2),$$

$$\int_0 (\phi^2+\psi^2)\, dx = \tfrac{1}{2}\pi\int_0^\infty (1-\cos\tfrac{1}{4}x^2)\frac{dx}{x^2} = \tfrac{1}{4}\pi\int_0^\infty \sin\tfrac{1}{4}x^2\, dx = \tfrac{1}{4}\pi\sqrt{(\tfrac{1}{2}\pi)},$$

$$\int_0^\infty (C^2+S^2)\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)},$$

which is Schuster's formula.

Correction of an error in a paper on certain integral equations

G. H. HARDY *and* E. C. TITCHMARSH.

We take this opportunity of correcting a mistake in our paper published in Vol. 23 of the *Proceedings*.† The mistake occurs on page 8, line 5, where we say: "in each case it is easily seen that the integral with respect to t is uniformly convergent over a finite range of values of x." We assert, in fact, that

$$\int_0^\infty \frac{\chi(t)}{t}\{f(x+t)-f(x-t)\}\, dt$$

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* See M. Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289–335; E. C. Titchmarsh, "Hankel transforms", *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473; M. Plancherel, "Sur les formules d'inversion de Fourier et de Hankel", *Proc. London Math. Soc.* (2), 24 (1925), 62–70.

† G. H. Hardy and E. C. Titchmarsh, "Solutions of some integral equations considered by Bateman, Kapteyn, Littlewood, and Milne", *Proc. London Math. Soc.* (2), 23 (1924), 1–26.

is uniformly convergent over any such interval. This is obvious enough (f being of integrable square) when $\chi(t)$ is $J_1(t)$, as in Kapteyn's equation, but anything but obvious when $\chi(t)$ is e^{-t}, as in Milne's equation. Our argument was written out originally with Kapteyn's equation in mind, and it is no doubt this which accounts for our having overlooked an obvious difficulty. But even in this case there is one point where our argument requires modification.

It seems necessary, if we are to correct the argument without introducing additional hypotheses concerning $f(x)$, to make use of the theory of "conjugate functions." It is known* that, if $f(x)$ is of integrable square, the function

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt$$

exists (though not generally as a Lebesgue integral) for almost all x, and is itself of integrable square. Further, if $f(x)$ is (as, apart from a constant factor, it is in our argument) the Fourier cosine transform of $\lambda(x)$, then $-g(x)$ is the sine transform of the same function.

Suppose then that $\chi(t) = e^{-t}$, so that

$$f'(x) = \tfrac{1}{2}\int_0^\infty e^{-t}\frac{f(x+t)-f(x-t)}{t}\,dt,$$

the existence of this integral for all values of x being one of our data. Then

$$\tfrac{1}{2}\pi g(x)-f'(x) = \tfrac{1}{2}\int_0^\infty \psi(t)\frac{f(x+t)-f(x-t)}{t}\,dt,$$

where $\psi(t) = 1-e^{-t}$; and this integral is uniformly convergent over any finite interval in x. Hence we have

$$f(x+t)-f(x-t) = -2\frac{d}{dx}\int_0^\infty \frac{(1-\cos xu)\sin tu}{u}\lambda(u)\,du,$$

$$(1)\qquad f'(x)-\tfrac{1}{2}\pi g(x) = \int_0^\infty \frac{\psi(t)}{t}\left\{\frac{d}{dx}\int_0^\infty \frac{(1-\cos xu)\sin tu}{u}\lambda(u)\,du\right\}dt$$

$$= \frac{d}{dx}\int_0^\infty \frac{\psi(t)}{t}\,dt\int_0^\infty \frac{(1-\cos xu)\sin tu}{u}\lambda(u)\,du.$$

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* See M. Plessner, "Zur theorie der konjugierten trigonometrischen Reihen", Giessen, 1922; G. H. Hardy, "Notes on some points in the integral calculus", *Messenger of Math.*, 54 (1924), 20–27, 81–88; and other memoirs there referred to.

Now $f(x)$ and $-g(x)$ are, apart from the same constant factor in each case, the cosine and sine transform of $\lambda(u)$; so that

$$(2) \qquad -g(x) = \frac{d}{dx}\int_0^\infty \frac{1-\cos xu}{u}\lambda(u)\,du$$

$$= \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{1-\cos xu}{u}\lambda(u)\,du\int_0^\infty \frac{\sin tu}{t}\,dt$$

$$= \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{dt}{t}\int_0^\infty \frac{(1-\cos xu)\sin tu}{u}\lambda(u)\,du,$$

if we may invert the order of integration. The u-integral is uniformly (and absolutely) convergent over any finite interval $(0, T)$ in t, and*

$$\lim_{T\to\infty}\int_0^\infty \frac{1-\cos xu}{u}\lambda(u)\,du\int_T^\infty \frac{\sin tu}{t}\,dt = 0.$$

Hence the inversion is legitimate, and (2) is proved. From (1) and (2) we derive

$$(3) \qquad f'(x) = -\frac{d}{dx}\int_0^\infty \frac{e^{-t}}{t}\,dt\int_0^\infty \frac{(1-\cos xu)\sin tu}{u}\lambda(u)\,du.$$

Finally, we have to prove that this integral is absolutely convergent, so that the order of integration may be inverted. For this, we use the inequalities

$$1-\cos xu \leqslant 2, \qquad 1-\cos xu \leqslant \tfrac{1}{2}(xu)^2,$$

$$|\sin tu| \leqslant (tu)^a \quad (0<a<\tfrac{1}{2}).$$

Our double integral may then be compared with the sum of the integrals

$$\tfrac{1}{2}x^2\int_0^\infty t^{a-1}e^{-t}dt\int_0^1 u^{a+1}|\lambda(u)|\,du,$$

$$2\int_0^\infty t^{a-1}e^{-t}dt\int_1^\infty u^{a-1}|\lambda(u)|\,du,$$

each of which is convergent; and from this point the discussion proceeds as before.

This last argument applies equally when $\chi(t)$ is $J_1(t)$, or any of our other kernels. But, even when dealing with Kapteyn's equation, we must use $1-\cos xu$ in place of the $-\cos xu$ of our original argument.†

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* By "bounded convergence" (*l.c.*, p. 10).

† So that, foot-note †, p. 8, is inaccurate.

Thursday, March 12th, 1925.

Prof. A. L. DIXON, President, in the Chair.

There were present twenty-one members.

Prof. G. H. Hardy read a paper, written in collaboration with Mr. J. E. Littlewood and Prof. G. Pólya, entitled "The maximum of a certain bilinear form."

The following papers were communicated by title from the Chair :—

The Tritangent Planes of the same System of the Curve of Intersection of a Cubic Surface and a Quadric Cone : W. P. Milne.

On Integrals with a Nucleus : J. M. Whittaker.

*A definite integral which occurs in physical optics : G. H. Hardy.

The First and Second Variations of the Length-Integral in Riemannian Space : J. L. Synge.

Polar Properties of Cubics through Six Points on a Conic : W. L. Marr.

*A Class of Null Series : C. Fox.

NOTES AND ABSTRACTS.

Polar Properties of Cubics through Six Points on a Conic

W. L. MARR.

If PA, PB, PC, PD, PE, PF be tangents to a cubic at A, B, etc., and PL, PM be tangents at L and M to the polar conic $ABCDEF$, L (or M) is the polar point of M (or L) with respect to the binary sextic $ABCDEF$. Hence it can be shown that, if A, B, C, D, E, F be six given points on a conic, there are ten positions of P such that PA, PB, etc., are tangents to a cubic at A, B, etc.

If two of the six points, say E and F, coincide at E, and the tangent at E to the conic is also a tangent to the cubic, there are three positions of P such that PA, PB, etc., are tangents to a cubic with a flex at E. These three positions are collinear, and lie on the tangent at E to the conic, being the points where the tangents to the conic at the three polar points of E with respect to the binary quartic $ABCD$ meet the tangent at E to the conic.

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* Printed in this volume.

If two of the six points coincide to form a node at E, there are six positions of P lying two by two on the three lines which join E to the polar points of E with respect to the binary quartic $ABCD$.

This problem may also be considered from a geometrical point of view. The locus of a point P which moves so that PA, PB, PC, PD, PE are tangents to a cubic at A, B, C, D, E is known to be a quintic touching the conic $ABCDE$ at the five points, and passing through the intersections of the lines which join these five points in pairs. The intersections of two quintics include the ten poles of the general case already mentioned. Incidentally, the other two cases of three and six poles respectively are also interpreted. By means of two degenerate quintics the number of poles is obtained when the six points lie three by three on two straight lines, and is found to be two.

Note on a Certain Expansion

By T. W. CHAUNDY.

IF the differential equation

$$\frac{d^2y}{dx^2} = (1-x^2)y$$

is solved in series, we obtain the two expansions : an even series

$$1+\frac{x^2}{2!}-\frac{x^4}{4!}-(2^2+3^2)\frac{x^6}{6!}+(1+4^2)\frac{x^8}{8!}+5(7^2+10^2)\frac{x^{10}}{10!}$$
$$-5(6^2+11^2)\frac{x^{12}}{12!}-5^3.13(5^2+6^2)\frac{x^{14}}{14!}+5.13(12^2+23^2)\frac{x^{16}}{16!}$$
$$+5.13.17(95^2+112^2)\frac{x^{18}}{18!}+5^2.13.17.1891.\frac{x^{20}}{20!}$$
$$-5^2.13.17(557^2+1152^2)\frac{x^{22}}{22!}-\ldots,$$

and an odd series

$$x+\frac{x^3}{3!}-5.\frac{x^5}{5!}-5^2.\frac{x^7}{7!}+5.(1+6^2)\frac{x^9}{9!}+5(6^2+19^2)\frac{x^{11}}{11!}$$
$$-5(37^2+48^2)\frac{x^{13}}{13!}-5^2(64^2+95^2)\frac{x^{15}}{15!}+5^3(110^2+127^2)\frac{x^{17}}{17!}$$
$$+5^2 13^2(7^2+148^2)\frac{x^{19}}{19!}-\ldots.$$

It seems to me rather remarkable that the numerators in the coefficients are all expressible as sums of two squares, with the exception of the coefficient of x^{20} in the even series. In explanation of this exception we may notice that the succession of signs $+ + - - + + \ldots$ has been broken at this term, and that the numerator, which ought to be of the form $4n+1$, has instead the form $-4n+1$, *i.e.* effectively $4n-1$, and so cannot be the sum of two squares.

I cannot give any explanation of this property of the numerators: it seems too systematic to be mere chance coincidence. It is possible to state it in other forms.

If we denote the two series by

$$\sum_0^\infty A_n x^{2n}/(2n)!, \quad \sum_0^\infty B_n x^{2n+1}/(2n+1)!,$$

the numerators obey the recurrence-formulae

$$A_{n+1} = A_n - 2n(2n-1)A_{n-1}, \qquad B_{n+1} = B_n - 2n(2n+1)B_{n-1},$$

and the property of the numerators can therefore be stated, more intrinsically, as a property of these recurrence-formulae.

If we associate these numerators with other denominators, and consider the two series

$$\sum_0^\infty A_n x^n/n!, \quad \sum_0^\infty B_n x^n/n!,$$

we find that these are the expansions of the functions

$$(1+4x^2)^{-\frac{1}{2}\pm\frac{1}{4}} \exp(\tfrac{1}{2}\tan^{-1}2x),$$

i.e.

$$|(1+2ix)^{-\frac{1}{2}(1+i)}|, \qquad |(1+2ix)^{-\frac{1}{2}(3+i)}|,$$

where $i = \sqrt{(-1)}$, and the property can thus be ascribed to the numerators in the expansion of the moduli of these two functions.

The Inversion Formulae of Hardy and Titchmarsh

RICHARD G. COOKE.

The object of this paper is to find sufficient conditions for the truth of two inversion formulae given recently by Hardy* and by Titchmarsh.†

Some further particular cases of the formulae are also given.

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* G. H. Hardy, "Some formulae in the theory of Bessel Functions", *Proc. London Math. Soc.* (2), 23 (1925), lxi–lxiii (*Records* for June 12, 1924).

† E. C. Titchmarsh, "An Inversion Formula involving Bessel Functions", *Proc. London Math. Soc.* (2), 23 (1925), xxii–xxiv (*Records* for January 17, 1924).

*Some properties of fractional integrals**

G. H. HARDY *and* J. E. LITTLEWOOD.

1. The idea of an integral or derivative, of arbitrary non-integral order, was introduced into analysis by Liouville and Riemann.† Such integrals and derivatives may be, and have been by different writers, defined in a variety of manners, and different systems of definitions may be the most useful in different fields of analysis. The most usual definition is as follows. Suppose that we are considering $f(x)$ in an interval (A, B), that $a \leqslant A$, that $f(x)$ is integrable‡ in (a, B), and $0 < \alpha < 1$, $A \leqslant x \leqslant B$. Then we define $f_\alpha(x)$, the α-th integral, or integral of order α, of $f(x)$, by the equation

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{1.1}$$

It is easy to see that $f_\alpha(x)$ exists for almost all x in (A, B), and is integrable. The definition may then be extended formally to values of α outside $(0, 1)$ by ordinary integration or differentiation.

The definition involves an arbitrary *origin of integration* a. It is usual to suppose a finite. It has, however, been pointed out by Weyl§ that for some purposes, and particularly in the theory of trigonometrical series, when $f(t)$ is periodic, it is more appropriate to take $a = -\infty$. The integral (1.1) will then usually be non-absolutely convergent at the lower limit.

We state here a number of theorems concerning fractional integrals which appear to be new and to have a number of interesting applications. We postpone a more detailed account of our researches, since we are at present by no means certain that we have found the most appropriate methods of proof.

2. We say that f belongs to the class L^p, where $p > 1$, if f and $|f|^p$ are integrable, and that f satisfies a Lipschitz condition of order r if

$$f(x+h) - f(x) = O(|h|^r),$$

* Read at the meeting of May 15th, 1924.

† See G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series", *Cambridge Math. Tracts* (18), 22, and the writings there referred to.

‡ In the sense of Lebesgue.

§ H. Weyl, "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Ges. in Zürich*, 62 (1917), 296-302.

uniformly throughout the interval (A, B). We write

$$p' = \frac{1}{p}, \quad q' = \frac{1}{q}.$$

Our starting point is a theorem concerning series of positive terms, proved by Prof. G. Pólya and ourselves in a paper presented recently to the Society.

THEOREM 1.—*Suppose that* a_m *and* b_n *are positive,*

$$p > 1, \quad q > 1, \quad p'+q' > 1,$$

and

$$\lambda = 2-p'-q';$$

and that Σa_m^p *and* Σb_n^q *are convergent. Then*

$$\Sigma\Sigma \frac{a_m b_n}{|m-n|^\lambda} \quad (m \neq n) \tag{2.1}$$

is convergent.

This theorem may be compared with others already known. If we replace $|m-n|$ by $m+n$ in (2.1), we obtain a simple generalization of a well known theorem of Hilbert.* This theorem, however, lies less deep than Theorem 1, and is true when

$$p'+q' = 1, \quad \lambda = 1,$$

when Theorem 1 becomes false.†

From Theorem 1 we deduce

THEOREM 2.—*If* f *and* g *belong to* L^p *and* L^q *respectively, and* p *and* q *satisfy the conditions of Theorem* 1, *then*

$$\iint \frac{f(x)\,g(y)}{|x-y|^\lambda}\,dx\,dy$$

is convergent.

THEOREM 3.—*If* f *belongs to* L^p *and* $0 < \alpha < p'$, *then* f_α *belongs to* L^ϖ, *where*

$$\varpi = \frac{p}{1-p\alpha}.$$

THEOREM 4.—*If* f *belongs to* L^p *and* $p' < \alpha < p'+1$, *then* f_α *satisfies a Lipschitz condition of order* $\alpha-p'$.

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* See G. H. Hardy, "Note on a theorem of Hilbert concerning series of positive terms", *Proc. London Math. Soc.* (2), 23 (1925), xlv–xlvi (*Records* for April 24, 1924); and memoirs there referred to.

† The series

$$\Sigma\Sigma \frac{a_m a_n}{m-n}$$

is then convergent (as has been shown by Marcel Riesz), but not necessarily absolutely.

THEOREM 5.—*If f belongs to L^p and $0<\alpha<1$, then*

$$\int_A^B |f_\alpha(x)-f_\alpha(x-h)|^p\,dx = O(|h|^{p\alpha}).$$

It is understood here that $f(x)=0$ outside (A, B).

THEOREM 6.—*If*

$$\int_A^B |g(x)-g(x-h)|^p\,dx = O(|h|^{p\alpha}),$$

then g possesses derivatives, belonging to L^p, of all orders less than α.

The last two theorems were suggested by the theorems of Weyl* that (1) the α-th integral of a continuous function satisfies a Lipschitz condition of order α, and (2) that a function which satisfies such a condition has continuous derivatives of all orders less than α.

3. THEOREM 7.—*If f belongs to L^p, and*

$$1<p\leqslant 2, \quad p\leqslant q\leqslant \frac{p}{p-1},$$

then

$$\Sigma n^{-\kappa}(|a_n|^q+|b_n|^q),$$

where

$$\kappa=\frac{p+q-pq}{p},$$

and a_n and b_n are the Fourier constants of f in (A, B), is convergent.

When $q=p/(p-1)$, Theorem 5 reduces to a theorem of Hausdorff;† and the general theorem is deducible from Hausdorff's theorem and the special result for $q=p$. It may be observed that, if we assert only that

$$\Sigma n^{-\kappa-\epsilon}(|a_n|^q+|b_n|^q)$$

is convergent for every positive ϵ, then the condition $q\geqslant p$ becomes unnecessary. This less precise theorem is, however, not new, for Young‡ has proved the convergence of

$$\Sigma n^{-\lambda}(|a_n|+|b_n|) \quad (\lambda>p'),$$

and the result follows from this and Hausdorff's theorem.

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* H. Weyl, *l.c.*

† F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163–167.

‡ W. H. Young, "On the multiplication of successions of Fourier constants", *Proc. Royal Soc.* (A), 87 (1912), 331–339.

There is an analogue of Theorem 7 for *moment constants*, viz.

THEOREM 8.—*If f belongs to L^p, $p \leqslant q$, and*

$$a_n = \int_0^1 x^n f(x)\,dx,$$

then $$\Sigma n^{-\kappa} a_n^q$$

is convergent.

This is a much easier theorem, which does not involve the difficulties characteristic of such theorems as 2, 3, or 7.

Theorems 7 and 8 may be generalized in various directions, to series involving products of Fourier or moment constants of several functions, to multiple series of any order, and so forth.

4. We conclude by quoting a theorem of a somewhat different character.

THEOREM 9.—*Suppose that $f(z) = \Sigma a_n z^n$ is an analytic function of $z = re^{i\theta}$, regular for $r < 1$, and that*

$$\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

is bounded for $r < 1$. Then $$\Sigma \frac{|a_n|}{n}$$

is convergent.

An equivalent theorem, stated in terms of functions of a real variable, is

THEOREM 10.—*Suppose that the two conjugate series*

$$\Sigma(a_n \cos n\theta + b_n \sin n\theta), \qquad \Sigma(b_n \cos n\theta - a_n \sin n\theta),$$

are both Fourier series. Then

$$\Sigma \frac{|a_n| + |b_n|}{n}$$

is convergent.

Finally, Theorem 9 may be generalized in a different direction.

THEOREM 11.—*If $0 < p \leqslant 1$ and*

$$\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta$$

is bounded, then each of the series

$$\Sigma n^{-1/p}|a_n|, \quad \Sigma n^{p-2}|a_n|^p$$

is convergent.

The last result may be regarded as an extension, to values of p not greater than 1, of the extreme case of Theorem 7 in which $q = p$.

Thursday, April 23rd, 1925.

Prof. A. L. DIXON, President, in the Chair.

There were present thirty-four members.

Dr. H. Jeffreys delivered a lecture on "The Cooling of the Earth."

The following papers were communicated by title from the Chair :—

On Parseval's Theorem : S. Pollard.

The Functions involved in the Theory of a Thin Elastic Rectangular Plate, clamped at the Edges : A. C. Dixon.

On Certain Series for the Complete Elliptic Integral : H. B. C. Darling.

Laplace's Equation and the Inversion of Coordinates : D. M. Wrinch.

*Note on the Area of a Surface : M. Fréchet.

NOTES AND ABSTRACTS.

On the Realization of Rotation with the Aid of Lorentzian Coordinates

J. BRILL.

If we consider the finite continuous group having the differential expression $dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2$ for an absolute invariant, we find that its

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* Printed in this volume.

infinitesimal transformations are given by the equations

$$\bar{x}_1 = x_1+\delta\tau(a+\omega_{24}x_3-\omega_{34}x_2+\omega_{23}x_4), \quad \bar{x}_2 = x_2+\delta\tau(b+\omega_{34}x_1-\omega_{14}x_3+\omega_{31}x_4),$$

$$\bar{x}_3 = x_3+\delta\tau(c+\omega_{14}x_2-\omega_{24}x_1+\omega_{12}x_4), \quad \bar{x}_4 = x_4+\delta\tau(d+\omega_{23}x_1+\omega_{31}x_2+\omega_{12}x_3). \tag{1}$$

We will consider the motion of a set of points as governed by the consideration that the variables x_1, x_2, x_3, x_4, and the coefficients a, b, c, d, ω_{23}, etc., may all be expressed as continuous functions of a parameter τ; and will further consider the displacement given by equations (1) as corresponding to a small motion of the system. Then the expression $(\bar{x}_r - x_r)/\delta\tau$ becomes a differential coefficient, which may be expressed by the symbol x_r', and we have a set of four equations of the type

$$x_1' = a+\omega_{24}x_3-\omega_{34}x_2+\omega_{23}x_4, \quad \text{etc.}$$

Suppose x_{r0} to refer to some particular point of the moving system, and write $\xi_r = x_r - x_{r0}$. We have then a set of equations of the type

$$\xi_1' = \omega_{24}\xi_3-\omega_{34}\xi_2+\omega_{23}\xi_4, \quad \text{etc.},$$

from which we readily deduce

$$\frac{d}{d\tau}(\xi_4^2-\xi_1^2-\xi_2^2-\xi_3^2) = 0.$$

To ascertain whether we have anything corresponding to uniform rotation, we will consider the case in which the ω's are constant and ξ_3' always zero. This requires that

$$\omega_{14} = \omega_{24} = \omega_{12} = 0,$$

and our equations become

$$\xi_1' = -\omega_{34}\xi_2+\omega_{23}\xi_4, \qquad \xi_2' = \omega_{34}\xi_1+\omega_{31}\xi_4, \qquad \xi_4' = \omega_{23}\xi_1+\omega_{31}\xi_2. \tag{2}$$

Thus we have $$\omega_{31}\xi'-\omega_{23}\xi_2'+\omega_{34}\xi_4' = 0,$$

which on integration becomes

$$\omega_{31}\xi_1-\omega_{23}\xi_2+\omega_{34}\xi_4 = 0.$$

With the aid of this we deduce from equations (2)

$$\xi_1''+\xi_1(\omega_{34}^2-\omega_{23}^2-\omega_{31}^2)=0, \qquad \xi_2''+\xi_2(\omega_{34}^2-\omega_{23}^2-\omega_{31}^2)=0.$$

This seems to be the most general type of uniformly periodic motion of which the theory admits. Thus, taking the more general form as derived from equations (1), we may seek a solution of these equations by the method of multipliers, on the assumption that the ω's are constant. We then have the set of equations, for determining the multipliers, in the form

$$\mu_3\omega_{24}-\mu_2\omega_{31}+\mu_4\omega_{23}=\lambda\mu_1, \qquad \mu_1\omega_{34}-\mu_3\omega_{14}+\mu_4\omega_{31}=\lambda\mu_2,$$

$$\mu_2\omega_{14}-\mu_1\omega_{24}+\mu_4\omega_{12}=\lambda\mu_3, \qquad \mu_1\omega_{23}+\mu_2\omega_{31}+\mu_3\omega_{12}=\lambda\mu_4;$$

and the equation for determining λ is

$$\lambda^4+\lambda^2(\omega_{14}^2+\omega_{24}^2+\omega_{34}^2-\omega_{23}^2-\omega_{31}^2-\omega_{12}^2)-(\omega_{14}\omega_{23}+\omega_{24}\omega_{31}+\omega_{34}\omega_{12})^2=0. \quad (3)$$

Now, in general, two of the values given by equation (3) will be real and two purely imaginary. Further, we cannot have purely periodic motion without making the real roots zero. This can only be effected by making

$$\omega_{14}\omega_{23}+\omega_{24}\omega_{31}+\omega_{34}\omega_{12}=0,$$

in which case the other values of λ will be given by

$$\lambda^2+\omega_{14}^2+\omega_{24}^2+\omega_{34}^2-\omega_{23}^2-\omega_{31}^2-\omega_{12}^2=0.$$

This is in agreement with the particular case discussed above. It is to be noted that, as the theory breaks down to the ordinary one on making ω_{23}, ω_{31}, ω_{12} evanescent, it is reasonable to suppose the value of λ^2 as given above to be negative. It is further to be noted that ξ_4 is periodic, for this last quantity does not express a time but a time-difference.

The finite equations of the group with which we started may be written in the form

$$X_r=a_r+a_{1r}x_1+a_{2r}x_2+a_{3r}x_3+a_{4r}x_r \quad (r=1, 2, 3, 4),$$

where the coefficients are subject to the conditions

$$a_{r1}^2+a_{r2}^2+a_{r3}^2-a_{r4}^2=\epsilon_r \quad (r=1, 2, 3, 4),$$

$$a_{r1}a_{s1}+a_{r2}a_{s2}+a_{r3}a_{s3}-a_{r4}a_{s4}=0 \quad (r, s=1, 2, 3, 4;\ r\neq s),$$

$$\epsilon_r=1 \quad (r=1, 2, 3), \qquad \epsilon_4=-1.$$

Thus the equations belonging to the new coordinates are of the form

$$\Xi_r = a_{1r}\xi_1 + a_{2r}\xi_2 + a_{3r}\xi_3 + a_{4r}\xi_4,$$

accompanied by the same conditions; and it is in this latter form that the Lorentz transformation is usually given. It can be proved that the expressions

$$\omega_{14}\omega_{23} + \omega_{24}\omega_{31} + \omega_{34}\omega_{12}, \qquad \omega_{14}^2 + \omega_{24}^2 + \omega_{34}^2 - \omega_{23}^2 - \omega_{31}^2 - \omega_{12}^2,$$

are invariants of the group so determined. It would appear that, for a purely periodic motion, the first of these invariants must vanish, while the period is determined by the other.

On Certain Sequences of Polynomials

G. PÓLYA.

Suppose that $P_1(z)$, $P_2(z)$, ..., $P_n(z)$, ... are polynomials, and that there are two constants A, B such that

$$|P_n(z)| \leqslant Ae^{B|z|}$$

for all values of z and n. Define the function $p(\phi)$ by the following property: for any fixed positive value of ϵ, $P_n(re^{i\phi})e^{-\{p(\phi)+\epsilon\}r}$ is bounded and $P_n(re^{i\phi})e^{-\{p(\phi)-\epsilon\}r}$ is unbounded, r varying through all positive and n through all positive integral values. We have, of course, $p(\phi+2\pi) = p(\phi)$, $p(\phi) \leqslant B$. We can prove by the methods of Phragmén and Lindelöf, or by the Borel-Laplace transformation, that $p(\phi)$ is the "Stützfunktion" of a bounded convex domain P.

I consider now a special sequence,* putting

$$P_n(z) = \left(1 - \frac{z}{n}\right)^n.$$

Then the convex domain P is identical with the set of the points s such

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin (1925), the problems quoted in Vol. 2, p. 407, in connexion with the partial products of $\sin z$.

that $$1+\frac{z}{1!}se^{s}+\frac{z(z-2)}{2!}s^{2}e^{2s}+\ldots+\frac{z(z-n)^{n-1}}{n!}s^{n}e^{ns}+\ldots$$

converges to the sum e^{zs} for arbitrary z. This domain P and its "Stützfunktion" $p(\phi)$ are intimately connected with a curious problem of interpolation, considered by Halphén.* Let $F(z)$ be an integral function and $H_n(z)$ a polynomial, of degree not exceeding n, determined by the equations

$$H_n(0) = F(0), \quad H_n'(1) = F'(1), \quad \ldots, \quad H_n^{(n)}(n) = F^{(n)}(n).$$

Then for the validity of the equation

$$\lim_{n\to\infty} H_n(z) = F(z),$$

it is necessary that $$F(re^{i\phi}) = o(r^{2}e^{rp(\phi)})$$

and sufficient that $$F(re^{i\phi}) = O(r^{-1-\alpha}e^{rp(\phi)})$$

for some positive constant value of α; both conditions are supposed to be satisfied for $r\to\infty$ and uniformly in ϕ. We obtain a short proof by using the Laplace-Borel transformation. The proof applies in the same way to the analogous case of the ordinary interpolation in the points $-n, -n+1, \ldots, -1, 0, 1, \ldots, n$, but does not give quite so exact results as those obtained by Nörlund.†

A Note on the Integration of Fourier Series

B. M. WILSON.

Let $f(x)$ be an integrable and periodic function with period 2π, such that

(1) $$\int_0^{2\pi} f(x)\,dx = 0,$$

* G. H. Halphén, *Œuvres II*, Paris (1918), 520.

† N. E. Nörlund, *Differenzenrechnung*, Berlin (1924).

and let

$$(2) \qquad \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x)$$

denote the Fourier series of $f(x)$. Then Young has shown that, *if* $g(x)$ *is of bounded variation in the interval* $(0, \infty)$ *and tends to zero when* x *tends to infinity, then*

$$(3) \qquad \int_0^{\infty} f(x)\, g(x)\, dx = \Sigma \int_0^{\infty} A_n(x)\, g(x)\, dx.*$$

The conditions imposed on $g(x)$ are, of course, not necessary for the validity of the term-by-term integration performed in equation (3); it has, in fact, been shown by Hardy that, if λ is real and non-integral, and if we write

$$g(x) = \frac{\sin \lambda x}{x},$$

when $g(x)$ is not of bounded variation in $(0, \infty)$, equation (3) still holds.†

We may note, as an immediate consequence of the above theorems, that, if λ is real and non-integral, and if $f(x)$ is subject to the same conditions as in Young's theorem,

$$(4) \qquad \int_0^{\infty} f(x) \frac{1-\cos \lambda x}{x}\, dx = \tfrac{1}{2}\pi \sum_{1 \leqslant n < |\lambda|} b_n + \tfrac{1}{2} \sum_{n=1}^{\infty} a_n \log \left| \frac{n^2-\lambda^2}{n^2} \right|.$$

Equation (4) thus establishes the validity of term-by-term integration for a particular function $g(x)$, which is non-negative and of unbounded variation in $(0, \infty)$. To prove the equation write

$$F(x) = \int_0^x f(t)\, dt,$$

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* The simplest proof of Young's theorem has been given by Hardy, *Messenger of Math.*, 51 (1922), 186–192 (190–191). Hardy there (187) gives references to the relevant papers by Young.

† Hardy, *Messenger of Math.*, 52 (1922), 49–53 (Theorem 2). Condition (1) is not required in the proof of this theorem, which therefore holds for any integrable and periodic function $f(x)$.

so that, in virtue of (1), $F(x)$ is periodic. Then, by integration by parts,

$$\int_0^\infty f(x)\,\frac{1-\cos\lambda x}{x}\,dx = \int_0^\infty F(x)\left\{2\left(\frac{\sin\frac{1}{2}\lambda x}{x}\right)^2 - \frac{\lambda\sin\lambda x}{x}\right\}dx = I_1 - I_2,$$

say. The function $F(x)$ does not, in general, satisfy condition (1); but, since

$$\left(\frac{\sin\frac{1}{2}\lambda x}{x}\right)^2$$

is of bounded variation and absolutely integrable in $(0, \infty)$, the integral I_1 may, in virtue of a theorem of Young similar to the one quoted at the beginning of this note, but independent of condition (1), be evaluated by formal term-by-term integration.* The integral I_2 may also be so evaluated, in consequence of Hardy's equation already referred to, in virtue of the remark in foot-note † on page vi. On performing these formal integrations we are led to (4).

With the same assumptions concerning $f(x)$ and λ as above, Hardy has given the formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos\lambda(t-x)}{t-x}\,dt = \sum_{1\leqslant n<|\lambda|}(b_n\cos nx - a_n\sin nx).\dagger$$

A simpler proof of this equation is obtained by the method used above than by that (depending upon the determination of the integral as, in the first instance, a principal value) which Hardy himself seems to indicate.

Lastly, it may be observed that the integrals considered in this note are, in one direction, more general than those of Young and Hardy to which we have referred. For the proof given depends only on the integrability and periodicity of $F(x)$, together with the validity of an integration by parts; it is therefore unnecessary to assume the existence of the function $f(x)$ having the Fourier series (2). It is thus seen that we have proved the following theorem :—

If $F(x)$ is of bounded variation in the interval $(0, 2\pi)$ and is periodic with period 2π, and if λ is real and non-integral, the Stieltjes integrals

$$\int_a^\infty \frac{1-\cos\lambda(t-x)}{t-x}\,dF(t) \quad (a = 0 \text{ or } \infty),$$

which then certainly exist, may be evaluated by formal term-by-term differentiation of the Fourier series of $F(t)$, followed by formal term-by-term integration of the resulting series.

* See Hardy, *ibid.*, 51 (1922), 186–192 (187).

† Hardy, *ibid.*, 52 (1922), 49–53 (Theorem 3).

Note on the Area of a Surface (Extract from a Letter to Prof. W. H. Young from Prof. M. Fréchet)

In your paper "On the triangulation method of defining the area of a surface," *Proc. London Math. Soc.* (2), 19 (1921), 117-152, you obtain a remarkable extension of a theorem quoted from the *Encykl. der Math. Wiss.*, 3 (3), 64.

Let triangles fill a region R of a plane, and correspond to a polyhedron inscribed in a surface U (the image of R). And let the parametric representation of U be effected by coordinates whose partial derivates are continuous. Call δ the maximum side of the triangles in the plane. Then you have proved that, whenever δ converges to zero, the area of the polyhedron converges to the well known integral (extended over R) expressing the area of U, provided that there exists a number γ $(0 < \gamma < \pi)$ such that, whatever be δ, one at least of the angles of every triangle of R lies between γ and $\pi - \gamma$.

Now I suggest putting this last condition under a different form, which should make it all the more natural. Strictly speaking, it is a slightly more general condition, in that it is not a condition independent of δ, but a limiting condition when δ tends to zero.

The condition to be fulfilled by the angles of the triangles in R is that the greatest of their limits, when δ tends to zero, be smaller than π.

Thursday, May 14th, 1925.

Prof. A. L. DIXON, President, in the Chair.

Present twenty members.

Miss G. K. Stanley, Mr. M. Hanna, Mr. M. A. Higab, Prof. W. A. Hurwitz, and Dr. O. Szász were nominated for election.

The President announced that the meeting of June 11th would be set aside for the celebration of the sixtieth anniversary of the Society, and would take place at University College, London. At 5.30, Mr. J. E. Littlewood would deliver a lecture on "Solved and unsolved problems in the Theory of Prime Numbers." There would be a dinner at 7 o'clock, and, after dinner, Dr. J. W. L. Glaisher would give an informal lecture on "The early history and personalities of the Society."

Mr. J. E. Littlewood communicated a paper by Mr. S. Pollard, "On Parseval's Theorem."

Mr. E. C. Titchmarsh read a paper on "The zeros of certain integral functions."

Lt.-Col. Allan Cunningham made an informal communication on "A new divisor of one of Fermat's Numbers."

The following papers were communicated by title from the chair :—

Some Properties of Methods of Evaluation of Divergent Sequences : W. A. Hurwitz.

On the Invariant Theory of Mixed Quaternary Forms : H. W. Turnbull.

NOTES AND ABSTRACTS.

On the Invariant Theory of Mixed Quaternary Forms

H. W. TURNBULL.

Two symbolic methods have been in wide use for developing the algebraic theory of projective invariants, or of subgroups of the projective group : (1) the *ordinary* or Clebsch-Aronhold symbolism, and (2) the *complex* symbolism. Ordinary symbols, a, combine as numbers; for example

$$a_i a_j = a_j a_i.$$

Complex symbols, ϵ, alternate ; for example

$$\epsilon_i \epsilon_j = -\epsilon_j \epsilon_i.$$

These are various umbral expressions for the same actual coefficient, say

$$a_{ij},$$

of a quadratic form.

Hitherto no writer seems to have followed the lead which Clebsch gave for attacking the line complex by ordinary symbolic methods, although much elaborate work has been done by use of the complex symbols. I show in this investigation that the Clebsch symbols are well adapted for the problem, once the difficulties of the fundamental theorem for the general mixed form, in point, line, and plane coordinates, have been surmounted. Examples are given for the linear and quadratic complexes, concluding with the application, first noticed by Weitzenbock, to the invariant theory of the Weyl transformations of general relativity. The Riemann-Christoffel curvature tensor is, in fact, structurally identical with the quadratic complex of line geometry.

Note on the inversion of a repeated integral

G. H. HARDY.

There are not very many simple and genuinely different examples in which the repeated integrals

$$\int_a^b dx \int_c^d f(x, y)\, dy, \quad \int_c^d dy \int_a^b f(x, y)\, dx$$

both exist and have different values. The example

$$\int_0^1 dx \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy = \tfrac{1}{4}\pi \quad \int_0^1 dy \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx = -\tfrac{1}{4}\pi$$

goes back to Cauchy.* A somewhat simpler example of the same type, viz.

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy = 1, \quad \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx = -1,$$

is given, as an exercise in calculation, in my *Course of Pure Mathematics*.†

Another simple example is

$$\int_0^\infty dx \int_0^\pi f(x, y)\, dy = 0, \quad \int_0^\pi dy \int_0^\infty f(x, y)\, dx = \tfrac{1}{2}\pi,$$

where $$f(x, y) = e^{-x\cos y} \cos(y - x \sin y).$$‡

It is worth while to add to the number of such examples, and I have done so recently as follows. Suppose that $f(u)$ and $g(u)$ are continuous for $u \geqslant 0$, that

$$\int_0^\infty \frac{f(u)-g(u)}{u} \log u\, du$$

is convergent (not necessarily absolutely), and suppose also that

$$\int_0^\infty \frac{f(u)-g(u)}{u}\, du = 0.$$

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* A. L. Cauchy, "Mémoire sur les intégrales définies", *Œuvres* (1), 1, 394–396.

† 356 (Ex. 38).

‡ See G. H. Hardy, "On differentiation and integration of divergent series", *Trans. Camb. Phil. Soc.*, 19 (1902), 296–321 (f.n., 314).

It may then easily be verified, a and b being positive, that

$$\int_0^a \frac{dx}{x}\int_b^\infty \frac{f(xy)-g(xy)}{y}\,dy = \int_0^{ab} \frac{f(u)-g(u)}{u}\log u\,du + \log ab \int_{ab}^\infty \frac{f(u)-g(u)}{u}\,du,$$

$$\int_b^\infty \frac{dy}{y}\int_0^a \frac{f(xy)-g(xy)}{x}\,dx$$

$$= -\int_{ab}^\infty \frac{f(u)-g(u)}{u}\log u\,du + \log ab \int_{ab}^\infty \frac{f(u)-g(u)}{u}\,du;$$

so that the repeated integrals of

$$\{f(xy)-g(xy)\}/xy,$$

over $(0, a; b, \infty)$, exist and have values differing by

$$\int_0^\infty \frac{f(u)-g(u)}{u}\log u\,du.$$

If we take $a = b = 1$,

$$f(u) = \frac{1}{1+u}, \quad g(u) = \frac{1}{1+u^2},$$

and then replace y by $1/y$, we are led to the example

$$\int_0^1 dx \int_1^\infty \frac{(x-y)\,dy}{(x+y)(x^2+y^2)} = \frac{\pi^2}{16}, \quad \int_0^1 dy \int_0^1 \frac{(x-y)\,dx}{(x+y)(x^2+y^2)} = -\frac{\pi^2}{16}.$$

Another interesting example is obtained by taking

$$f(u) = pue^{-pu}, \quad g(u) = que^{-qu},$$

where p and q are positive. We have, in fact,

$$\int_0^1 dx \int_1^\infty (pe^{-pxy} - qe^{-qxy})\,dy = \int_0^1 \frac{e^{-px}-e^{-qx}}{x}\,dx,$$

$$\int_1^\infty dy \int_0^1 (pe^{-pxy} - qe^{-qxy})\,dx = -\int_1^\infty \frac{e^{-py}-e^{-qy}}{y}\,dy;$$

and the difference between the two integrals is

$$\int_0^\infty \frac{e^{-px}-e^{-qx}}{x}\,dx = \log\frac{q}{p}.$$

Some Remarks on Dirichlet's Series

EINAR HILLE.

The old problem of determining when a Dirichlet's series defines an entire (or integral) function is attacked in the present paper by means of the Cahen-Perron integral

$$D(s\,;\,\lambda) = \frac{1}{\Gamma(s/\nu)} \int_0^\infty x^{(s/\nu)-1}\, D(x\,;\,l^\nu)\, dx,$$

where $$D(s\,;\,\lambda) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad D(u\,;\,l^\nu) = \Sigma a_n e^{-l_n^\nu u}.$$

This integral is also used for the purpose of summing a class of divergent Dirichlet's series. From the integral an expression for $D(s\,;\,\lambda)$ is obtained as the sum of an entire function and a factorial series. A necessary and sufficient condition in order that this representation shall be valid for all values of s is obtained. It is sufficient that $D(u\,;\,l^\nu)$ is holomorphic at the origin. A sufficient condition that $D(s\,;\,\lambda)$ shall define an entire function is that $\lim_{x\to+0} D^{(n)}(x\,;\,l^\nu)$ exists and is finite for every $n \geqslant 0$.

SIXTIETH ANNIVERSARY MEETING.

Thursday, June 11th, 1925.

Prof. A. L. DIXON, President, in the Chair.

The Society met in University College, by kind permission of the Committee of the College, to celebrate the Sixtieth Anniversary of its foundation.

There were present sixty-five members and eight guests.

Miss G. K. Stanley, Mr. M. Hanna, Mr. M. A. Higab, Prof. W. A. Hurwitz, and Dr. O. Szász were elected members of the Society.

Mr. J E. Littlewood delivered a lecture entitled "Solved and unsolved problems in the theory of prime numbers."

Prof. E. Landau (hon. member of the Society) spoke briefly on the subject of the lecture.

The following papers were communicated by title from the Chair :—

*The Zeros of Riemann's Zeta-function on the Critical Line : M. Fekete.

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* Printed in *Journal London Math. Soc.*, 1 (1926).

On the Summability (C, k) of Series of Type $\Sigma a_n f_n$: W. L. Ferrar.

(i) The Expansion of Zero as a Schlömilch Series; (2) The Expression of Hypergeometric Series in Terms of Similar Series; (iii) Some Further Contributions to the Theory of Null Series and their connexion with Null Integrals: C. Fox.

On the mean values of power series: J. E. Littlewood.

Identification of the Coefficients in a Trigonometrical Integral: S. Pollard.

A Theory of Relativity, in which the Dynamical Manifold can be conformally represented upon the Manifold: G. Temple.

The Society then adjourned to dinner. There were present the following members:—Rev. W. E. Baker, F. Bath, Dr. W. E. H. Berwick, J. Brill, Dr. R. Bryant, Prof. S. Chapman, Prof. C. E. Cullis, R. J. Dallas, W. R. Dean, Prof. A. L. Dixon (*President*), Prof. L. N. G. Filon, R. H. Fowler, C. Fox, P. Fraser, Dr. J. W. L. Glaisher, W. M. H. Greaves, S. L. Green, Prof. G. H. Hardy, Prof. H. R. Hassé, A. C. Heath, Prof. P. J. Heawood, Dr. J. Henderson, Prof. M. J. M. Hill, Prof. H. Hilton, Prof. E. W. Hobson, Miss Hilda P. Hudson, A. E. Ingham, L. Isserlis, C. G. F. James, Prof. G. B. Jeffery, Dr. H. Jeffreys, G. S. Le Beau, J. E. Littlewood, Prof. A. E. H. Love, Dr. Arthur Lynch, W. H. Macaulay, Prof. E. A. Milne, Prof. L. J. Mordell, Prof. E. H. Neville, H. O. Newboult, Dr. J. W. Nicholson, A. Oppenheim, Prof. G. Pólya, Dr. E. G. C. Poole, Prof. A. R. Richardson, Dr. W. F. Sheppard, S. W. P. Steen, Miss M. Stimson, R. Stoneley, Miss Lorna M. Swain, E. C. Titchmarsh, Prof. G. N. Watson, Dr. A. E. Western, F. J. W. Whipple, F. Puryer White, Miss F. M. Wood, T. L. Wren, Dr. D. M. Wrinch; and the following guests:—Dr. W. Ackermann, Sir M. Sheldon Amos, The Astronomer Royal (representing the Royal Astronomical Society), Mr. C. E. G. Bailey, Major A. G. Church, Prof. Winifred C. Cullis, Mr. A. C. Davies, Dr. P. Dienes, Dr. Major Greenwood, Miss H. R. Hicks, Mr. T. W. Hodgson (representing Messrs. C. F. Hodgson & Sons), Miss W. M. Hudson, Dr. J. Jackson, Mr. E. S. Keeping, Miss M. D. Kennedy, Prof. E. Landau (hon. member of the Society), Mr. G. K. Lester, Mr. E. H. Linfoot, Prof. C. Siegel, Mrs. R. V. Southwell, Sir J. J. Thomson, Master of Trinity College, Cambridge (representing the Royal Society), Mr. F. H. Vallancey (representing Mr. Francis Hodgson).

Congratulatory messages from the President and the Hon. Secretary of the American Mathematical Society, and from Prof. A. Einstein, Prof. H. Lebesgue, Prof. A. Sommerfeld, and Prof. V. Volterra (hon. members of the Society), were read by the President.

The toast of the Society was proposed by Sir J. J. Thomson, and replied to by the President and Prof. A. E. H. Love.

After dinner Dr. J. W. L. Glaisher gave an informal address* on "The Early History and Personalities of the Society."

NOTES AND ABSTRACTS.

Note on the Numerical Evaluation of a Bessel Function Derivative

P. R. ANSELL *and* R. A. FISHER.

As early as 1870, J. W. L. Glaisher had distinctly formulated a scheme for the systematic tabulation of a group of fundamental functions, in addition to the algebraic, logarithmic, and exponential functions. His purpose was to choose such fundamental functions as commended themselves by the elegance of their properties, and their naturally central position as the products of the simplification of integral expressions. His table of the error function appeared in 1871 [*Phil. Mag.* (4), 42, 436], following on the publication of the sine integral, the cosine integral, and the exponential integral in 1870 [*Phil. Trans.* 160, 367]. How well chosen was this group of functions will be familiar to all who have had occasion to require the numerical values of integral expressions.

The present note calls attention to the existence of a previously unsuspected connexion between the derivative of the Bessel function

$$\frac{\partial}{\partial \nu} J_\nu(x)$$

and the sine and cosine integrals, which enables the values of the former, when ν is the half of an odd integer, to be calculated directly from tables of the latter. In any expression of the form

$$f_1(\nu)+f_2(\nu)\,J_\nu$$

if, for any value of ν, f_1, f_2 tend to infinity, while the whole expression remains finite, we are liable to require the numerical value of the derivative. The case, of this sort, which led to the following results, occurred in the investigation of the efficiency in small samples of certain statistics ["Theory of Statistical Estimation," *Proc. Camb. Phil. Soc.*, 22 (1925), 700–725].

The identity may perhaps best be derived from Hankel's formula

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}} \cos zt\, dt,$$

* Printed in *Journal London Math. Soc.*, 1 (1926).

by direct differentiation and the substitution $\nu = \frac{1}{2}$. Then

$$\frac{\partial}{\partial \nu} J_\nu(z) = J_\nu(z) \left\{ \log \frac{z}{2} - \frac{\Gamma'(\nu+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \right.$$

$$\left. + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \log(1-t^2) \cos zt\, dt, \right.$$

which, when $\nu = \frac{1}{2}$, reduces to

$$\sqrt{\left(\frac{2}{\pi z}\right)} \left\{ \sin z \left(\log \frac{z}{2} + \gamma \right) + \tfrac{1}{2} z \int_{-1}^{1} \log(1-t^2) \cos zt\, dt \right\}.$$

The latter term may be reduced successively to

$$\int_{-1}^{1} \log(1+t) \cos zt\, z\, dt = \int_{-z}^{z} \log \frac{z+t}{z} \cos t\, dt$$

$$= \int_0^{2z} \log \frac{u}{z} \cos(u-z)\, du.$$

Now

$$\int_\epsilon^{2z} \log \frac{u}{z} \cos(u-z)\, du = \left[\log \frac{u}{z} \sin(u-z) \right]_\epsilon^{2z} - \int_\epsilon^{2z} \sin(u-z) \frac{du}{u},$$

and when ϵ tends to zero, may be reduced to

$$\sin z \left(\log 2 + \log \frac{\epsilon}{z} \right) - \cos z\, Si(2z) + \sin z \{ Ci(2z) - Ci(\epsilon) \},$$

where $$Si(x) = \int_0^x \sin x \frac{dx}{x}, \quad Ci(x) = -\int_x^\infty \cos x \frac{dx}{x};$$

and since, when $\epsilon \to 0$ through real values,

$$Ci(\epsilon) - \log \epsilon \to \gamma,$$

we have

$$\int_0^{2z} \log \frac{u}{z} \cos(u-z)\, du = \sin z \left(\log \frac{2}{z} - \gamma \right) - \cos z\, Si(2z) + \sin z\, Ci(2z),$$

and consequently

$$\left[\frac{\partial}{\partial \nu} J_\nu(z) \right]_{\nu=\frac{1}{2}} = \sqrt{\left(\frac{2}{\pi z}\right)} \{ \sin z\, Ci(2z) - \cos z\, Si(2z) \}$$

$$= J_{\frac{1}{2}}(z)\, Ci(2z) - J_{-\frac{1}{2}}(z)\, Si(2z) \qquad \text{(I)}$$

The corresponding formula for $\nu = -\frac{1}{2}$ may be obtained from

$$\frac{\partial}{\partial z} J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z),$$

whence $$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \nu} J_\nu(z)\right] = \frac{\partial}{\partial \nu} J_{\nu-1}(z) - \frac{1}{z} J_\nu(z) - \frac{\nu}{z} \frac{\partial}{\partial \nu} J_\nu(z).$$

Putting $\nu = \frac{1}{2}$, we have

$$\left[\frac{\partial}{\partial \nu} J_\nu(z)\right]_{\nu=-\frac{1}{2}} = \frac{\partial}{\partial z}\left\{\sqrt{\left(\frac{2}{\pi z}\right)}\left(\sin z\, Ci(2z) - \cos z\, Si(2z)\right)\right\}$$

$$+ \frac{1}{z}\sqrt{\left(\frac{2}{\pi z}\right)} \sin z + \frac{1}{2z}\sqrt{\left(\frac{2}{\pi z}\right)}\left(\sin z\, Ci(2z) - \cos z\, Si(2z)\right)$$

$$= \frac{1}{z}\sqrt{\left(\frac{2}{\pi z}\right)} \sin z$$

$$+ \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos z\, Ci(2z) + \sin z\, Si(2z) + \frac{\sin z \cos 2z - \cos z \sin 2z}{z}\right\};$$

therefore

$$\left.\begin{aligned}\left[\frac{\partial}{\partial \nu} J_\nu(z)\right]_{\nu=-\frac{1}{2}} &= \sqrt{\left(\frac{2}{\pi z}\right)}\left\{\cos z\, Ci(2z) + \sin z\, Si(2z)\right\} \\ &= J_{-\frac{1}{2}}(z)\, Ci(2z) + J_{\frac{1}{2}}(z)\, Si(2z)\end{aligned}\right\}. \qquad \text{(II)}$$

Using the general recurrence formula

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{z}{2\nu} J_\nu(z),$$

the values of $\partial J_\nu(z)/\partial \nu$ may be obtained for all values of ν equal to the half of an odd integer.

Glaisher's tables of the sine integral give 18 decimal places for values of x proceeding by intervals of 0·01 from 0 to 1, and 11 places proceeding by intervals of 0·1 from 1 to 5; both parts of the table may be interpolated to the full number of figures given, using second, fourth, and sixth differences. Unfortunately, the central difference formulae were not known to Glaisher, and he gives the first three differences only. The most serious gap in his tables is from 5 to 15, where he gives 11 places for integer values only, and direct interpolation is useless. The table of the cosine integral is not so satisfactory, since in the range from 0 to 1 accurate interpolation is no longer possible, and it would have been wiser to tabulate $Ci(x) - \log x$. Beyond 15 the asymptotic formulae are good for seven places for both functions.

Proof of an Inequality

G. PÓLYA.

We have $$\sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

if we suppose that $a_n \geqq 0$, and that the series on the right-hand side converges to a sum which is greater than zero. We get a false statement if we replace e by any smaller constant. This theorem is due to T. Carleman.*

1. Define the positive numbers $c_1, c_2, c_3, \dots$ by

$$c_1 c_2 \dots c_n = (n+1)^n \quad (n = 1, 2, 3, \dots).$$

Using the inequality between the arithmetic and the geometric means, we get

$$\sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{1/n} = \sum_{n=1}^{\infty} \frac{(a_1 c_1 \cdot a_2 c_2 \dots a_n c_n)^{1/n}}{(c_1 c_2 \dots c_n)^{1/n}}$$

$$\leqq \sum_{n=1}^{\infty} \frac{a_1 c_2 + \dots + a_k c_k + \dots + a_n c_n}{n (c_1 c_2 \dots c_n)^{1/n}} = \sum_{k=1}^{\infty} a_k c_k \sum_{n=k}^{\infty} \frac{1}{(c_1 c_2 \dots c_n)^{1/n}}$$

$$= \sum_{k=1}^{\infty} a_k \frac{(k+1)^k}{k^{k-1}} \frac{1}{k} < e \sum_{k=1}^{\infty} a_k.$$

2. Put $a_n = c_n^{-1}$ for $n \leqq N$ and $a_n = 0$ for $n > N$. Then

$$\frac{\sum_{n=1}^{\infty} (a_1 a_2 \dots a_n)^{1/n}}{\sum_{n=1}^{\infty} a_n} = \frac{\sum_{n=1}^{N} (n+1)^{-1}}{\sum_{n=1}^{N} n^{n-1} (n+1)^{-n}} \sim \left(\frac{N+1}{N}\right)^{N-1} \to e$$

for $N \to \infty$.

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* T. Carleman, "Conférences faites au 5me congrès des mathematiciens scandinaves", Helsingfors 1923, 181–196. See also G. H. Hardy, *Messenger of Math.*, 54 (1925), 150–156; and the proof, by J. E. Littlewood, on p. 186 of G. Valiron's treatise *The General Theory of Integral Functions* (Toulouse, 1923).

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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SUCCESSIONS WITH SUBSEQUENCES CONVERGING TO AN INTEGRAL.

By W. H. Young.

[Received and Read January 18th, 1923.]

1. In a Note* presented to the Royal Society in 1915, I proved the existence of converging sequences in certain oscillating successions of functions. It is frequently of importance† to know, not merely the convergence, but the property of the limiting function of being an *integral;* this is, in particular, the case when the given succession consists itself of *integrals*, that is, of *absolutely convergent* (*Lebesgue*) *integrals.*

The main theorem of the present communication gives a simple criterion in the case considered. This criterion, which is sufficient, enables us not only to pronounce definitively that the subsequences exist and converge to integrals, but enable us to read off what may be called the *type of supersummability* of the limiting functions, this being, in fact, the same as that of the generating functions of the succession. By *type of supersummability of* $f(x)$, I mean the class of functions $Q(u)$ which, when we put $u = f(x)$, become summable functions of x. Such functions are in the simplest cases u^2, u^{1+p} (where p is fractional or integral), $u \log u$, $u \log\log u$, $u \log\log\log u$, and so forth. Generally the types which I consider are those for which I formulated, in another communication to the Royal Society,‡ certain inequalities corresponding to and generalising what is known as Schwarz's inequality for the functions u^2 and v^2, where, in my case, we put $v = 1$. For convenience I give here the proof of the

* W. H. Young, "Note on the Existence of Converging Sequences in Certain Oscillating Successions of Functions", *Proc. Royal Soc.* (A), 92 (1915), 353–356. See below.

† In fact, one of the chief difficulties in the application in practice of the theory of restricted Fourier series to the study of other series is that of proving that a certain series, known to converge, converges to an integral, integral of an integral, etc.

‡ "On Classes of Summable Functions and their Fourier Series", *Proc. Royal Soc.* (A), 87 (1912), 225–229.

inequality used in the proof of the main theorem of the present paper. I have moreover gone carefully into the proof of the theorem above referred to on converging sequences, partly because I am able to formulate the theorem in more general form, and partly because the present proof is an improvement on the former.

The type-functions $Q(u)$ are supposed to be themselves integrals, viz.

$$Q(u) = \int_0^u U(z)\,dz,$$

where $U(u)$ is, for all positive values of the variable u, a positive monotone increasing function of u, which tends to $+\infty$ as $u \to +\infty$. The result obtained in the main theorem and corollary is then that, *if*

$$\int_a^b Q\{|f_n(x)|\}\,dx$$

exists for all values of n, and is a bounded function of n, then, in the succession of integrals

$$\int_a^x f_n(x)\,dx,$$

or in any of its subsuccessions, there is a sequence converging to an integral,

$$\int_a^x f(x)\,dx$$

say, where the integrand $f(x)$ has the same type of supersummability as the generating functions, i.e.

$$\int_a^b Q\{|f(x)|\}\,dx$$

exists.

I have placed the enunciation and proof of this result first in the following communication, and have, afterwards, developed the accessory theorems involved in the proof.

In the theory of normal functions, and in particular in the theory of Fourier series or restricted Fourier series, we obtain from the main theorem the following result:—

If $u_1, u_2, \ldots$ be a succession of bounded normal functions, and $f_1, f_2, \ldots$ denote the partial Cesàro summations (index integral or fractional) of the series

$$a_1 u_1 + a_2 u_2 + \ldots,$$

then, if $\int_a^b Q\,|f_n(x)|\,dx$ is bounded in the closed interval (a, b) in which the conditions of normality hold, the series is associated with a function $f(x)$ of the same type of supersummability as $f_n(x)$.

2. The main result is embodied in the following theorem and corollary:—

THEOREM 1.—*If in the interval* $(a \leqslant x \leqslant b)$

$$\int_a^x f_n(x)\,dx \to F(x) \quad (n \to \infty),$$

and if

$$\int_a^b Q\{|f_n(x)|\}\,dx$$

exists for all values of n, *and is bounded as* n *assumes all integral values, where*

$$Q(u) = \int_0^u U(z)\,dz \quad (0 \leqslant u),$$

and $U(u)$ *is for all positive values of* u *a positive monotone increasing function of* u, *which tends to* $+\infty$ *as* $u \to +\infty$, *and possesses a differential coefficient* $U(u)$ *which is nowhere zero,* (1) *the limiting function* $F(x)$ *is an absolutely continuous function of* x, *that is an integral with respect to* x, *and* (2) *it is the integral of a function* $f(x)$ *such that* $Q\{|f(x)|\}$ *is a summable function of* x *in* (a, b).

We begin by remarking that, since $\int_a^x Q\{|f_n(x)|\}\,dx$ represents a monotone increasing function of x, which, as n and x vary, remains, in virtue of the hypothesis, bounded, it forms, when x is fixed and n varies, a bounded succession of monotone increasing functions. Consequently,* there is a succession of integers, $n_1, n_2, \ldots, n_i, \ldots$, for which the functions converge. Let us write

$$\int_a^x Q\{|f_{n_i}(x)|\}\,dx = P_i(x),$$

and

$$\operatorname*{Lt}_{i \to \infty} P_i(x) = P(x),$$

and also

$$\int_a^x f_{n_i}(x)\,dx = F_i(x).$$

Then, by the known inequality proved below,†

$$Q\left\{\int_x^{x+h} |f_{n_i}(x)|\,dx \Big/ h\right\} \leqslant \int_x^{x+h} Q\{|f_{n_i}(x)|\}\,dx/h,$$

* See below, Theorem 4*a* and accessory Theorem ii.

† See below, § 10, Lemma 2.

which gives

$$|h|Q\{|[F_i(x+h)-F_i(x)]/h|\} \leqslant P_i(x+|h|)-P_i(x).$$

Let us take any division of the interval (a, b) into a finite number of parts, and, denoting the generic part by $(x, x+|h|)$, let us sum the inequalities of the above form, for each part, from a to b; the result may be written as follows:

$$\sum_a^b |h|Q\{|[F_i(x+h)-F_i(x)]/h|\} \leqslant P_i(b)-P_i(a).$$

Letting $i \to \infty$, we get

$$\sum_a^b |h|Q\{|[F(x+h)-F(x)]/h|\} \leqslant P(b)-P(a).$$

But this is the known condition* that $F(x)$ may be the integral of a function having the property (*type of supersummability*) required by the enunciation. This, therefore, proves the theorem.

COR.—*If the given succession of integrals* $\int_a^x f_n(x)\,dx$ *do not converge, but* $\int_a^b Q\{|f_n(x)|\}\,dx$ *is, as before, bounded, we can find a subsequence converging to a function* $F(x)$ *having the same properties as in the above theorem.*

For, if we divide the segment (a, b) into the set E of points at which $f_n(x)$ is numerically greater than $Q|f_n(x)|$, and the complementary set F, we shall have in the set E,

$$Q\{|f_n(x)|\}/|f_n(x)| < 1,$$

and therefore $f_n(x)$ is certainly always less than a fixed constant B, since, when $u > 2k$,

$$Q(u)/u = \left\{\int_0^u U(z)\,dz/u\right\} > \left\{\int_k^u U(z)\,dz/u\right\}$$

$$> U(k)\{1-k/u\} > \tfrac{1}{2}U(k),$$

which tends to ∞ as $k \to \infty$, *since, by hypothesis* $U(k) \to +\infty$. There is therefore a finite upper bound B to the values of u for which

$$\{Q(u)/u\} < 1.$$

---

* See below, § 11.

We have therefore

$$\int_a^b |f_n(x)|\,dx = \int_E |f_n(x)|\,dx + \int_F |f_n(x)|\,dx < B(b-a) + \int_a^b Q\,|f_n(x)|\,dx.$$

Thus $\int_a^x |f_n(x)|\,dx = F_n(x)$ describes a bounded succession.

Therefore, since $\int_a^x f_n(x)\,dx$ describes a succession of functions of bounded variation, whose variations are bounded in their ensemble,* we can extract a sequence of our integrals $\int_a^x f_n(x)\,dx$ converging to a function of bounded variation which is, by the above theorem, an integral, and moreover the integral of a function of the required type of supersummability.

3. We can now apply the main theorem to the theory of normal functions, and obtain the following:—

THEOREM 2.—*If* $u_1(x)$, $u_2(x)$, ... *be a succession of bounded normal functions, and* $f_1(x)$, $f_2(x)$, ... *denote the partial Cesàro summations of the series*

$$a_1u_1(x) + a_2u_2(x) + \ldots,$$

then if $\int_a^b Q\{|f_n(x)|\}\,dx$ *is bounded in the closed interval* (a, b) *in which the conditions of normality hold, we have*

$$a_p = \int_a^b f(x)\,u_p(x)\,dx,$$

for all values of p, *where* $f(x)$ *is a function such that* $\int_a^b Q\{|f(x)|\}\,dx$ *exists.*

For, by the corollary to the main theorem, we can find a succession of integers $n_1, n_2, \ldots, n_i, \ldots$ for which the integrals $\int_a^b f_{n_i}(x)\,dx$ converge to the integral of a function $f(x)$ having the type of supersummability required, [viz. $Q\{|f(x)|\}$ is summable]. Since $u_p(x)$ is a bounded function,

* See below, § 9, Theorem vi.

we have therefore

$$\int_a^b f_n(x)\, u_p(x)\, dx \to \int_a^b f(x)\, u_p(x)\, dx, \quad (n = n_1, n_2, \ldots, n_i, \ldots) \quad (1)$$

But, by definition,

$$f_n(x) = a_1 u_1 + \frac{n-1}{n} a_2 u_2 + \frac{n-2}{n} a_3 u_3 + \ldots + \frac{n-p+1}{n} a_p u_p + \ldots + \frac{1}{n} a_n u_n,$$

and the conditions of normality are

$$\int_a^b u_r u_s dx = 0 \quad (r \neq s),$$

$$= 1 \quad (r = s).$$

Therefore (1) becomes

$$\frac{n-p+1}{n} a_p \to \int_a^b f(x) u_p(x) dx \quad (n = n_1, n_2, \ldots, n_i, \ldots),$$

whence the required result at once follows.

The corresponding statement is true for a trigonometrical series. This is as follows:—

COR. 1.—*If $f_1, f_2, \ldots$ denote the partial Cesàro summations of the trigonometrical series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

then, provided that, as n varies,

$$\int_{-\pi}^{\pi} Q\{|f_n(x)|\}\, dx$$

remains bounded, the series is a Fourier series, and its associated function $f(x)$ is such that

$$\int_{-\pi}^{\pi} Q\{|f(x)|\}\, dx$$

exists.

COR. 2.—*The theorem is true if the Cesàro summation in question has index k, where k is any fraction (or integer) greater than or equal to* 0.

Indeed we had assumed the index of Cesàro summation to be unity:

the argument is not affected if we replace the coefficients

$$\frac{n-1}{n}, \quad \frac{n-2}{n}, \quad \ldots$$

by their k-th powers $(k \geqslant 0)$; we then have, for $k > 0$, the k-th Cesàro summation or, when $k = 0$, the ordinary summation.

4. We now occupy ourselves with the auxiliary theorems used in the proof of the main theorem. Those which follow in §§ 5–9 have already in part been given in the "Note on the Existence of Converging Sequences" quoted above.

During the early part of the War I was, owing to absence in Asia, unable to correct the proofs, and a revision of one part, found to be necessary, was made by my wife, and, as was notified in a foot-note, incorporated in the paper. The correction made, at the bottom of p. 354, however, is itself in need of correction.* The theorem (Theorem 3 of the Note cited), is correct and the mode of reasoning employed also. As I require this theorem in the present paper, I take this opportunity of giving the proof in correct form. At the same time, I have been able to give an addition to the enunciation which enhances its interest. Whereas the theorem as originally stated merely affirmed the existence of a sequence of functions in every subsuccession, provided all the lower functions of the given succession and its subsuccessions† were lower semi-continuous on the right and upper semi-continuous on the left, we are now able to omit the latter condition and to add that the limiting function of the sequence may, if we desire, be taken to be any one of these lower functions, or to coincide with any one of the upper functions except at a countable set of points. *It is hardly necessary to add that "right" may be exchanged with "left" in the theorems without affecting their validity*: this is merely equivalent to the change of $h > 0$ into $h < 0$ in the demonstrations.

I do not in the present paper use this new result, but it may, I imagine, be used to simplify the proofs of several known theorems in the Theory of Functions of a Real Variable, and perhaps to extend them to functions of more than one variable.

* It is there inferred, because the lower functions are lower semi-continuous on the right and upper semi-continuous on the left, that the same is true of the upper functions. This is true on the right, but does not follow on the left. Theorem 2, identical with Theorem 1 of the present paper is, in fact, misapplied. The property, as seen below, is, however, not required.

† By *the lower function of a succession of functions* $f_1, f_2, \ldots$, we mean that function whose value at any point x is the lowest possible limit of $f_n(x)$ as $n \to \infty$.

5. The following known theorem* is fundamental :—

THEOREM 3.—*If at any particular point all the lower functions of a succession of functions and its subsuccessions are lower semi-continuous on the right, the same is true of all the upper functions.*

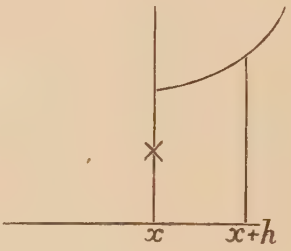
For, if x be any point, there is a subsuccession of the functions which converges at x to the value $u(x)$ of the upper function of the whole succession there. Let $l_1(x)$ be the lower function of this subsuccession. Then, since at this point x

$$l_1(x) = u(x), \qquad (1)$$

and $$u(x+h) \geqslant l_1(x+h) \quad (h > 0),$$

all the limits of $u(x+h)$ are greater than or equal to all those of $l_1(x+h)$, that is

$$\underset{h\to 0}{\text{Llt}}\, u(x+h) \geqslant \underset{h\to 0}{\text{Llt}}\, l_1(x+h), \qquad (2)$$



but, since $l_1(x)$ is lower semi-continuous on the right, we have

$$\underset{h\to 0}{\text{Llt}}\, l_1(x+h) \geqslant l_1(x).$$

Therefore by (1) and (2)

$$\underset{h\to 0}{\text{Llt}}\, u(x+h) \geqslant u(x) \quad (h > 0),$$

that is $u(x)$ is lower semi-continuous on the right.

We have here taken the whole succession, but the argument is the same for any of its subsuccessions. This therefore proves the theorem.

6. We now prove the following :—

THEOREM 4 *a*.—*If each of the lower functions of a succession of functions and its subsuccessions is lower semi-continuous on the right, then in any subsuccession we can find a sequence of the functions, whose unique limiting function is the lower function of the subsuccession.*

For since the value $l(x)$ of any function does not lie *below* the lower limit of $l(x+h)$, as $0 < h \to 0$, except possibly at a countable set of points,† we can choose a countable everywhere dense set of points $P_1, P_2, \ldots, P_n, \ldots$, such that these include the exceptional points just

* See § 4 *supra*.

† W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity", *Quart. J. of Math.*, 39 (1907), 92, Theorem 6.

referred to. We shall then have, if $l(x)$ be the lower function of the whole succession,

$$l(x) = \underset{h \to 0}{\text{lower limit}}\ l(x+h),$$

at each of the points x, except $P_1, P_2, \ldots, P_n, \ldots$, in virtue of the hypothesis made.

Now there is a subsuccession of the given functions converging at any given point to any one of the limits there, in particular, there is a subsuccession

$$f_{1,1}, f_{2,1}, \ldots, f_{n,1}, \ldots \tag{1}$$

converging at P_1 to the value of $l(x)$ there, namely $l(P_1)$.

Similarly, omitting the first member $f_{1,1}$, there is a subsuccession of (1), say,

$$f_{2,2}, \ldots, f_{n,2}, \ldots \tag{2}$$

converging at P_2 to $l(P_2)$. Repeating this process *ad infinitum*, the succession formed by the omitted first members,

$$f_{1,1}(x), f_{2,2}(x), \ldots, f_{n,n}(x), \ldots \tag{3}$$

converges at each of the points P_n, to the corresponding value $l(P_n)$, since it is a subsuccession of each of the subsuccessions preceding it.

Let $U(x)$ and $L(x)$ be the upper and lower functions of this last succession (3). Then, by the hypothesis, $L(x)$ is lower semi-continuous on the right, and, by the preceding theorem, the same is true of $U(x)$. Therefore, for $h > 0$,

$$\underset{h \to 0}{\text{lower limit}}\ L(x+h) \geqslant L(x) \quad \text{and} \quad \underset{h \to 0}{\text{lower limit}}\ U(x+h) \geqslant U(x). \tag{4}$$

But if we move up to x along points $x+h$ *belonging to the exceptional set* $P_1, P_2, \ldots, P_n, \ldots$, which is possible since this set is dense everywhere, we shall have

$$L(x+h) = l(x+h) = U(x+h), \tag{5}$$

since the succession (3) converges to $l(x+h)$ at these points. Therefore, if x do not itself belong to the exceptional set, $L(x+h)$, $U(x+h)$, and $l(x+h)$ will in this way as $h \to 0$ have the same lower limit $l(x)$. If, however, we move up to x along any other set of points, we shall obtain limits which are not less than this, since, at *any* point $x+h$,

$$U(x+h) \geqslant L(x+h) \geqslant l(x+h), \tag{6}$$

in virtue of the fact that l is the lower function of the whole succession, and L of a subsuccession.

Thus the lower limits in (4) are equal to $l(x)$, and we have consequently

$$L(x) \geqslant L(x), \qquad l(x) \geqslant U(x),$$

which is again only possible if the sign of equality be taken. Thus

$$L(x) = l(x) = U(x), \tag{7}$$

at every point not belonging to the exceptional set, and therefore at every point, without exception. This proves the theorem, since in this argument any subsuccession might be taken in place of the whole succession.

7. Similarly we have an alternative or companion theorem.

THEOREM 4*b*.—*If each of the upper functions of a succession of functions and its subsuccessions is upper semi-continuous on the left, then in any subsuccession we can find a sequence of the functions, whose unique limiting function is the upper function of the subsuccession.*

8. Combining the preceding results we have, in particular, the following, which, as we shall see, includes many of the most important cases that occur in analysis.

THEOREM 4*c*.—*If all the lower functions of a succession of functions are lower semi-continuous on the right, and all the upper functions upper semi-continuous on the left, then in any subsuccession we can find a sequence of the functions. The corresponding unique limiting function is lower semi-continuous on the right and upper semi-continuous on the left, and can always, if we please, be chosen to be the lower function of the whole succession, or subsuccession, or, if we please, the upper function.*

9. The following accessory theorems are closely connected with the preceding;—

THEOREM 1.—*A function which is lower semi-continuous on the right and upper semi-continuous on the left is continuous, except at a countable set of points.**

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* "Note on the Existence", etc., *loc. cit.* Theorem 1. The proof is immediate, using the theorem given in my *Quarterly Journal* paper above quoted, Theorem 5, p. 77, that the upper and lower limits of approach of $f(x+h)$ and $f(x-h)$, $(0 < h \to 0)$ are the same, except at a countable set of points x.

THEOREM ii.—*A monotone increasing* function is lower semi-continuous on the right and upper semi-continuous on the left.*

THEOREM iii.—*The upper and lower functions of a succession of monotone increasing functions are monotone increasing functions.*

For if $f_n(x)$, the typical function of the succession, be monotone increasing in (a, b), and $l(x)$ and $u(x)$ be the lower and upper functions,

$$f_n(x+h) \geqslant f_n(x) \quad (a \leqslant x \leqslant x+h \leqslant b) \quad (n = 1, 2, \ldots).$$

Therefore, letting n increase indefinitely along first such a succession of integers that $f_n(x)$ has its greatest limit, and then along such that $f_n(x+h)$ has its least limit, we get successively the two following relations:—

$$u(x+h) \geqslant \operatorname*{Llt}_{n\to\infty} f_n(x+h) \geqslant u(x)$$

and

$$l(x+h) \geqslant \operatorname*{Llt}_{n\to\infty} f_n(x) \geqslant l(x).$$

That is $u(x)$ and $l(x)$ are monotone increasing functions. This proves the theorem.

THEOREM iv.—*An upper semi-integral is lower semi-continuous on the right and upper semi-continuous on the left.*

By definition, an upper semi-integral is the sum of an integral (continuous function) and a positive monotone increasing function; the increment of the integral tends to zero, and that of the monotone increasing function is positive on the right and negative on the left of the point. This proves the statement.

We see, moreover, that since an integral is not only continuous, but what is called "*absolutely continuous*," an upper semi-integral may be said to be "*upper-absolutely-continuous*," that is to say, taking any set of non-overlapping intervals (x_i, x_i+h_i) $(i = 1, 2, \ldots)$, the sum of the increments of an upper semi-integral $f(x)$ over them,

$$\Sigma\{f(x+h)-f(x)\} \equiv \sum_i \{f(x_i+h_i)-f(x_i)\} \quad (0 < h)$$

tends, as the content $\Sigma h = \sum_i h_i$ approaches zero, to a limit or limits which are positive or zero.

* Called by some writers monotone non-decreasing.

I showed in my original memoir on these functions* that, just as the property of absolute continuity is both necessary and sufficient in order that a function may be an integral, so that property of upper-absolute-continuity is both necessary and sufficient in order that a function may be an upper semi-integral. We remark now that, just as the property of being absolutely continuous includes that of being continuous, so the property of being upper-absolutely-continuous includes the characteristic property of the functions, considered by me in the present note and elsewhere, of being upper semi-continuous on the left and lower semi-continuous on the right. These functions are therefore, from this point of view, more properly to be considered as "half-continuous" than the semi-continuous functions of Baire.

THEOREM V.—*If a succession of functions* $f_1(x)$, $f_2(x)$, ... *is semi-integrable below, all the upper and all the lower functions of the succession of integrals* $\int f_1(x)dx$, $\int f_2(x)dx$, ... *are upper semi-integrals.*

This follows at once from the definition of *semi-integrable below*—namely, from the fact that the integral of the lower function

$$l(x) = \underset{n\to\infty}{\text{lower limit}}\ f_n(x)$$

is less than or equal to the lower limit of the integrals of the functions. Denoting the lower and upper limits of the integrals by $L(x)$ and $U(x)$, this gives us, integrating over any interval $(x, x+h)$,

$$\int_x^{x+h} l(x)\,dx \leqslant L(x+h)-L(x) \leqslant U(x+h)-U(x),$$

whence, summing over any set of non-overlapping intervals, and letting the content of this set tend to zero, the result follows at once by what has just been pointed out.

This last theorem includes the two important cases in which a succession of functions is known to be semi-integrable below; the first is *when the succession of functions is bounded below*, that is, all the functions are greater than a fixed constant; and the second *when they are all greater than or equal to a fixed summable function.* In both these cases, therefore, we can assert that there is a succession of the integrals

* "On Semi-integrals", *Proc. London Math. Soc.* (2), 9 (1911), 286-324.

which converges to the lower function (or to the upper function) of the succession of integrals, and that there is in every subsuccession of the integrals a sequence converging to an upper semi-integral.

Hence we immediately have the following result :—

THEOREM vi.—*If a succession of functions of bounded variation has its variations bounded in their ensemble, there is in every subsuccession of the functions a sequence converging to a unique limiting function, which is, of course, itself a function of bounded variation.**

10. We proceed, in conclusion, to re-enunciate and prove the condition for supersummability.

We require the following lemmas :—

LEMMA 1.—*If* $Q(x)$ *has at every point of the interval considered* $(a < x < b)$ *a positive (non-zero) second differential coefficient, then*

$$Q\{(a_1+a_2+\ldots+a_n)/n\} \leqslant \{Q(a_1)+Q(a_2)+\ldots+Q(a_n)\}/n$$

the sign of equality holding only when all the quantities $a_1, a_2, \ldots, a_n$ *are equal.*

For if we write $$nA = a_1+a_2+\ldots+a_n,$$

and, keeping A fixed, vary the n quantities a_i in any manner, writing

$$a_i = A+h_i \quad (h_i \neq 0),$$

we have, using the theorem of the mean,

$$Q(a_i)-Q(A) = h_i\, Q'(A+h_{i,1})$$

where $h_{i,1}$ lies between zero and h_i, so that $h_i h_{i,1} > 0$. Therefore also, using again the theorem of the mean,

$$Q(a_i)-Q(A)-h_i Q'(A) = h_i h_{i,1}\, Q''(A+h_{i,2}),$$

where $h_{i,2}$ lie between zero and $h_{i,1}$. Since $Q'' > 0$, the right-hand side is positive (not zero). Summing for all values of i, and remembering that

$$h_1+h_2+\ldots+h_n = 0,$$

* "Note on the Existence", etc., *loc cit.* Theorem 6.

we see that, unless all the quantities h_i are zero,

$$Q(a_1)+Q(a_2)+\ldots+Q(a_n)-nQ(A)>0.$$

Hence the required result at once follows.

From this, we have easily the following

COR.—*If* $b_1, b_2, \ldots$ *be any positive quantities whose sum is finite,*

$$Q\{\Sigma a_i b_i/\Sigma b_i\} \leqslant \Sigma b_i Q(a_i)/\Sigma b_i.$$

The proof is precisely similar to the above, using A to mean

$$\Sigma a_i b_i/\Sigma b_i$$

so that

$$\Sigma b_i h_i = 0.$$

LEMMA 2.—*Under the same condition as the preceding Lemma,*

$$Q\left\{\int |f(x)|\,dx \Big/ \int dx\right\} \leqslant \int Q\{|f(x)|\}\,dx \Big/ \int dx.$$

This is obvious when $f(x)$ is a constant, and hence, using the above lemma, when $f(x)$ is constant in stretches. By the method of monotone sequences it then follows for any summable function $f(x)$.

COR.—*Under the conditions of the preceding Corollary,*

$$Q\left\{\int |f(x)|\,dg(x) \Big/ \int dg(x)\right\} \leqslant \int Q\{|f(x)|\}\,dg(x) \Big/ \int dg(x)$$

where $g(x)$ *is any monotone increasing function.*

It follows in particular that

$$Q\left\{\int_E |f(x)|\,dx/E\right\} \leqslant \int_E Q\{|f(x)|\}\,dx/E,$$

where E stands alike for a set of points and its content.

11. THEOREM 5.—*If* $U(t)$ *is a positive monotone increasing function of* t, *defined for all positive values of* t, *and such that it possesses everywhere a differential coefficient* $U'(t)$ *which is positive (never zero), and in addition* $U(t)$ *tends to infinity as* t *increases indefinitely, then, if we write* $Q(t)$ *for the integral of* $U(t)$,

$$Q(t) = \int_a^t U(t)\,dt,$$

the necessary and sufficient condition that $F(x)$ *should be the integral of a function* $f(x)$, *such that*

$$Q\{|f(x)|\};$$

is a summable function of x in some fundamental interval $a \leqslant x \leqslant b$, is (i) *that, when we take any set of non-overlapping intervals inside that fundamental interval, and denote the length of the generic interval by h, and its left-hand end-point by x,*

$$\Sigma hQ\{|F(x+h)-F(x)|/h\} \leqslant B,$$

where B is a finite constant, independent of the particular set of intervals chosen.

Moreover, (ii) *the constant B can have any value greater than or equal to*

$$\int_a^b Q\{|f(x)|\}\,dx \quad \text{where} \quad F(x) = \int_a^x f(x)\,dx,$$

*but no smaller value.**

In particular, in any interval to the right of a certain fixed point, $Q(x)$ may be any of the following functions :—

$$x^{1+p}, \quad x\log x, \quad x\log\log x, \quad x\log\log\log x, \quad \text{etc.},$$

the corresponding values of U, being

$$(1+p)x^p, \quad \log x+1, \quad \log\log x+1/\log x, \quad \log\log\log x+1/\log\log x, \quad \text{etc.},$$

satisfying the requirements.

For instance, in any interval (a, b), where $1/e < a < b$, $Q(x)$ may be $x\log x$.

To prove the necessity of (i) and (ii), we assume that $\int_a^b Q\{|f(x)|\}\,dx$ exists, and write

$$F(x+h)-F(x) = \int_x^{x+h} f(x)\,dx.$$

Then we have, since $Q(t)$ is a monotone increasing function of t, and $h > 0$,

$$hQ\{|F(x+h)-F(x)|/h\} \leqslant hQ\left\{\int_x^{x+h} |f(x)|\,dx/h\right\},$$

which, using the preceding lemma, is less than or equal to

$$\int_x^{x+h} Q\{|f(x)|\}\,dx.$$

---

* See also concluding note to the present paper.

Summing over any set of non-overlapping intervals, the result is therefore less than or equal to

$$\int_a^b Q\{|f(x)|\}\,dx,$$

which is by hypothesis a finite constant. This proves the necessity of the condition.

Next, to prove the sufficiency of the condition, we proceed in three steps :—

STEP 1.—*To show that* $F(x)$ *is a continuous function of* x.

By hypothesis,

$$\Sigma hQ\{|F(x+h)-F(x)|/h\} < B,$$

therefore this must be true, and therefore have a meaning, when there is only one term in the summation, and for every interval of length, however small. If, therefore, $Q(t)$ is a function which only exists in a finite interval $(\alpha \leqslant t \leqslant \beta)$, the incrementary ratio $\{F(x+h)-F(x)\}/h$ is bounded, and therefore, by known theorems, $F(x)$ is an absolutely continuous function (*i.e.* an integral), which includes the second step in the proof. If, however, $Q(t)$ exists in the infinite interval $(0 < t < \infty)$, we easily see that the increment $F(x+h)-F(x)$ must tend to zero with h, whether x be fixed or not: for, if this were not so, we could give a succession of such increments which remained greater than some positive quantity k, as $h \to 0$, and therefore, since $Q(t)$ is monotone increasing, for such values of h,

$$hQ\{k/h\} < B,$$

that is

$$h\int_a^{k/h} U(t)\,dt < B,$$

which is impossible, since, by hypothesis, $U(t) \to \infty$, as $t \to \infty$, so that, for sufficiently small h,

$$h\int_a^{k/h} U(t)\,dt > h\int_{\frac{1}{2}k/h}^{k/h} U(t)\,dt > hM/h > M,$$

where M is as large as we please. Thus the increment always tends to zero, and therefore $F(x)$ is a continuous function of x.

STEP 2.—$F(x)$ *is an absolutely continuous function of* x.

As already remarked, we only still need to prove this, when $Q(t)$ is defined for every finite positive value of t. But since the condition of

absolute continuity,

$$\sum_E |F(x+h)-F(x)| \to 0 \quad \text{when} \quad E \to 0$$

(where the same letter E is used to denote the set of intervals and its content), is such that it evidently holds for all sets of intervals if it is known to hold for all such sets containing only a *finite* number of intervals, and, by reason of the continuity of $F(x)$, already proved, it is sufficient to prove the condition for rational values of h only, we shall confine our attention to finite sets of intervals of rational length.

In this case, however, we may, without loss of generality, assume that all the intervals of the set E are equal; for we can, if they are unequal, subdivide them into equal intervals, since they are rational in length and finite in number; and the summation over the equal intervals so formed will not be less than it was before, since the modulus of a sum is not greater than the sum of the moduli of the parts.

Let n be the number of the intervals and h their common length. Then, by hypothesis,

$$h \Sigma Q\{|F(x+h)-F(x)|/h\} < B$$

and therefore, dividing both sides by n, and using the first of our two lemmas,

$$hQ\left\{\frac{1}{nh}\Sigma|F(x+h)-F(x)|\right\} < B/n.$$

That is,

$$EQ\left\{\frac{1}{E}\Sigma|F(x+h)-F(x)|\right\} < B.$$

As in the proof of Step 1, it now follows that the summation must tend to zero with E, which proves that $F(x)$ is an absolutely continuous function of x.

STEP 3.—*To prove that $f(x)$, the integrand of $F(x)$, is such that $Q\{|f(x)|\}$ is summable and that $\int_a^b Q\{|f(x)|\}$ is the upper bound of the summations.*

Let in the first instance $f(x)$ denote a positive summable function whose integral is $F(x)$, and satisfies the condition

$$\Sigma hQ\{[F(x+h)-F(x)]/h\} \leqslant B.$$

What we have to show is that $Q\{f(x)\}$ is summable, and that its integral is less than or equal to B. Since this then holds for every B, $\int Q\{f(x)\}dx$ will be the upper bound of the summations, in virtue of the necessity, already proved, of the condition.

If $f(x)$ is a constant, $Q\{f(x)\}$ is also a constant, and therefore of course summable, and, the condition being satisfied,

$$\int_a^b Q\{f(x)\}\,dx = \Sigma \int_x^{x+h} Q\{f(x)\}\,dx = \Sigma Q\{f(x)\}\,.\,h$$

$$= \Sigma hQ\left\{\int_x^{x+h} f(x)dx/h\right\} \leqslant B.$$

Again, if $f(x)$ is a simple function, constant in stretches, $(x, x+h)$, we have, in virtue of the condition we know to be satisfied,

$$\Sigma \int_x^{x+h} Q\{f(x)\}\,dx = \Sigma hQ\left\{\int_x^{x+h} f(x)\,dx/h\right\} \leqslant B.$$

But the former summation is by definition the integral of $Q\{f(x)\}$ from a to b, which accordingly exists. Thus the theorem is true when f is a simple function.

Next suppose $f(x)$ is the limit of a monotone ascending sequence

$$f_1(x) < f_2(x) < \ldots < f_n(x) < \ldots \to f(x),$$

of positive functions (simple or otherwise), for which the theorem has been proved true. Then, denoting the integrals by capital letters,

$$F_1(x) \leqslant F_2(x) \leqslant F_3(x) \ldots \to F(x),$$

and, since $Q(x)$ is a monotone ascending function of x and is continuous,

$$Q\{f_1(x)\} \leqslant Q\{f_2(x)\} \leqslant \ldots \leqslant Q\{f_n(x)\} \leqslant \ldots \to Q\{f(x)\}\,.$$

Therefore the necessary and sufficient condition that the function $Q\{f(x)\}$ should be summable is that the limit of $\int_a^b Q\{f_n(x)\}\,dx$ should be finite and this limit will be the integral of $Q\{f(x)\}\,dx$.

But since by hypothesis the condition

$$\Sigma hQ\{[F(x+h)-F(x)]/h\} \leqslant B$$

holds, therefore also, since Q is monotone ascending and continuous,

$$\Sigma hQ\left\{\int_x^{x+h} f_n(x)\,dx/h\right\} \leqslant B.$$

By our supposition this proves that $Q\{f_n(x)\}$ is summable, and that we then have

$$\int_a^b Q\{f_n(x)\}\,dx \leqslant B.$$

Since this is true for all integers n, it remains true in the limit, whence, as pointed out, the truth of our theorem holds in this case for the function $f(x)$.

The theorem is therefore true for l functions, as being the limits of monotone ascending sequences of simple functions. It is immediately evident from the conditions of the problem that it is true for a bounded u-function, as being expressible as the difference between a constant and a positive l-function. Hence it is true for lu and ul functions, and hence for all positive summable functions, bounded or unbounded. But if $f(x)$ be not a positive function, we only write

$$f(x) = f_1(x) - f_2(x),$$

where $f_1(x) = 0$ where $f(x) \geqslant 0$, and $f_2(x) = 0$ where $f(x) < 0$. Then the given condition is satisfied for $f_1(x)$ and $f_2(x)$ separately, so that since

$$Q\{|f(x)|\} = Q\{f_1(x)\} + Q\{f_2(x)\}$$

the required result follows at once.

[*Note added during passage through press,* October 10th, 1924.— Without using monotone sequences, we can in the following way prove Step 3. Divide (a, b) into a finite number of parts $(x_{n,r}, x_{n,r}+h_{n,r})$ at successive stages $n = 1, 2, \ldots$, so that the greatest of the quantities $h_{n,r}$, for fixed n, tends to zero when $n \to \infty$. We then define a function $g_n(x)$, constant in stretches, so that, in the interval $(x_{n,r}, x_{n,r}+h_{n,r})$ for the particular n in question,

$$g_n(x) = \{|F(x_{n,r}+h_{n,r}) - F(x_{n,r})|\}/h_{n,r}.$$

Hence, using the given condition,

$$\int_a^b Q\{g_n(x)\}\,dx \leqslant B.$$

We see also that, except at the countable set of points $x_{n,r}$ and the set of content zero at which $F'(x)$ does not exist, or is not equal to $f(x)$,

$$\operatorname*{Lt}_{n\to\infty} g_n(x) = |f(x)|.$$

Hence the positive functions $Q\{g_n(x)\}$ form a succession which converges almost everywhere to $Q\{|f(x)|\}$. Since, as we saw, the integrals of this succession are less than or equal to B, the succession is semi-integrable below, that is, the lower function is summable, and its integral is less than, or equal

to, the limit of the integrals. But, as we already saw (p. 15 *supra*), since $\int_a^b Q\{|f(x)|\}\,dx$ exists, the integrals of our succession do not exceed it, and must therefore tend to it as limit.

Thus $$\int_a^b Q\{|f(x)|\}\,dx = \operatorname*{Lt}_{h\to 0} \sum_a^b hQ\{|F(x+h)-F(x)|/h\}$$

and is at the same time the upper bound of this summation.

The artifice used in this proof was already introduced in the original proof of the theorem in a joint paper by my wife and myself on the Riesz-Fischer Theorem [*Quart. J. of Math.*, 44 (1912), 66–68]. It is, perhaps, here the place to point out a curious error on p. 67, *loc. cit.* Of course we cannot assume that $Q \geqslant Au^2+Bu+C$. An extra condition, to replace this, has to be inserted in the enunciation, and this has been done, in the present paper, by postulating that $U(u)\to\infty$, which leads, as on p. 4 *supra*, to the result required on p. 67, *loc. cit.* This ensures at the same time the same property for the inverse function $V(v) = u$, where $v = U(u)$.

It should also be remarked that the inequality of Lemma 2 *supra*, when the integration is not over a whole interval, but over a set of points E, does not follow immediately from Lemma 1, as assumed on p. 67, *loc. cit.*, but, as here pointed out, from the corollary to Lemma 1.]

ON STOKES'S THEOREM.

By W. H. Young.

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1. The necessity of establishing rigorously, under the most general conditions obtainable, the validity of the fundamental theorems of mathematical analysis has been recognised for many years. Sometimes it has been found possible to give one or more sets of necessary and sufficient conditions under which such a theorem is true. But in the majority of cases, including some of the most interesting, it has not only been found impracticable to discover such a set, but it lies in the nature of things that no such conditions exist, other than a mere rewording of the theorem. It may indeed happen that both conditions which are sufficient can be found, and others which are necessary. The problem would then be to find on the one hand the most wide set of necessary conditions possible, and, on the other, the most narrow set of sufficient conditions.

In certain cases this problem may seem to have a final solution in this sense, that we may arrive at a last term in the series of successive approximations to which in the natural progress of research we may be led. In others, the facility may consist in establishing a scale of an infinite number of terms, or an infinite number of scales with each an infinite number of terms, as, for example, in the theory of the convergence of series.

In some domains very little progress has been made up to the present. Thus, for a function to possess a first differential it is necessary that the various partial differential coefficients should exist. It is sufficient that one of these should be continuous at the point considered with respect to the ensemble of all the n independent variables, a second with respect to the ensemble of the $(n-1)$ variables omitting the variable of differentiation of the preceding differential coefficient, a third with respect to the ensemble of the $(n-2)$ variables omitting both the preceding variables of differentiation, and so on as far as the last but one, which is continuous with respect to a pair of variables, while the last differential coefficient needs only to exist at the point.

I am not aware that any progress has been made in the direction of improving this set of conditions. I do not indeed feel sure that even any attempt has been made to solve the more general problem of sufficient conditions under which we may employ the expression for a first differential, for example, under the integral sign, when less is known than is implied in the above conditions, a problem of which a solution is given in § 29 below. In this connexion another remark occurs to me. There is a certain repugnance in the mathematical mind to a lack of symmetry, as in life there is to discord; and in the search after truth it is the symmetrical and harmonious rather than the unsymmetrical and inharmonious which is felt to possess finality. From this point of view the conditions for a first differential are unsatisfactory. Not only is favouritism, so to speak, shown in a marked degree, but in the conditions there is a sudden jump at the end. All but one of the partial differential coefficients have to satisfy a condition of continuity which, if we impose it on the corresponding derivates, ensures the existence at the point of the differential coefficient in question; but the harmonious descent of the scale of continuity connected with the first $(n-1)$ differential coefficients is broken off at the last step, the mere existence of the n-th being demanded, and not its continuity with respect to the single remaining variable. The same reproach must be made to the conditions given below under which it is allowable to write

$$X = \int \left\{ \frac{dX}{dx}\frac{dx}{dt} + \frac{dX}{dy}\frac{dy}{dt} + \frac{dX}{dz}\frac{dz}{dt} \right\} dt,$$

when X need not satisfy the above conditions for a differential. These conditions are as follows :—

(i) *The derivates dx/dt, dy/dt, and dz/dt are summable functions of t;*

(ii) *The partial derivates of X are bounded functions of (x, y, z) on the curve*

$$x = x(t),\ y = y(t),\ z = z(t)$$

considered;

(iii) *dX/dx exists as a differential coefficient at each point of the curve;*

dX/dy exists and is a bounded partial differential coefficient with respect to x throughout a certain linear neighbourhood $(x-H, x+H)$, y and z being constant, of each point of the curve;

dX/dz exists throughout a certain plane neighbourhood of each point (x, y) of the curve $(x-H, y-K;\ x+H, y+K)$, z being constant, and in

this neighbourhood dX/dz is either a bounded differential coefficient with respect to x and a bounded differential coefficient with respect to y, or it is the bounded differential coefficient with respect to x of a bounded differential coefficient with respect to y.

In generalising a known theorem in the sense of relaxing the conditions under which its employment is recognised to be legitimate, we must carefully bear in mind the possibility of such a want of symmetry in the conditions obtained, and we must not regard the question as prejudged even when in the enunciation as well as in sets of conditions already in use a certain symmetry exists.

In the theorem which I take as the subject of the present communication, due, in so far as the formula

$$\int_C \{X(x, y, z)\,dx + Y(x, y, z)\,dy + Z(x, y, z)\,dz\}$$
$$= \iint_S \left\{ l\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + m\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + n\left(\frac{dY}{dx} - \frac{dX}{dy}\right) \right\} dS$$

is concerned, to Stokes,* the enunciation is symmetrical in form, and the conditions, even as stated as recently as last year,† conform to the law of symmetry. These conditions possessed the following characteristics:—

(i) they were symmetrical with respect to the dependent and the independent variables simultaneously;

(ii) all the functions and the partial differential coefficients were continuous with respect to the ensemble of all the independent variables;

(iii) the surface S was supposed to have a continuously varying tangent plane, and its bounding curve C a continuously varying tangent line.

This theorem is so classical and so constantly in request as to belong almost to the platitudes of mathematical analysis, and it might be supposed that little remained to be done in the way of generalising it, and that any formulation of fresh conditions which render the theorem or its extensions valid would preserve at any rate the property of symmetry. As far as I am aware, however, the only novelty that has been introduced is in con-

* Question 8, Smith's Prize Paper (1854), Cambridge University Calendar (1854).

† L. E. J. Brouwer, "Remark on Multiple Integrals", *Amsterdam Proceedings*, 22 (1919), 150–154. In addition to the above this author has a condition further restricting the generality of the surface and of its bounding curve. The present paper was completed in manuscript form in 1921, but was delayed till the present, various causes having prevented its being put into finished form for publication.

nexion with the extension of Stokes's theorem to manifolds, not necessarily of 2 dimensions, in space of n dimensions, a generalisation in which, for the rest, the sole difficulty is one of conception, more especially in that generalisation of the idea of "sense" on a line, which has been called *the indicatrix*.

In this paper I propose in the first place to indicate how the class of surfaces on which Stokes's theorem may be used can be extended so as to include many of the very general surfaces for which I have shown the term *area of the surface* to have a meaning. In the course of the work we shall, in the second place, see how contingent the fulfilment of the conditions of symmetry is. Mere accidents in the mode of demonstration are responsible for the subjection of all the partial differential coefficients to the same type of condition. This does not lie at all in the nature of things: quite the contrary. The model we have to keep in our minds is indeed precisely that of the varying types of conditions to be satisfied by the differential coefficients in order that the first differential should exist. More precisely, the conditions resemble those required for us to be able to use the expression for the first differential under the integral sign.

This connexion, indeed, is not merely one of form. In that method of proof which on the purely formal side may be regarded as due to Maxwell, the first differentials of the functions appear, implicitly in the old proofs, explicitly in that given below. In the considerably more powerful method of proof that I have devised and developed in Part III below, the artifice is precisely that which applied to the discussion of integration of the first differential leads to conditions which are parallel to those which I obtain for the validity of Stokes's theorem.

I have not in the present communication followed the order of the ideas as they presented themselves to my mind. It was to Stokes's theorem that I first applied the artifice referred to, without any reference to the theory of differentials. It seems to me, however, desirable to begin by attacking the problem of differentials, as in this way the inevitable character of the conditions for the validity of Stokes's theorem, as subsequently discussed by the same method, is brought to light. I deal briefly with the Maxwell form of proof before passing to the direct application of the artifice in question to Stokes's theorem; all this in the second and third parts of the paper.

In the first part of the paper I obtain the generalisation of the nature of the surface already referred to, using for this purpose the general methods I have explained elsewhere dealing with areas of surfaces and what I have called *directed areas in the plane*, these being defined by bounding curves which may cut themselves in any manner subject only to

the conditions of a very slight nature imposed. Such curves naturally present themselves, for example, as the projections on a plane of curves without double points, forming circuits or cuts on a surface.

The conditions imposed on the surface are the following:—

(1) *The surface is bifacial;*

(2) *the equations to the surface may be written*

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

$$(a \leqslant u \leqslant c; \quad b \leqslant v \leqslant d).$$

where $x(u, v)$, $y(u, v)$, *and* $z(u, v)$, *regarded as functions of one of the variables* (u, v), *the other being constant, are absolutely continuous;*

(3) *the partial differential coefficients* dx/du, dy/du, dz/du *are each numerically less than a summable functions* $\lambda(u)$ *of* u *alone, at the points where these differential coefficients exist and are finite, and the same holds with* u *and* v *interchanged.*

Condition (3) imposed on the partial derivates may be termed *a condition of infrasummability with respect to the variable of differentiation*, Each of the coordinates x, y, and z is *an infrasummable function* with respect to each of the parameters u, v, being besides, in virtue of the condition (2), the integral of either of its partial derivates. I propose to term such a surface *an infrasummable surface.*

The two proofs here given of Stokes's theorem (§ 32 and §§ 34–36) are neither long nor difficult, when we have cleared the ground by proving all the required auxiliary properties of the functions which enter. I may call attention in this connexion to the rules given in Part II, under which we may say that a function of any number of variables is the differential coefficient of its multiple integral at a certain point, rules which may be extended to unbounded functions by the same method.

The proof, which I have called the Maxwell proof, being due in its original form to James Clerk Maxwell, is familiar in form to all students; what I have been able to do is to render it definite and rigid for an infrasummable surface and under conditions explicitly stated as follows:—

(*a*) $X(x, y, z)$ *is a continuous function of* x;

(*b*) *the partial derivates of* X, *of* Y, *and of* Z *are bounded functions of* (x, y, z) *on the surface;*

(*c*) *the same is true in a closed neighbourhood of the surface, omitting from consideration* $\partial X/\partial x$, $\partial Y/\partial y$, *and* $\partial Z/\partial z$;

(*d*) *each of the functions X, Y, and Z has a first differential on the surface with respect both to u, and to v, separately, or satisfies any of the sets of conditions proved sufficient for us to treat it as if it had a first differential, under the integral sign.*

Under these circumstances Stokes's formula, as given above, may be used.

By means of the second method of proof (Part III) we are able to dispense with any reference to the partial differential coefficients, or derivates, which do not appear explicitly in Stokes's equation, and we need not refer to the possible existence of a differential. Retaining the condition (*c*) above, by which *those partial derivates of X, Y, and Z which occur in Stokes's formula are bounded functions of* (x, y, z) *in a certain closed neighbourhood containing the surface*, we only require *X to be a continuous function of x* [*when* (y, z) *is constant*], *or, if not, to be a bounded function of* (x, y, z), *whose integrals with respect to y and z,* $\int X\,dy$ *and* $\int X\,dz$, *are continuous functions of* (x, y, z), *and similar conditions for Y and Z.**

* In a paper "On Stokes's Formula and the Maxwell-Lorentz Equations," written after the present paper was completed in manuscript, and published last year in *Aberystwyth Studies*, Vol. IV, pp. 181–189, I applied the method of Part II of the present paper to the case of four-dimensional space.

The formula in the "partial Stokes form" is then

$$\iint_S P\frac{d(x, y)}{d(u, v)}\,du\,dv = \iiint_\Omega \left\{\frac{dP}{dz}\frac{d(x, y, z)}{d(p, q, r)} + \frac{dP}{dt}\frac{d(t, x, y)}{d(p, q, r)}\right\} dp\,dq\,dr, \tag{1}$$

P being a function of (x, y, z, t), Ω a three-dimensional manifold, and S its bounding surface. The "complete Stokes formula," involving three such functions $P_{i,j}\,(i, j = 1, 2, 3)$, may then be written in a handy form by using the notation

$$Q_i = \frac{dP_{i1}}{dx_1} + \frac{dP_{i2}}{dx_2} + \frac{dP_{i3}}{dx_3}$$

where

$$P_{i,j} = -P_{j,i}.$$

The complete Stokes formula becomes then

$$\iint_S \{P_{12}d(x_3, x_4) + P_{23}d(x_1, x_4) + P_{31}d(x_2, x_4) + P_{14}d(x_2, x_3) + P_{24}d(x_3, x_1) + P_{34}d(x_1, x_2)\}$$
$$= \iiint \{Q_1 d(x_2, x_3, x_4) + Q_2 d(x_3, x_1, x_4) + Q_3 d(x_1, x_2, x_4) - Q_4 d(x_1, x_2, x_3)\} \tag{3}$$

where the differential bracket symbol stands for the product of the Jacobian of the coordinates in the bracket into the differentials of the parameters.

The theorem which I gave is accordingly as follows:—

Stokes's complete formula holds when, for each of the indices $i = 1, 2, 3, 4$, *each function* Q_i *of the four variables* (x_1, x_2, x_3, x_4) *is such that, regarded as a function of the single variable* x_i, *it is a differential coefficient. We also assume that the partial differential coefficients of*

PART I.

2. In the present part of the paper we fix our attention on the types of surfaces on which Stokes's theorem may be used. We exclude once and for all any surface which is *unifacial*, such as the well known ribbon of Möbius, the fundamental property of which –in popular language this property is that a little triangle, simply pushed about on the surface and brought back to where it was, may be found to have been turned inside out—is clearly incompatible with the applicability of the formula.

Confining ourselves, then, to *bifacial* surfaces, we start with the plane. Regarded as determined, for instance, by an equation in Cartesian co-ordinates,

$$lx+mx+nz-p=0,$$

the plane is not yet defined as bifacial. It divides, however, space into two parts, which for convenience we shall call the *positive and negative*

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the functions $P_{i,j}$ that actually appear in the formula are bounded functions of the ensemble of all the coordinates, though it may be remarked that the method employed may be shown to lead to somewhat more general results. No assumption is made as to the existence of partial differential coefficients which do not appear; but we assume finally that $P_{i,j}$ is a bounded function, and, regarded as depending on the remaining pair of coordinates, when x_i and x_j are supposed constant, is a mixed differential coefficient of the second order [and therefore the mixed differential coefficient in the right order of its repeated integral with respect to the two remaining coordinates when x_i and x_j are constant. There is an obvious slip in the indices in this statement in the *Aberystwyth Studies*, p. 183, lines 11 and 12].

I then compared the equations (2) with the Maxwell-Lorentz equations of electro-dynamics

$$\frac{dN}{dy}-\frac{dM}{dz}=\frac{dX}{dt}+A, \qquad \frac{dZ}{dy}-\frac{dY}{dz}=-\frac{dL}{dt}$$

with two pairs of equations got from these by cyclical interchange of the letters (X, Y, Z) and (A, B, C), together with the pair of equations

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=\rho, \qquad \frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}=0.$$

These are in fact two sets of four equations of the type (2). In the second set the Q's are all zero, and therefore satisfy our requirements. In the first set, however, the Q's are the quantities A, B, C, and ρ; so that, to bring this into line with our theorem, we require that *A should be a bounded differential coefficient with respect to x, B with respect to y, C with respect to z, and ρ with respect to t; and that the partial differential coefficients that actually occur in these eight equations should exist and be bounded.*

When we take the two systems of four equations together, no more is required, since the further conditions of our theorem are then fulfilled of themselves.

Under these conditions the Maxwell-Lorentz equations may be accordingly replaced by the corresponding integral equations

$$\iint_S (X\,dy\,dz+Y\,dx\,dz+Z\,dx\,dy-L\,dx\,dt-M\,dy\,dt-N\,dz\,dt) = \iiint \{A\,dy\,dz\,dt+B\,dz\,dx\,dt+C\,dx\,dy\,dt+\rho\,dx\,dy\,dz\}$$

and

$$\iint_S \{L\,dy\,dz+M\,dx\,dz+N\,dx\,dy+X\,dx\,dt+Y\,dy\,dt+Z\,dz\,dt\}=0.$$

parts, such that any continuous path from a point of one part to a point of the other part contains at least one point of the plane. If we draw a normal to the plane there will be two possible senses to it: the sense which proceeds from the plane towards the positive part of space, or, which is the same thing, towards the plane from the negative parts of space, this we shall call the *positive sense of the normal*, and the opposite, the *negative*, sense. Consider now a right-handed screw round the positive direction of the normal; this will determine a circular sense round the foot of the normal in the plane, a so-called *indicatrix of the point*. Viewed from the positive part of space, this indicatrix appears to be anti-clockwise, and, viewed from the negative part, clockwise, in the plane. The ensemble of the analytical point (x, y, z) of the plane together with this indicatrix is said to constitute *a point of the bifacial plane*, and indeed *a point of its positive face*. The points with the opposite indicatrix, that which viewed from the positive part of space seems clockwise, constitute the points of the other, or *negative*, face of the bifacial plane.

In accordance with this understanding, we may speak of the points of the positive face as belonging to the positive part of space, in so far as they can be regarded as the traces, or feet, of the positive normals in the positive part of space; and it is in this strictly mathematical sense that we may say that a point on the positive face can be reached, for the purpose of drawing or marking it, by a pencil in the positive part of space, or, as we need to do in the present connexion, that a triangle *is drawn* on the positive face of the plane.

Conversely, since the distinction of positive and negative hitherto used is purely relative, having defined what we mean by a triangle being drawn on one of the faces of our bifacial plane, we may call the face on which it is drawn the positive, or outside face, and the corresponding normal the positive or outward normal. This we shall in future do.

3. Denote then by D, or by ABC, a triangle, with a definite sense, indicated by the order of the points ABC, drawn on the positive, or outside, face of a certain plane, the sense being positive if the points follow one another in anti-clockwise order, when viewed from the positive part of space, and negative in the contrary case, *the formula we have to prove is now*

$$\int_D X(x, y, z)\, dx = \iint_D \left(m \frac{dX}{dz} - n \frac{dX}{dy}\right) dS,$$

where X is a function of (x, y, z) which, as well as its partial differential coefficients dX/dy and dX/dz, is continuous with respect to the ensemble

(x, y, z) *throughout the closed triangle* D. The integral on the left is a contour integral, round the perimeter of the triangle in the prescribed sense ABC, and the integral on the right is a surface integral throughout the surface of the triangle. The quantities m and n which occur on the right belong to the trio (l, m, n) of the direction cosines of the outward normal to the plane.

We might now assume this formula proved, since our object is to pass on to the case where the surface is of a complicated character. As, however, the proof in the general case rests on this as basis, and the proof, though simple, requires some care as to signs, it seems better to give it here.

4. Let our plane $$lx+my+nz-p=0$$

cut the axes respectively in the points L, M, and N, and O be the origin. Then, if the direction cosines of the line MN, drawn from M to N, be $(0, \beta, \gamma)$,

$$\beta = MO/MN = -n/\sqrt{(1-l^2)},$$

$$\gamma = ON/MN = m/\sqrt{(1-l^2)}.$$

Through the point L, on the axis of x, and the perpendicular p through the origin, draw the plane LOK, meeting our plane in the line LK, perpendicular to MN, K being the point in which this plane meets MN. Then OLK is a triangle in this plane, right-angled at O, and p is the perpendicular from O on the hypotenuse LK. *Therefore* OLK *is the acute angle between the axis of* x (*i.e.* OL) *and the direction in our original plane perpendicular to* MN (*i.e.* KL), and is complementary to the acute angle between the axis of x and the perpendicular p on our plane. Thus

$$OLK = \sin^{-1} l = \cos^{-1}\sqrt{(1-l^2)}.$$

5. Now consider the triangle D, or ABC, drawn on the positive face of the plane LMN, and cut it by a plane parallel to that of (y, z). Let the line of intersection, in the same sense as MN, enter the triangle at P_1 and emerge at Q_1; and, when the cutting plane advances a distance h in the direction of increase of x let P_2 and Q_2 be the new points of entry and egress; also let P and Q be the middle points of P_1P_2 and Q_1Q_2.

Now the sense of ABC may be positive or negative, and we shall assume, for clearness in the argument, that it is positive, *i.e.* anti-clockwise when viewed from the positive part of space. The sense of P_1P_2 and of Q_2Q_1 may be the same, or it may be the contrary to the sense of ABC,

but in either case the sign to be attributed will be the same as that to be attributed to the area of the quadrilateral $P_1P_2Q_2Q_1$ in the course of the ensuing argument. Consequently the argument will be perfectly general, without any confusion as to signs, if we assume during it that P_1P_2 and Q_2Q_1 are also in the positive sense of the periphery.

6. In the following, as throughout in the present paper, the symbol θ is to be interpreted as some indeterminate quantity, or function, numerically less than or equal to unity, and not always the same in different expressions or equations.

Consider the contour integral of our function $X(x, y, z)$ with respect to x round the triangle ABC. We have already said that X as well as its partial differential coefficients with respect to y and z are taken to be continuous with respect to (x, y, z), and we shall assume for the moment that the triangle D, or ABC, is so small that the oscillation of each of these three continuous functions in it is less than an arbitrary e.

In consequence of what was said in the preceding article, we may disintegrate the contour integral into the sum of the contour integrals round the quadrilaterals* $P_1P_2Q_2Q_1$ which fill up the triangle. As x is constant along two of the sides of this quadrilateral, and in the other two sides we may give X a constant value, equal to its value at any convenient point of the triangle, to *e près*, the contribution of the quadrilateral $P_1P_2Q_2Q_1$ may be expressed as follows:

$$\{X(Q)-X(P)\}h+2\theta eh.$$

But, by the Theorem of the Mean,

$$\frac{X(Q)-X(P)}{PQ}=\frac{dX}{ds},$$

the differential coefficient on the right being taken at some point of PQ with respect to that line, we may therefore replace it by

$$\beta\frac{dX}{dy}+\gamma\frac{dX}{dz}+\theta e\beta+\theta e\gamma,$$

the partial differential coefficients being taken at any point we please of the triangle.

Hence the contribution of our quadrilateral may, using the values found

* If P_1P_2 is not in the same sense as ABC, we should need here to write $P_2P_1Q_1Q_2$ in order that the quadrilateral might have the same sense as the triangle. This is the first place in the argument where the remark of § 5 is necessary.

for β and γ, be written

$$\left(-n\frac{dX}{dy}+m\frac{dX}{dz}\right)PQ\,.\,h\Big/\sqrt{(1-l^2)}+4\theta eh.$$

But, by article 4, $\sqrt{(1-l^2)}$ is the cosine of the acute angle between the axis of x and the perpendicular distance between the parallel sides of the quadrilateral, so that $PQ\,.\,h/\sqrt{(1-l^2)}$ is the area of the quadrilateral; also multiplying this by n or m we get its projection on the coordinate plane (x, z) or (x, y). When we add* for all the quadrilaterals, we therefore get the projections D_y and D_z of the triangle D on the same coordinate planes. Moreover the h of the last term in our expression adds up to something not exceeding the height H of the triangle D. Thus the result of our addition gives

$$\int_D X(x, y, z)\,dx = D_y\frac{dX}{dz}-D_z\frac{dX}{dy}+4\theta eH.$$

But we may in the surface integral also replace the integrand by its value at any point we please of the triangle, if we add the error. Thus we may write

$$\iint_D\left(m\frac{dX}{dz}-n\frac{dX}{dy}\right)dS = D_y\frac{dX}{dz}-D_z\frac{dX}{dy}+2\theta eD.$$

These two relations give us

$$\int_D X(x, y, z)\,dx = \iint_D\left(m\frac{dX}{dz}-n\frac{dX}{dy}\right)dS-2\theta eD+4\theta eH.$$

7. Now let us drop the restriction as to the size of the triangle D. It may then be subdivided up into a number of such little triangles, in each of which the equation last written down is valid. Adding all these, we get an equation of the same form, in which however only the quantity e remains small, indeed, as small as we please: D and H are the area and height of our triangle. By the usual argument we may then omit altogether the terms involving e, and we get *Stokes's theorem for a perfectly general triangle* D,

$$\int_D X(x, y, z)\,dx = \iint_D\left(m\frac{dX}{dz}-n\frac{dX}{dy}\right)dS,$$

under the assumption that X, dX/dz, *and* dX/dy *are continuous with respect to* (x, y, z) *throughout the closed triangle* D.

* This is the second place in the argument where the relative sense of P_1P_2 to ABC comes in, and compensates the former.

8. In passing from the preceding result to the case when the surface is one formed of a finite number of triangles, we have no difficulty if all the triangles lie in a single plane, or if the number of triangles is at most four; but in the general case we have to stipulate that *the surface formed is bifacial*, and this condition is then not only necessary but sufficient for the validity of Stokes's theorem. Indeed, as Möbius showed,* such a surface, a Δ-*faced polyacron*, as Cayley termed it, or *trigonal polyhedron*, to use Möbius's terminology, *being bifacial*, is such that the senses of the Δ-faces can be so assigned that the two senses which appear connected with each edge are opposite. Here, of course, the surface considered is a closed surface, that is to say each edge is a side of precisely two of the Δ-faces, and there is no boundary. In the application to Stokes's theorem we are isolating a portion of such a surface, bounded by a single polygon, formed of edges of the polyacron. *The Law of Edges* (*Kantengesetz*) given by Möbius, and mentioned above, by which each edge of the polyacron has connected with it the sense + when regarded as belonging to one face and — when belonging to the other, then ensures that the sides of the bounding polygon, as determined by the triangles of our portion, follow one another in one and the same sense all round the bounding parameter. This being so, we merely have to add together corresponding sides of the equation proved in the preceding article for each of the Δ-faces to get the equation required; the part of each contour integral due to any side not belonging to the bounding polygon will appear once in one sense and once in the other, so that it will disappear from the final result, while the part due to each side of the bounding polygon will range itself in its place in the integral round the perimeter, in consequence of what has been remarked as to the senses of these edges. Denoting the polyacron by P, and its bounding polygon by p, our resulting equation is

$$\int_p X(x, y, z)\, dx = \iint_P \left(m \frac{dX}{dz} - n \frac{dX}{dy}\right) dS.$$

It should be remarked that there is no ambiguity here as to the sign of m or n, and that the Law of Edges is *sufficient* in order to ensure the requisite harmony.

On the other hand, if we were to omit the condition as to the bifaciality of the Δ-faced polyacron, no process of assigning the senses of the perimeters of the faces could lead to the elimination from the final result of

* A. F. Möbius, *Ges. Werke*, 2 (1886), 475 *seq.*

the integrals along every one of the edges except those forming the bounding polygon. In this sense, therefore, we say that the condition of bifaciality is *necessary* for the validity of Stokes's theorem. It is moreover implied in the formula itself, since the sign of the normal of a unifacial surface is completely ambiguous.

9. We pass now to the general form of surface satisfying the following conditions :—

(i) *The surface is bifacial ;*

(ii) *the equations to the surface may be written*

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v), \quad (a \leqslant u \leqslant c;\ b \leqslant v \leqslant d)$$

where x, y, z, regarded as functions of one of the variables (u, v), the others being constant, are absolutely continuous ;

(iii) *the partial differential coefficients dx/du, dy/du, dz/du are each numerically less than a summable function $\lambda(u)$ of u alone, at the points where these differential coefficients exist and are finite : the same holds with u and v interchanged.*

Condition (iii) may be expressed by saying that the partial differential coefficients of $x(u, v)$, $y(u, v)$, and $z(u, v)$ *satisfy a condition of infrasummability with respect to the variable of differentiation,* holding wherever these differential coefficients exist and are finite; that is, in consequence of (ii), by which $x(u, v)$, $y(u, v)$, and $z(u, v)$, are the integrals of their derivates—almost everywhere with respect both to u and v. I am therefore tempted to term these surfaces "*infrasummable surfaces.*"

The assumption of these conditions enables me to apply to our present problem the results of my previous work on areas and integrals. We have, however, in order to avoid any misconceptions, to go through certain preliminary details which may seem tedious, since they are of the same nature as in all the similar questions which I have hitherto treated by this method.

10. We take the fundamental rectangle (a, b, c, d) in the (u, v)-plane, and divide it in the usual way into triangles, that is, semi-rectangles, by first drawing parallels to the sides, and then inserting the diagonals sloping down from left to right in each small rectangle. The peripheries of these semi-rectangles are then all provided with senses, in such a way

that in the sides parallel to the axes of v the sense is that in the direction of increase of the variable v which is not constant on the line considered. The sense in the periphery of the fundamental rectangle is accordingly clockwise, and along every other line occurring in the figure there are opposite senses, according as it is regarded as a side of one or the other triangle to which it belongs. The vertices of the semi-rectangles determine on the surface an array of corresponding points, and, joining these by stretches, marked with the senses determined by the corresponding senses in the (u, v)-plane, we get a Δ-faced polyacron inscribed in our surface, with the senses assigned obeying the Law of Edges. We shall call this the polyacron related to the semi-rectangles; it has the property given in the following lemma.

LEMMA.—*If the conditions* (ii) *and* (iii) *are satisfied, the sum of the areas of the related* Δ*-faces, each taken positively, is less than*

$$2\int_a^c \lambda(U)\,dU \int_b^d \lambda(V)\,dV.$$

Consider the semi-rectangle which has its right angle at (u, v) and its other vertices at $(u+h, v)$ and $(u, v+k)$ where h and k are positive. Then, if D be the related Δ-face, and D_z its projection on the plane of (x, y),

$$2D_z = \{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\}$$
$$-\{x(u, v+k)-x(u, v)\}\{y(u+h, v)-y(u, v)\}$$
$$= \int_u^{u+h} \frac{dx}{dU}(U, v)\,dU \int_v^{v+k} \frac{dy}{dV}(u, V)\,dV$$
$$-\int_v^{v+k} \frac{dx}{dV}(u, V)\,dV \int_u^{u+h} \frac{dy}{dU}(U, v)\,dU.$$

Therefore $$|2D_z| \leqslant 2\int_u^{u+h} \lambda(U)\,dU \int_v^{v+k} \lambda(V)\,dV.$$

Summing for all the triangles for which h and k are positive, and then for all for which h and k are negative, we get the result enunciated in the lemma.

11. The Δ-face related to any one of the semi-rectangles in the (u, v)-plane is inscribed in that parts of the surface which is the image of the same semi-rectangle. This image is a curvilinear triangle, two of whose

sides are given by $u = \text{const.}$ and $v = \text{const.}$ respectively, and in consequence of the conditions satisfied by the surface, it possesses an area* given by the double integral

$$\iint V\,du\,dv$$

over the semi-rectangle, where v is the positive value of

$$\sqrt{\left\{\left(\frac{d(y,z)}{d(u,v)}\right)^2+\left(\frac{d(z,x)}{d(u,v)}\right)^2+\left(\frac{d(x,y)}{d(u,v)}\right)^2\right\}}.$$

12. The direction cosines of the normal to the surface are given in absolute value, of course, by

$$l:m:n:\pm 1=\frac{d(y,z)}{d(u,v)}:\frac{d(z,x)}{d(u,v)}:\frac{d(x,y)}{d(u,v)}:V,$$

and the two faces of the surface are distinguishable by the ambiguity of sign here specified. We shall take that face of the surface for which *the outward normal* is the positive direction of the normal, that is, we take throughout the sign + in the preceding relation. Taking a right-handed screw round the directed axis so determined, the face which we choose, and designate as *the positive or outer face of the surface*, is that one on which the trace of the screw, or indicatrix, is anti-clockwise round the point (trace of the axis on the surface) when looking back on the surface along the reverse direction of the axis.

13. We shall now again, as in proving the formula for a triangle, assume that the portion of three-dimensional space in which our operations take place is so confined that the oscillations of the continuous functions X, dX/dy and dX/dz should be less than an arbitrary small positive quantity ϵ. This is indeed possible, since the points of our surface, being in continuous correspondence with a closed rectangle, form a closed set of points in space. Therefore X, dX/dy and dX/dz are not only continuous, but uniformly continuous on the surface, and therefore we can assign η so that, in any cube of side not exceeding η, with faces parallel to the co-ordinate planes, containing points of the surface, the oscillations of X, dX/dy, and dX/dz are less than a chosen small quantity ϵ.

We can now choose the mesh of the network of lines in the (u, v)-plane so fine that the points (x, y, z) on the surface corresponding to those of any one of the sub-rectangles in the (u, v)-plane all lie inside a cube of the

* W. H. Young, "On the Area of Surfaces", *Proc. Royal Soc.* (A), 96 (1919), 77; also "On a Formula for an Area", *Proc. London Math. Soc.* (2), 18 (1919), 362-364.

type we have just been considering. Any triangle will then lie in the same cube, provided its vertices are among those points (x, y, z).

Taking, then, first of all such a sub-rectangle as our fundamental rectangle, and confining ourselves to such a cube in the (x, y, z)-space, we only need to prove Stokes's formula for the portion of surface which appears; we then can, as before, pass by summation to the same formula extended over the whole surface with which we started.

14. Dividing up our little fundamental rectangle as before, we may apply Stokes's formula to the related Δ-faced polyacron, and in the double integral we may regard dX/dy and dX/dz as having fixed values, namely, their values at any convenient point of the surface provided we account for the error, which by the lemma is less than

$$2\epsilon\int_a^c \lambda(U)\,dU\int_b^d \lambda(V)\,dV.$$

We may then write our equation in the form

$$\int_p X\,dx = P_y\frac{dX}{dz} - P_z\frac{dX}{dy} + 2\theta\epsilon\int_a^c\int_b^d \lambda(U)\,\lambda(V)\,dU\,dV,$$

the contour integral being round the bounding polygon of the polyacron (which is a polygon inscribed in the boundary of our surface), and P_y and P_z denoting the areas of the projections of the surface of the polyacron on the coordinate planes of (z, x) and (x, y) respectively.

But the conditions imposed on the surface, being transferred to these coordinate planes, are such that the projection C_y of our bounding curve C on the plane of (z, x) is rectifiable, and its area is given by the expression

$$\int_a^c\int_b^d \frac{d(z, x)}{d(U, V)}\,dU\,dV.$$

Moreover this is the limit of the area R_y, when the mesh becomes indefinitely fine.

Similarly for the other projection, C_z.

Thus we get in the limit, when the mesh becomes indefinitely fine, for the right-hand side of our formula

$$\frac{dX}{dz}\int_a^c\int_b^d \frac{d(z, x)}{d(U, V)}\,dU\,dV - \frac{dX}{dy}\int_a^c\int_b^d \frac{d(x, y)}{d(U, V)}\,dU\,dV,$$

which, with an error of at most

$$2\epsilon \int_a^c \int_b^d \lambda(U)\lambda(V)\, dU\, dV,$$

may be written

$$\int_a^c \int_b^d \left\{ \frac{dX}{dz} \frac{d(z, x)}{d(U, V)} - \frac{dX}{dy} \frac{d(x, y)}{d(U, V)} \right\} dU\, dV.$$

15. We now consider the left-hand side of our formula for the polyacron.

For this purpose we must bear in mind that x round the bounding curve C, image of the periphery of (a, b, c, d), is an absolutely continuous function, and therefore a function of bounded variation, of a single variable, which is u on two of the sides of the rectangle and v on the other two. Therefore the contour integral,

$$\int_C X(x, y, z)\, dx,$$

in which the integrand is an absolutely continuous function of the single independent variable in question, is the limit of the "Riemann" summation, using as points of division on the periphery of $(a, b;\ c, d)$ the points of our network. Also, in consequence of the fineness of the mesh, we may replace this "Riemann" summation by the contour integral round the bounding polygon of our polyacron, provided we take account of the error, which will not exceed ϵ times the perimeter of the polygon. Ultimately, as the mesh becomes indefinitely fine, the perimeter of the polyacron tends to a unique limit, which is the length of the curve C, since the curve is rectifiable.

We thus get, as limit, when the mesh becomes indefinitely fine, the equation

$$\int_C X(x, y, z)\, dx + \theta\epsilon C = \int_a^c \int_b^d \left\{ \frac{dX}{dz} \frac{d(z, x)}{d(U, V)} - \frac{dX}{dy} \frac{d(x, y)}{d(U, V)} \right\} dU\, dV + \theta\epsilon \int_a^c \int_b^d \lambda(U)\lambda(V)\, dU\, dV.$$

16. We can now pass by summation to our original fundamental rectangle and our original surface, removing the restriction as to size. The form of the equation is the same as that just written down, since the various integrals add up to integrals of the same form, and the lengths of

the images of the perimeters of the sub-rectangles add up to the length of the image of the perimeter of the large fundamental rectangle, in virtue of the opposite senses in which the internal stretches of the figure in the (u, v)-plane are described.

Since ϵ is as small as we please, this gives us Stokes's formula for our surface,

$$\int_\sigma X(x, y, z)\, dx = \int_a^c \int_b^d \left\{ \frac{dX}{dz} \frac{d(z, x)}{d(U, V)} - \frac{dX}{dy} \frac{d(x, y)}{d(U, V)} \right\} dU\, dV,$$

this being, in accordance with the values given in articles 11 and 12, the same as the usual form,

$$\int_C X(x, y, z)\, dx = \iint_S \left\{ m \frac{dX}{dz} - n \frac{dX}{dy} \right\} dS.$$

The formula has thus been proved for any infrasummable surface, under the assumption that X, dX/dy and dX/dz are continuous functions of (x, y, z) at every point of the surface, and for all points within a certain fixed distance of it, this distance being, however, as small as we please.

PART II.

On the Integration of Differentials and the Maxwell Proof of Stokes's Theorem.

17. Maxwell's proof* of Stokes's theorem was, I believe, the first to appear in print; the enunciation is given in terms independent of analytical formulæ in the following form:—

A line integral, taken round a closed curve, may be expressed in terms of a surface integral, taken over a surface bounded by the curve.

The conditions under which this and other early proofs of this theorem hold have to be inferred from the context. Maxwell's account as it stands does not lend itself to criticism, for the mathematical reasoning is for the most part omitted. Various restrictions must be imposed in order that the transformations which Maxwell imposed on the surface integral may be legitimate. Maxwell, of course, assumed all the differential coefficients

* The proof may be consulted in J. Clerk Maxwell's *Treatise on Electricity and Magnetism*, or in Thomson and Tait's *Treatise on Natural Philosophy*, § 190, p. 143, of the second edition (1888).

to exist and to be continuous, including those of the first order which do not occur in the formula, and the mixed differential coefficients of x, y, and z with respect to u and v. The continuity of these latter is not actually involved, but the method would require the mixed differential coefficients to be the same in whichever order we perform the two differentiations, and the partial differential coefficients of x, y, and z with respect to u and v must be not merely but absolutely continuous with respect to the variable of differentiation in each case, in order to justify the process of integration by parts, used by Maxwell towards the end of his analysis.

Maxwell's proof is, however, interesting as well as classical, and I propose now to bring it up to date as regards reasoning. It then gives us a perfectly valid proof of Stokes's theorem for the general form of surface introduced in Part I, and under very light restrictions as to the functions to be integrated. I have been able to avoid integration by parts, using results which I have myself published; and by the assumption that X has a first differential, Maxwell's use of the formula

$$\frac{dX}{du} - \frac{dX}{dx}\frac{dx}{du} - \frac{dX}{dy}\frac{dy}{du} - \frac{dX}{dz}\frac{dz}{du} = 0$$

is rendered legitimate.

I remarked, however, that this latter assumption is unnecessarily stringent. The left-hand side of the last equation only occurs in our calculation under the double integral sign, viz.,

$$\int_a^c \int_b^d \frac{dx}{dv}\left\{\frac{dX}{du} - \frac{dX}{dx}\frac{dx}{du} - \frac{dX}{dy}\frac{dy}{du} - \frac{dX}{dz}\frac{dz}{du}\right\} du\, dv = 0,$$

multiplied by the function dx/dv, which, for our particular form of surface, being numerically less than a summable function of v alone, can be left out during the first integration, that with respect to u. Thus all we require is conditions under which

$$\int \left\{\frac{dX}{du} - \frac{dX}{dx}\frac{dx}{du} - \frac{dX}{dy}\frac{dy}{du} - \frac{dX}{dz}\frac{dz}{du}\right\} du = 0.$$

We proceed therefore, as a preliminary, to discuss such conditions. Integrating out the first term of the integrand, the equation to be satisfied takes the form

$$X = \int \left\{\frac{dX}{dx}\frac{dx}{du} + \frac{dX}{dy}\frac{dy}{du} + \frac{dX}{dz}\frac{dz}{du}\right\} du,$$

and *the conditions for the validity of this result which we obtain are the following :—*

(i) *dX/dx exists* at the points considered ;*

(ii) *dX/dy exists and is a bounded partial differential coefficient with respect to x throughout a certain linear neighbourhood $(x-\epsilon, x+\epsilon)$ of the point (y and z being constant) ;*

(iii) *dX/dz exists and is either*

(iii *a*) *a bounded differential coefficient with respect to x of a differential coefficient with respect to y throughout a plane neighbourhood of the point considered $(x-\epsilon, y-\eta ; x+\epsilon, y+\eta)$, z being constant, or*

(iii *b*) *in such a plane neighbourhood it is a bounded differential coefficient with respect to x, and also one with respect to y.*

18. These conditions include as a special case the known conditions for the existence of a differential, viz. :

$$\frac{dX}{dt} = \frac{dX}{dx}\frac{dx}{dt} + \frac{dX}{dy}\frac{dy}{dt} + \frac{dX}{dz}\frac{dz}{dt}, \tag{A}$$

provided

(i) *the partial derivates of $X = X(x, y, z)$ with respect to x are continuous with respect to (x, y, z) at the point in question, and those with respect to y are continuous there with respect to (y, z), while the partial differential coefficient dX/dz exists at the point (the existence there of dX/dx and dX/dy being thus already ensured);*

(ii) *dx/dt, dy/dt, and dz/dt exist at the point.*

We proceed to the discussion of the conditions under which the formula (A) may only be employed when we integrate both sides of the equation with respect to t, and write

$$X = \int \left\{\frac{dX}{dx}\frac{dx}{dt} + \frac{dX}{dy}\frac{dy}{dt} + \frac{dX}{dz}\frac{dz}{dt}\right\} dt. \tag{B}$$

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* *I.e.* exists as a differential coefficient. That is to say the symbol dX/dx, by which hitherto we have been understanding any derivate of X with respect to x, is here, a differential coefficient, all these derivates coinciding at the point considered.

It will *prima facie* suffice for (i) and (ii) to hold with the exception of points forming a set of values of t of content zero, provided X is an absolutely function of t, so that we are sure that we may write

$$X = \int \frac{dX}{dt}\, dt.$$

This will be the case if X is absolutely continuous with respect to x, to y, and to z, and x, y and z are absolutely continuous with respect to t*; the condition (ii) can then be omitted, and (i) only hypothecated to hold almost everywhere with respect to t.

More generally, however, we have to discuss how the condition (i) can be modified at points not belonging to a set of values of t of content zero, so that (B) may be valid.

19. For this purpose we introduce the auxiliary function

$$R(x, y, z;\ h, k) = \int_x^{x+h} \int_y^{y+k} \int_z^{z+q} X(x, y, z)\, dx\, dy\, dz/hkq,$$

and make first of all no hypothesis with respect to X except that it is a bounded function,† and, regarded as a function of x, continuous. We may then change the order of integration; moreover, by the Lebesgue theory for a single variable,

$$\frac{dR}{dx} = \int_y^{y+k} \int_z^{z+q} \big(X(x+h, y, z) - X(x, y, z)\big)\, dy\, dz/hkq,$$

with similar expressions for dR/dy and dR/dz.

By Lemma 1 of the ensuing article, these differential coefficients are continuous functions of (x, y, z), so that the conditions for the validity of (B) are satisfied by the function R.

Thus we may write

$$R = \int \left\{ \frac{\partial R}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial t} \right\} dt.$$

* Proved explicitly below, § 32.

† I do not explicitly say "measurable function," but this is to be understood when I say "function."

20. The lemma required in the preceding article is the following:—

LEMMA 1.—*If $X(x, y, z)$ is a bounded function of (x, y, z) and a continuous function of x, its double integral*

$$F(x, y, z) = \int^{y} \int^{z} X(x, y, z)\, dy\, dz$$

is a continuous function of (x, y, z).

For

$$\begin{aligned} &F(x+h, y+k, z+q) - F(x, y, z) \\ &\quad = \int^{y+k} \int^{z+q} X(x+h, y, z)\, dy\, dz - \int^{y} \int^{z} X(x, y, z)\, dy\, dz \\ &\quad = \int^{y+k} \int_{z}^{z+q} X(x+h, y, z)\, dy\, dz + \int_{y}^{y+k} \int^{z} X(x+h, y+z)\, dy\, dz \\ &\qquad\qquad + \int^{y} \int^{z} \Big(X(x+h, y, z) - X(x, y, z) \Big)\, dy\, dz. \end{aligned}$$

The integrands of all three integrals being bounded, the first and second are respectively less in absolute value than a finite multiple of q and of k, while the limit of the third integral may be evaluated by term-by-term integration, giving zero, since X is continuous with respect to x. Thus the whole right-hand side tends to zero, as h, k, and q diminish indefinitely. This proves the required result.

The following is proved in the same way:—

LEMMA 2.—*If the derivates of $X(x, y, z)$ with respect to y and z are bounded functions of (x, y, z), then $X(x, y, z)$ is a continuous function of (y, z), and if $X(x, y, z)$ is also continuous with respect to x it is continuous with respect to (x, y, z).*

For

$$\begin{aligned} &| X(x+h, y+k, z+q) - X(x, y, z) | \\ &\quad = \Big| \Big(X(x+h, y+k, z+q) - X(x+h, y+k, z) + X(x+h, y+k, z) \\ &\qquad\qquad - X(x+h, y, z) + X(x+h, y, z) - X(x, y, z) \Big) \Big| \\ &\quad = \Big| \int_{z}^{z+q} \frac{dX(x+h, y+k, z)}{dz}\, dz + \int_{y}^{y+k} \frac{dX(x+h, y, z)}{dy}\, dy \\ &\qquad\qquad + X(x+h, y, z) - X(x, y, z) \Big| \\ &\quad < Aq + Ak + \epsilon, \end{aligned}$$

where A is greater than the upper bounds of the moduli of the partial derivates of X with respect to y and z, and ϵ is a quantity which is zero if x is kept constant (so that h is zero), and is as small as we please, provided h is less than a certain quantity δ determined by the smallness of ϵ, in the case when X is continuous with respect to x. Hence as $(h, k, q) \to (0, 0, 0)$ in any manner whatever, the increment of the function X tends to zero, so that X is continuous with respect to (y, z) in the first case, and (x, y, z) in the second.

This proves the lemma.

21. We shall now suppose *the partial derivates of* $X(x, y, z)$ *with respect to* y *and* z *to be bounded functions of* (x, y, z); retaining the condition already used, that X *should be continuous with respect to* x (§ 18), it will then, by the preceding lemma, be a continuous function of (x, y, z). Also, by Dini's theorem, the incrementary ratios of X,

$$\{X(x, y+h, z)-X(x, y, z)\}/h \quad \text{and} \quad \{X(x, y, z+q)-X(x, y, z)\}/q,$$

regarded as functions of each of the variables y and z separately, are bounded. We shall require to investigate certain properties of these partial derivates, and to prove a series of convenient lemmas applying to them, in obtaining which we shall replace them by a general bounded function $f(x, y, z)$, and use the following symbolism :—

$$f = \frac{d}{dx}$$

without mention of any particular neighbourhood, means that, in a certain linear neighbourhood $(x-h, x+h)$ *of the point considered,* y *and* z *being constant,* f *is the differential coefficient of an unspecified function. Similarly*

$$f = \frac{\partial}{\partial y} \quad \text{or} \quad f = \frac{d}{dx}\frac{d}{dy},$$

means that in a two-dimensional neighbourhood, z *being constant,* f *is the partial differential coefficient with respect to* y, *or this mixed differential coefficient, of an unspecified function. Thus again*

$$f = \frac{d}{dx} \quad \text{and} \quad \frac{d}{dy}$$

means that in such a two-dimensional neighbourhood, f *is a differential coefficient with respect to* y *of an unspecified function of* (x, y) *and in such a linear neighbourhood* $(x-h, x+h)$, *for fixed* y, f *is a differential coefficient with respect to* x *of an unspecified function of* (x, y).

This notation will now be clear however many variables occur ; *the variables which do not occur explicitly in the symbolic equation are to be supposed constant, and those which do occur to be confined to a certain neighbourhood of the point considered.*

If, however, we require the relation to hold in a more extended neighbourhood, we must mention this explicitly each time.

22. We shall use the symbol

$$\operatorname*{Llt}_{(h(k(l)))\to 0} f(x, y, z ; h, k,$$

to denote the complete set of limits of a function f, when first l, then k, and finally h, tend to zero, reserving the symbol Lt for the case when these limits are known to coincide *a priori*. The symbolism employed for mixed differential coefficients will exactly correspond to the above notation for repeated limits. Thus, for instance, when the set of limits at each stage of the limiting process reduces to a single one, we have

$$\operatorname*{Lt}_{(h(k(l)))\to 0} \int_x^{x+h}\int_y^{y+k}\int_z^{z+l} f\,dx\,dy\,dz/hkl = d^3F/dx\,dy\,dz,$$

where

$$F = \iiint f\,dx\,dy\,dz.$$

23. LEMMA 1.—*If the bounded function f of any number n of variables is a differential coefficient with respect to one of them, x, in a certain linear neighbourhood containing x, the remaining variables $(y, z, \ldots)$ being fixed, then f is the differential coefficient of its integral at the point $(x, y, \ldots)$ considered,* that is*

$$dF/dx = f \quad \textit{where} \quad F = \int f\,dx.$$

Though formally different, this is concomitant to the well-known result that the indefinite integral is the primitive function.†

24. LEMMA 2.—*If a bounded function f is both a differential coefficient with respect to x and also with respect to y, or if it is a mixed differential coefficient in the order x, y, in a certain plane neighbourhood containing (x, y) the remaining variables being constant, then f is at the*

* And therefore throughout a linear neighbourhood.

† H. Lebesgue, *Leçons sur l'intégration* (1904), 120.

point $(x, y, \ldots)$* *the mixed differential coefficient of its repeated integral, the differentiation being in the order* (x, y), *that is*

$$d^2F/dx\,dy = f, \quad \textit{where} \quad F = \iint f\,dx\,dy.$$

In fact, since f is bounded, change of order of integration is allowable Therefore

$$\int_x^{x+h}\int_y^{y+k} f\,dx\,dy/hk = \int_x^{x+h}\left\{\int_y^{y+k} f\,dy/k\right\}dx/h$$

$$= \int_y^{y+k}\left\{\int_x^{x+h} f\,dx/h\right\}dy/k.$$

Now if $f = d/dy$, we have, by Lemma 1,

$$\operatorname*{Lt}_{k\to 0}\int_y^{y+k} f\,dy/k = f,$$

for every value of x in the plane neighbourhood implied by the symbolic equation $f = d/dy$.

We may therefore write this equation in the form

$$\operatorname*{Lt}_{k\to 0}\int_0^1 f(x+ht_1,\ y+kt_2)\,dt_2 = f(x+ht_1,\ y).$$

Integrating term-by-term with respect to t_1, which is allowable, since, f being bounded, the integral whose limit is considered is a bounded function,

$$\operatorname*{Lt}_{k\to 0}\int_0^1\left\{\int_0^1 f(x+ht_1,\ y+kt_2)\,dt_2\right\}dt_1 = \int_0^1 \operatorname*{Lt}_{k\to 0}\int_0^1 f(x+ht_1,\ y+kt_2)\,dt_2\Big\}\,dt_1$$

$$= \int_0^1 f(x+ht_1,\ y)\,dt_1,$$

that is

$$\operatorname*{Lt}_{k\to 0}\int_x^{x+h}\left\{\int_y^{y+k} f\,dy/k\right\}dx/h = \int_x^{x+h}\operatorname*{Lt}_{k\to 0}\left\{\int_y^{y+k} f\,dy/k\right\}dx/h = \int_x^{x+h} f\,dx/h.$$

Supposing, then, that $f = d/dx$, the integral on the extreme right, has, by Lemma 1, the unique limit f, therefore the same is true of the expression on the extreme left, when $h \to 0$, that is, we have

$$\operatorname*{Lt}_{(h(k))\to 0}\int_x^{x+h}\left\{\int_y^{y+k} f\,dy/k\right\}dx/h = f,$$

* And therefore throughout a plane neighbourhood.

and therefore, since we may remove the brackets,

$$\underset{h(k))\to 0}{\mathrm{Lt}} \int_x^{x+h}\int_y^{y+k} f\,dx\,dy/hk = f.$$

This proves the required result under the first of the two alternative sets of conditions.

If the second set of alternative conditions holds, we use the other form of the integral, given by changing the order of integration. We then have

$$\int_x^{x+h} f\,dx/h = \int_0^1 f(x+ht_1,\ y)\,dt_1$$

is certainly a differential coefficient with respect to y, since $f = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial y}$ and therefore $\int f\,dx = dG/dy$, f being bounded, where G is a certain function, and therefore

$$\int_x^{x+h} f\,dx/h = \frac{d}{dy}\left\{\frac{G(x+h,\ y)-G(x,\ y)}{h}\right\}.$$

Thus we may use Lemma 1, and write

$$\underset{k\to 0}{\mathrm{Lt}} \int_y^{y+k}\left\{\int_x^{x+h} f\,dx/h\right\} dy/k = \int_x^{x+h} f\,dx/h,$$

whence, as before, by a second application of Lemma 1, we get again the required result. This proves the lemma.

25. The proof given shows that a somewhat more general form of the lemma holds. In fact, as we saw, it is sufficient if $f = d/dx$ and $\int f\,dx = d/dy$. The former of these conditions only implies that f is for fixed (y, z) a differential coefficient with respect to x *at every point of a certain linear neighbourhood of our point*, and it follows, by Lemma 1, that *at every point of such a neighbourhood,*

$$f = \frac{d}{dx}\int f\,dx$$

and therefore that *at every point of this linear neighbourhood*

$$f = \frac{d}{dx}\frac{dG}{dy},$$

where G is the indeterminate function such that, by our supposition,

$$\int f\,dx = \frac{dG}{dy}$$

at every point of a plane neighbourhood (z only being constant). The hypothesis here made is therefore less restricted than that of Lemma 2, in so far as the neighbourhood involved in one of our conditions *need not be a plane one*. This is even more obvious in the case of the alternative condition. We have therefore the following more general form of the lemma:—

LEMMA 2*a*.—*We may assert that at the point* (x, y, z) *

$$f = d^2F/dx\,dy, \quad \text{where} \quad F = \iint f\,dx\,dy = \iint f\,dy\,dx,$$

provided f *is a bounded function, and*

(i) *at every point of a certain plane neighbourhood,* $z = \text{const.}$ *containing the point* (x, y, z) *considered, either* f *or* $\int f\,dx$ *is a differential coefficient with respect to* y*; and*

(ii) *at every point of a certain linear neighbourhood,* $(y, z) = \text{const.}$*, of the point,* f *is a differential coefficient with respect to* x.

These conditions are exactly expressed in abbreviated form by saying that $f = d/dx$, *and either* f *or* $\int f\,dx = d/dy$.

26. LEMMA 3.—*If throughout a three-dimensional neighbourhood of the point considered, the bounded function* $f(x, y, z)$ *satisfies any one of the following eight sets of alternative conditions, we shall have at the point*†

$$\frac{d}{dx}\frac{d}{dy}\frac{d}{dz}\iiint f\,dx\,dy\,dz = f;$$

(i) *if* $f = \dfrac{d}{dx}$ *and* $\dfrac{d}{dy}$ *and* $\dfrac{d}{dz}$; (i *a*) *if* $f = \dfrac{d}{dx}\dfrac{d}{dy}$ *and* $\dfrac{d}{dz}$;

(ii) *if* $f = \dfrac{d}{dx}$ *and* $\dfrac{d}{dy}\dfrac{d}{dz}$; (ii *a*) *if* $f = \dfrac{d}{dx}\dfrac{d}{dy}$ *and* $\dfrac{d}{dy}\dfrac{d}{dz}$;

(iii) *if* $f = \dfrac{d}{dy}$ *and* $\dfrac{d}{dx}\dfrac{d}{dz}$; (iii *a*) *if* $f = \dfrac{d}{dx}\dfrac{d}{dy}$ *and* $\dfrac{d}{dx}\dfrac{d}{dz}$;

(iv) *if* $f = \dfrac{d}{dx}\dfrac{d}{dz}$ *and* $\dfrac{d}{dy}\dfrac{d}{dz}$; (iv *a*) *if* $f = \dfrac{d}{dx}\dfrac{d}{dy}\dfrac{d}{dz}$.

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* And therefore necessarily in a linear neighbourhood $(x-H, x+H)$.

† And therefore throughout a three-dimensional neighbourhood.

In fact, by all these conditions the requirements of Lemma 2 are satisfied. We only need therefore to see that the first limiting process, that with respect to the increment of z, leads to

$$\underset{(h(k))\to 0}{\mathrm{Llt}} \int_x^{x+h}\int_y^{y+k} f\,dx\,dy.$$

This is evident in cases (i) and (i a) if we write the expression to be considered in the following form, using term-by-term integration:

$$\underset{(h(k))\to 0}{\mathrm{Llt}} \int_x^{x+h}\int_y^{y+k}\left(\underset{q\to 0}{\mathrm{Lt}}\int_z^{z+q} f\,dz/q\right)dx\,dy/hk.$$

In cases (ii) and (ii a), on the other hand, we write the expression in the form

$$\underset{(h(k))\to 0}{\mathrm{Llt}} \int_x^{x+h}\left[\underset{q\to 0}{\mathrm{Lt}}\int_z^{z+q}\left(\int_y^{y+k} f\,dy/k\right)dz/q\right]dx/h$$

where, since $f = \dfrac{d}{dy}\dfrac{d}{dz}$, we have $\int f\,dy = d/dz$, so that, by Lemma 1, we have what was required, in cases (iii) and (iii a), the same argument applies, interchanging x and y; in cases (iv) and (iv a) we use the form

$$\underset{(h(k))\to 0}{\mathrm{Llt}}\ \underset{q\to 0}{\mathrm{Llt}} \int_z^{z+q}\left[\int_x^{x+h}\int_y^{y+k} f\,dx\,dy/hk\right]dz/q.$$

27. The argument used in the proof of Lemma 3 consists effectively in permuting x, y, and z under the integral sign, and then integrating term by term. This might lead us to expect twelve instead of eight cases, according as we use for the first limit (that with respect to q, the increment of z) the appropriate form of Lemma 1, and then supplement one or other of the pair of conditions of Lemma 2. We see on inspection, however, that we thus get repetitious owing to the fact that $\int^x\int^y$ and $\int^y\int^x$ are the same.

If we use the somewhat more general Lemma 2 a, we get not only a somewhat more general form of the lemma, but also one which it is easier to generalise for any number of variables; it is as follows, when we use the abbreviations already explained, and remember that $f = d/dx$ *means that* f *is a differential coefficient with respect to* x *in a linear neighbourhood of the point, but* $f = d/dy$ *in a plane, and* $f = d/dz$ *in a three-dimensional neighbourhood.*

LEMMA 3 *a*.—*If the bounded function* $f(x, y, z)$ *satisfies any one of the following eight conditions, we may write*

$$\frac{d}{dx}\frac{d}{dy}\frac{d}{dz}\iiint f\,dx\,dy\,dz = f.$$

Conditions (i) *and* (i *a*),

$$\text{if } f = \frac{d}{dz} \text{ and } = \frac{d}{dx} \text{ and either } f = \frac{d}{dy} \text{ or } \int f\,dx = \frac{d}{dy}.$$

Conditions (ii) *and* (ii *a*),

$$\text{if } \int f\,dy = \frac{d}{dz},\ f = \frac{d}{dx}, \text{ and either } f = \frac{d}{dy} \text{ or } \int f\,dx = \frac{d}{dy}.$$

Conditions (iii) *and* (iii *a*).

$$\text{if } \int f\,dx = \frac{d}{dz},\ f = \frac{d}{dx}, \text{ and either } f \text{ or } \int f\,dx = \frac{d}{dy}.$$

Conditions (iv) *and* (iv *a*),

$$\text{if } \iint f\,dx\,dy = \frac{d}{dz},\ f = \frac{d}{dx}, \text{ and either } f \text{ or } \int f\,dx = \frac{d}{dy}.$$

28. We can conveniently express the whole series of results, exemplified in the preceding lemmas, in the following manner:—

(i) $\frac{d}{dx}\int f\,dx = f$, *if* $f = \frac{d}{dx}$;

(ii) $\frac{d}{dx}\frac{d}{dy}\iint f\,d(x, y) = f$, *if, in addition to* (i), f *or* $\int f\,dx = \frac{d}{dy}$;

(iii) $\frac{d}{dx}\frac{d}{dy}\frac{d}{dz}\iiint f\,d(x, y, z) = f$, *if, in addition to* (i) *and* (ii), f, *or* $\int f\,dx$, *or* $\int f\,dy$, *or* $\iint f\,d(x, y) = \frac{d}{dz}$;

(iv) $\frac{d}{dx}\frac{d}{dy}\frac{d}{dz}\frac{d}{dw}\iiint\int f\,d(x, y, z, w) = f$, *if, in addition to* (i), (ii), *and* (iii), f, *or* $\int f\,dx$, *or* $\int f\,dy$, *or* $\int f\,dz$, *or* $\iint f\,d(x, y)$, *or* $\iint f\,d(x, z)$, *or* $\iint f\,d(y, z)$, *or* $\iiint f\,d(x, y, z) = \frac{d}{dw}$;

and so on.

29. Returning now to the function

$$R(x, y, z;\ h, k, q) = \int_x^{x+h} \int_y^{y+k} \int_z^{z+q} X(x, y, z)\, dx\, dy\, dz/hkq,$$

X satisfying the conditions of § 21, we can at once write

$$\operatorname*{Lt}_{(h(k(q)))\to 0} R = X,$$

since X is then a continuous function of (x, y, z) by Lemma 2, § 20, and therefore certainly satisfies the required conditions of § 26.

Also, the conditions of Lemmas 1 and 2 of §§ 23 and 24 being satisfied, for the same reason

$$\operatorname*{Llt}_{(h(k(q)))\to 0} \frac{dR}{dx} = \operatorname*{Llt}_{(h(k(q)))\to 0} \int_y^{y+k} \int_z^{z+q} \Big(X(x+h, y, z) - X(x, y, z)\Big)\, dy\, dz/hkq$$

$$= \operatorname*{Llt}_{h\to 0} \Big(X(x+h, y, z) - X(x, y, z)\Big)/h$$

$$= \frac{dX}{dx},$$

provided this partial differential coefficient exists at the point.

For the same reason we can take the first limit (that with respect to q) in the expression given below, for dR/dy, and we can take the second limit (that with respect to k), because the integrand is bounded and has, as we shall hypothecate immediately, the limit dX/dy.

Thus changing the order of integration in the triple integral and using the result of Lemma 1 of § 23,

$$\operatorname*{Llt}_{(h(k(q)))\to 0} \frac{dR}{dy} = \operatorname*{Llt}_{(h(k(q)))\to 0} \int_x^{x+h} \int_z^{z+q} \Big(X(x, y+k, z) - X(x, y, z)\Big)\, dx\, dz/hkq$$

$$= \operatorname*{Llt}_{h\to 0} \int_x^{x+h} \frac{dX}{dy}\, dx/h,$$

provided the partial differential coefficient dX/dy exists in a certain linear neighbourhood $(x-H, x+H)$ of the point considered. If then, in addition, this function dX/dy is in the interval considered a differential

coefficient with respect to x, so that, with the notation already used,

$$\frac{dX}{dy} = \frac{d}{dx},$$

we may use Lemma 1 of § 22, and we then have

$$\underset{(h(k(q)))\to 0}{\text{Llt}} \frac{dR}{dy} = \frac{dX}{dy}$$

Again, provided the partial differential coefficient dX/dz exists in a two-dimensional neighbourhood of the point considered,

$$\underset{(h(k(q)))\to 0}{\text{Llt}} \frac{dR}{dz} = \underset{(h(k(q)))\to 0}{\text{Llt}} \int_x^{x+h}\int_y^{y+k} \Big(X(x, y, z+q) - X(x, y, z)\Big)\, dx\, dy/hkq$$

$$= \underset{(h(k))\to 0}{\text{Llt}} \int_x^{x+h}\int_y^{y+k} \frac{dX}{dz}\, dx\, dy/hk,$$

to which we may apply Lemma 2 of § 23, provided this function

$$dX/dz = \frac{\partial}{\partial x}\frac{\partial}{\partial y},$$

or $dX/dz = d/dx$ and d/dy, in the neighbourhood considered, that is, this function is a mixed differential coefficient first with respect to y, and then with respect to x, or it is both a differential coefficient with respect to x and a differential coefficient with respect to y. We may under either of these hypotheses write

$$\underset{(h(k(q)))\to 0}{\text{Llt}} \frac{dR}{dz} = \frac{dX}{dz}.$$

Under the hypotheses here made we shall therefore have

$$\underset{(h(k(q)))\to 0}{\text{Llt}} \left(\frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \frac{dR}{dz}\frac{dz}{dt}\right) = \frac{dX}{dx}\frac{dx}{dt} + \frac{dX}{dy}\frac{dy}{dt} + \frac{dX}{dz}\frac{dz}{dt},$$

as well as

$$\underset{(h(k(q)))\to 0}{\text{Llt}} \; R = X.$$

Integrating *term-by-term* along the curve, *i.e.* with respect to t, which is allowable *if all the derivates of* X *are bounded on the curve and those of* x, y, *and* z *are summable functions of* t, and using the final result of § 18, we have therefore the required result,

$$X = \int \left(\frac{dX}{dx}\frac{dx}{dt} + \frac{dX}{dy}\frac{dy}{dt} + \frac{dX}{dz}\frac{dz}{dt} \right) dt, \ \ldots,$$

provided in addition dX/dx *exists as a differential coefficient on the curve,* dX/dy *exists and is a partial differential coefficient with respect to* x, $[dX/dy = d/dx]$, *throughout a certain linear neighbourhood* $(x-H, x+H)$ *of each point of the curve, while, finally,* dX/dz *exists and is either a bounded differential coefficient with respect to* x *of a bounded differential coefficient with respect to* y *throughout a plane neighbourhood of each point of the curve* $(x-H, y-K\,;\ x+H, y+K)$, *or, in such a neighbourhood, it is a bounded differential coefficient with respect to* x *and also one with respect to* y. [*In symbols,* $dX/dz = \frac{d}{dx}\cdot\frac{d}{dy}$ *or* $= d/dx$ *and* d/dy.]

We notice also that under these conditions we may introduce under the integral sign any bounded function λ of t and equate the result to

$$\int \lambda(t) \frac{dX}{dt} . dt.$$

30. The preceding result may further be applied to give us the effect of proceeding to the triple repeated limit in a different order, *e.g.*

$$\operatorname*{Llt}_{(k(h(q)))\to 0} .$$

Since the order of integration is immaterial, it is evident that we only have to take the conditions already given and, in them, interchange x, y, and z in accordance with the transposition of h, k, and q.

31. The reasoning is perfectly general, and may be applied in conjunction with our lemmas to the case when the number of independent variables is greater than three. The auxiliary function R will, of course, take a corresponding form. Thus, for instance, in the case of four variables, R will be the quadruple incrementary ratio of the fourfold

integral of X. Passing to the limit, *the results for* dR/dx, dR/dy, dR/dz *are the same as before, and we now have in addition*

$$\operatorname*{Llt}_{(h(k(q(r))))\to 0} \frac{dR}{dw}$$

$$= \operatorname*{Llt}_{(h(k(q(r))))\to 0} \int_x^{x+h}\int_y^{y+k}\int_z^{z+q}\int_w^{w+r} \{X(x,y,z,w+r)-X(x,y,z)\}\,\frac{dx\,dy\,dr\,dw}{hkqr}$$

$$= \operatorname*{Llt}_{(h(k(q)))\to 0} \int_x^{x+h}\int_y^{y+k}\int_z^{z+q} \frac{dX}{dw}\,dx\,dy\,dz/hkl,$$

provided the partial differential coefficient dX/dw *exists and is bounded in a three-dimensional neighbourhood,* $w = \text{const.}$, *of the point considered. We have therefore*

$$\operatorname*{Llt}_{(h(k(q(r))))\to 0} \frac{dR}{dw} = \frac{dX}{dw},$$

if in the neighbourhood mentioned dX/dw *not only exists, but also satisfies any one of the conditions of Lemma* 3, *e.g.*

$$\frac{dX}{dw} = \frac{d}{dx} \quad \text{and} \quad = \frac{d}{dy} \quad \text{and} \quad = \frac{d}{dz},$$

or

$$\frac{dX}{dw} = \frac{d}{dx}\,\frac{d}{dy}\,\frac{d}{dz}.$$

If these conditions are satisfied we may write

$$X = \int \left(\frac{dX}{dx}\,\frac{dx}{dt} + \frac{dX}{dy}\,\frac{dy}{dt} + \frac{dX}{dz}\,\frac{dz}{dt} + \frac{dX}{dw}\,\frac{dw}{dt}\right) dt.$$

32. Before proceeding to the discussion of Stokes's theorem by Maxwell's method, we proceed to state and prove a simple auxiliary theorem.

Auxiliary Theorem.—*If* X *is a function of any number of variables, and is absolutely continuous with respect to each separately, while each of these variables is an absolutely continuous function of a parameter* u, *then, if the partial derivates of* X *are all numerically less than a fixed*

constant M, X is an absolutely continuous function of u. Also, if there are other parameters involved, and the partial derivates of the variables with respect to the parameter u are numerically less than a function $\lambda(u)$ independent of the remaining parameters, then the partial derivate of X with respect to u, when the remaining parameters remain constant, is numerically less than $M\lambda(u)$.*

For definiteness we shall take the case where there are three variables. Then, letting u increase by Δu, and in consequence x, y, and z by h, k, and q, and X by $X+\Delta X$, we have, as in the proof of Lemma 2 of § 19—since X is absolutely continuous with respect to x, to y, and to z—

$$\Delta X = \int_z^{z+q} \frac{dX}{dz}(x+h, y+k, z)\,dz + \int_y^{y+k} \frac{dX}{dy}(x+h, y, z)\,dy + \int_x^{x+h} \frac{dX}{dx}(x, y, z)\,dx.$$

Taking now any set of non-overlapping intervals $(u, u+\Delta u)$ on the u-line, of content less than ϵ, the sum of the corresponding increments taken positively $|q|$, or $|k|$, or $|h|$ will, by the fundamental property of an integral, be numerically less than a quantity η, chosen arbitrarily

* A referee points out that the remark that if a function $X(x, y\ldots)$ is absolutely continuous with respect to x, y, ... separately, while x, y, ... are each absolutely continuous with respect to u, $X(x(u), y(u), \ldots)$ is not *ipso facto* absolutely continuous with respect to u, was made to de la Vallée Poussin by Dunham Jackson in 1915, *Trans. Amer. Math. Soc.*, 16 (1915), 435–501 (462). The additional condition given in the text, though sufficient, and containing all that is required in Part 2 of the present paper, may be at once rendered more general by replacing the constant M by a summable function of the variable of differentiation; in other words, *infrasummability may be substituted for boundedness of the derivates of $X(x, y, \ldots)$, and the result remains true.* (This substitution is, however, insufficient for the validity of the second statement of the Auxiliary Theorem.)

In fact, if

$$\left|\frac{\partial X}{\partial z}\right| \leqslant \lambda(z), \quad \left|\frac{\partial X}{\partial y}\right| \leqslant \lambda(y), \quad \left|\frac{\partial X}{\partial x}\right| \leqslant \lambda(x)$$

the equation given in the text yields

$$\Sigma|\Delta X| \leqslant \Sigma\left\{\left|\int_z^{z+q}\lambda(z)dz + \int_y^{y+k}\lambda(y)dy + \int_x^{x+h}\lambda(x)dx\right|\right\} \leqslant 3\eta',$$

where η' is any chosen positive number, provided $\Sigma|q|$, $\Sigma|k|$, and $\Sigma|h|$ are each less than a certain η, depending on η', but not on the particular sets of intervals Σq, Σk, Σh. But, since x, y, and z are themselves absolutely continuous functions of u, this will be the case, provided the set of u-increments causing the increments Σq, Σk, Σh in question is less than a certain ϵ, depending only on η, and therefore on η' alone. Thus, given η', we have ϵ, so that $\Sigma|\Delta X| \leqslant \eta'$, provided $\Sigma|\Delta u| \leqslant \epsilon$, which proves that X is an absolutely continuous function of u.

[ϵ depending only on η, and not on the particular increments], and therefore ΔX will be numerically less than $3M\eta$. This proves that X is an absolutely continuous function of u.

Again, if the partial derivates of X with respect to x, y, and z are numerically less than M, it follows from the above that

$$\Delta X/\Delta u < M\{q/\Delta u + k/\Delta u + h/\Delta u\}$$

and therefore, letting Δu tend to zero,

$$\delta X/u < M\lambda(u).$$

This proves the lemma.

33. We proceed to the proof of Stokes's theorem by Maxwell's method brought up to date. We have first to enunciate the theorem in precise form.

THEOREM.—*Let S be an infrasummable surface, that is, defined by equations of the form*

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$
$$\{a \leqslant u \leqslant c\}, \quad \{b \leqslant v \leqslant d\}$$

such that

(i) *x, y, and z are absolutely continuous functions of the parameter u for each fixed value of v, and of v for each fixed value of u; and*

(ii) *the partial derivates of x, y, and z with respect to either of the parameters, the other being fixed, are numerically less than a summable function of that variable parameter alone;*

(iii) *and let X, Y, and Z be functions of (x, y, z) having the two following properties, when we omit from consideration on the surface S a certain set of points, which is the image of a set of content zero in the (u, v)-plane :—*

(iv) *the partial derivates of X with respect to y and z, of Y with respect to z and x, and of Z with respect to x and y are bounded functions of (x, y, z);*

(v) *$\partial X/\partial x$, $\partial Y/\partial y$, $\partial Z/\partial z$ are bounded on the surface itself;*

(vi) *each of the three functions has a first differential (or satisfies any of the sets of conditions proved sufficient for us to treat it, under the*

integral sign, as if it had one) on the surface, with respect to each of the parameters (u, v), *then the surface integral*

$$\iint_S \left(l\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right) dS$$

over a fragment of the surface S *bounded by the curve* C, *image of the perimeter of the rectangle* $(a, b\,;\ c, d)$ *in the* (u, v)*-plane, is equal to the contour integral*

$$\int_C X\frac{dx}{ds}\,ds$$

taken round the curve C; l, m, n *being the direction cosines of the normal to the surface in the "positive" sense, i.e. that determined by*

$$lV=\frac{d(y, z)}{d(u, v)}, \qquad mV=\frac{d(z, x)}{d(u, v)}, \qquad nV=\frac{d(x, y)}{d(u, v)}$$

$$(l^2+m^2+n^2=1, \quad V>0).$$

As already pointed out in § 11, we may use the result of § 6 of my paper "On the Area of Surfaces" and say that the surface S has an area, expressible as a double integral, namely,

$$S=\int_a^c\int_b^d V\,du\,dv,$$

where V has the same signification as in the enunciation of the theorem. Moreover, an equation of the same form holds for the area of each fragment of the surface corresponding to any partial rectangle cut by parallels to the axes of u and v.

The surface integral under discussion may therefore* be transformed into a double integral with respect to (u, v), over the fundamental rectangle. This integral is the sum of three, of which that involving X is

$$\iint_S \left(m\frac{dX}{dz}-n\frac{dX}{dy}\right) dS=\int_a^c\int_b^d \left(\frac{dX}{dz}\frac{d(z, x)}{d(u, v)}-\frac{dX}{dy}\frac{d(x, y)}{d(u, v)}\right) du\,dv.$$

The function of (u, v) to be here integrated may be written

$$\frac{dx}{dv}\left(\frac{dX}{dz}\frac{dz}{du}+\frac{dX}{dy}\frac{dy}{du}\right)-\frac{dx}{du}\left(\frac{dX}{dz}\frac{dz}{dv}+\frac{dX}{dy}\frac{dy}{dv}\right),$$

* "On Multiple Integrals", *Proc. Roy. Soc.* (A), 93 (1916), 33.

which by condition (vi) may, under the double integral sign,* be written

$$\frac{dx}{dv}\left(\frac{dX}{du}-\frac{dX}{dx}\frac{dx}{du}\right)-\frac{dx}{du}\left(\frac{dX}{dv}-\frac{dX}{dx}\frac{dx}{dv}\right)=\frac{d(X,\,x)}{d(u,\,x)}.$$

Thus
$$\iint_S\left(m\frac{dX}{dz}-n\frac{dX}{dy}\right)dS=\int_a^c\int_b^d\frac{d(X,\,x)}{d(u,\,v)}\,du\,dv.$$

Now, the conditions imposed upon the function (x, y, z) and the variables x, y, z as functions of the parameters (u, v) are such that we may apply the Auxiliary Theorem of § 32. It follows that the functions X and x of (u, v) satisfy the conditions which I have proved† sufficient to justify our replacing the double integral by the allied contour integral round the perimeter of the fundamental rectangle,

$$\int_a^c\int_b^d\frac{d(X,\,x)}{d(u,\,v)}\,du\,dv=\int X\,dx$$

$$=\int_C X\frac{\partial X}{\partial s}\,ds.$$

This proves the theorem as far as X is concerned. Similarly for the other two parts depending on Y and Z. This proves the theorem.

PART III.

A New Method of Treating Stokes's Theorem.

34. We now proceed to discuss the treatment of Stokes's theorem by a new method.‡ This consists in defining an auxiliary function which satisfies conditions under which Stokes's theorem has been shown to hold, connected with the function to which we wish to apply that theorem in such a way that we can pass from the formula involving the auxiliary function to that involving the desired function by a limiting process.

* "On a Formula for an Area", *Proc. London Math. Soc.* (2), 18 (1919), 343–4.

† For integration with respect to u, $\partial x/\partial v$ is, by hypothesis, a bounded function of u, and the same is true of $\partial x/\partial u$ for integration with respect to v.

‡ See the paper by myself, "On Stokes's Formula and the Maxwell-Lorentz Equations", quoted above, § 1. In it I refer to the present paper for the details of the method.

It will be convenient to prove the following preliminary theorem.

PRELIMINARY THEOREM.—*If* $X(x, y, z)$ *be a function having the following properties :—*

(i) *It is bounded with respect to* (x, y, z), *and its integrals with respect to* y *and* z *are continuous functions of* (x, y, z) ;*

(ii) *the derivates of* X *with respect to* y *and* z *are bounded functions of* (x, y, z) ; *then the function*

$$R(x, y, z\,;\ h, k) = \int_y^{y+k} \int_z^{z+q} X(x, y, z)\, dy\, dz$$

(*a*) *is a continuous function of* (x, y, z) :

(*b*) *and has partial differential coefficients with respect to* y *and* z, *which are also continuous with respect to* (x, y, z) ;

(*c*) *as* (k, q) *tends to* $(0, 0)$ *in any manner,* R *tends to* X *as unique double limit ;*

(*d*) *as first* $k \to 0$, *then* $q \to 0$, $\dfrac{dR}{dz}$ *tends to* $\dfrac{dX}{dz}$, *and* $\dfrac{dR}{dy}$ *to*

$$\frac{d}{dz}\int \frac{dX}{dy}\, dz,$$

wherever these limits exist as differential coefficients :

(*e*) *as first* $q \to 0$, *then* $k \to 0$, $\dfrac{dR}{dz}$ *tends to* $\dfrac{d}{dy}\int \dfrac{dX}{dz}\, dy$ *and* $\dfrac{dR}{dy}$ *to* $\dfrac{dX}{dy}$, *wherever these limits exist as differential coefficients.*

The approach to the limits is bounded.

In fact, writing

$$P(x, y, z) = \iint X(x, y, z)\, dy\, dz,$$

* By Lemma 2, § 0, we see that (i) is satisfied if X is continuous with respect to x, provided (ii) holds.

dP/dy and dP/dz are precisely the integrals which are, in (i), assumed to be continuous functions of (x, y, z). Since

$$R(x, y, z\,;\ h, k)$$

$$= \{P(x, y+k, z+q)-P(x, y+k, z)-P(x, y, z+q)+P(x, y, z)\}/kq,$$

this gives (b), from whence (a) at once follows.

To prove (c) we change the variables of integration, and write

$$R(x, y, z\,;\ k, q) = \int_0^1 \int_0^1 X(x, y+ku, z+qv)\, du\, dv.$$

Since the integrand is continuous* with respect to (y, z), it converges boundedly to $X(x, y, z)$, and may accordingly be integrated term-by-term with respect to (u, v). Thus

$$\underset{(k, q)\to(0, 0)}{\mathrm{Lt}} R(x, y, z\,;\ k, q) = \int_0^1 \int_0^1 X(x, y, z)\, du\, dv = X(x, y, z).$$

This proves (c).

To prove (d) we differentiate R, and write the result in the form

$$\frac{dR}{dz} = \frac{1}{k} \int_y^{y+k} \frac{1}{q} \Big(X(x, y, z+q)-X(x, y, z)\Big)\, dy.$$

Here the integrand is a *continuous* function of y by (ii), and a *bounded* function of (x, y, z, q), since, by Dini's theorem, it has for fixed (x, y) the same bounds as dX/dz and, by (ii), these are numerically less than a finite constant, independent of (x, y).

It follows that

$$\underset{k\to 0}{\mathrm{Lt}} \frac{dR}{dz} = \frac{1}{q} \Big(X(x, y, z+q)-X(x, y, z)\Big)$$

and

$$\underset{q\to 0}{\mathrm{Lt}} \frac{dR}{dz} = \left\{\frac{1}{k} \int_y^{y+k} \frac{dX}{dz}\, dy\right\},$$

in both cases the approach to the limit being bounded.

Hence, by the definition of a differential coefficient, the required repeated limits are found, namely,

$$\underset{(q(k))\to 0}{\mathrm{Lt}} \frac{dR}{dz} = \frac{dX}{dz}, \qquad \underset{(k(q))\to 0}{\mathrm{Lt}} \frac{dR}{dz} = \frac{d}{dy} \int^y \frac{dX}{dz}\, dy,$$

* By Lemma 2, § 20.

wherever $X(x, y, z)$ and $\int \frac{dX}{dz}\, dy$ have partial differential coefficients, the former with respect to z and the latter with respect to y.

Similarly, wherever $X(x, y, z)$ and $\int \frac{dX}{dy}\, dz$ have partial differential coefficients, the former with respect to y and the latter with respect to z, we have

$$\operatorname*{Lt}_{(q(k))\to 0} \frac{dR}{dy} = \frac{d}{dz}\int^z \frac{dX}{dy}\, dz, \qquad \operatorname*{Lt}_{(q(k))\to 0} \frac{dR}{dy} = \frac{dX}{dy}.$$

The approach to the limit is in each case, of course, bounded.

35. We have, in Part I, proved Stokes's theorem for a function whose partial differential coefficients with respect to y and z are continuous functions of (x, y, z). By the Preliminary Theorem of § 34, such a function is

$$R(x, y, z;\ k, q) = \int_y^{y+k}\int_z^{z+q} X(x, y, z)\, dy\, dz/kq$$

provided dX/dy and dX/dz are bounded functions of (x, y, z), whose double integrals with respect to (y, z) are continuous functions of (x, y, z). We have then

$$\int_C R(x, y, z;\ k, q)\, dx = \iint_S \left(m\frac{dR}{dz} - n\frac{dR}{dy}\right) dS,$$

where

$$\frac{dR}{dz} = \int_y^{y+k}\int_z^{z+q} \frac{dX}{dz}\, dy\, dz/kq.$$

Also, by the Preliminary Theorem, $R(x, y, z;\ k, q)$ tends boundedly to $X(x, y, z)$ as unique double limit, when $k \to 0$, $q \to 0$. Therefore we may integrate term-by-term with respect to the function of bounded variation $x(u, v)$ round the curve C, on which one of the variables (u, v) is always constant, while x is an absolutely continuous function of the other, since C is the image of the periphery of the fundamental (u, v)-rectangle. Thus *the contour integral of R tends to the contour integral of X.*

We have next to prove that the surface integral of R, which, by Stokes's theorem, is equal to the contour integral of the same function and therefore tends to the contour integral of X as *unique double limit*, has the surface integral of X as limit, when we make k and q tend to zero *in some convenient manner;* this will give us Stokes's theorem for our function $X(x, y, z)$.

36. We shall take each of the two terms of the integrand separately. The first gives us as relation to be proved

$$I = \int_c^c \int_b^d \left(\frac{dR}{dz} - \frac{dX}{dz}\right) \frac{d(z, x)}{d(u, v)}\, du\, dv \to 0.$$

We may write I as a quadruple integral in the form

$$I = \int_c^c du \int_b^d dv \int_0^1 dt_1 \int_0^1 dt_2 \left(\frac{dX}{dz}(x, y+kt_1, z+qt_2) - \frac{dX}{dz}(x, y, z)\right) \frac{d(z, x)}{d(u, v)}.$$

Now, the conditions imposed on the surface* were such that the Jacobian here occurring, as well as the two other such Jacobians, are summable functions of (u, v), and the conditions† imposed on the function X are such that the remaining factor of the integrand is a bounded function of all the four variables of integration; therefore the integral is an absolutely convergent quadruple integral, and we may accordingly change the order of integration, integrating with respect first to u, then v, then t_1, and finally t_2. We thus have

$$I = \int_0^1 \int_0^1 K(t_1, t_2)\, dt_1\, dt_2,$$

where

$$K(t_1, t_2) = \int_c^c \int_b^d \left(\frac{dX}{dz}(x, y+kt_1, z+qt_2) - \frac{dX}{dz}(x, y, z)\right) \frac{d(z, x)}{d(u, v)}\, du\, dv,$$

and, by a well-known theorem, any double integral of this type, in which the integrand is the product of a summable function into the increment of a bounded function, tends uniformly to zero as a unique double limit.

Integrating term by term with respect to t_1 and t_2, we get the required result.

Similarly we get the required result for the second term of the integrand of the surface integral.

We have thus proved that the surface integral of R tends to that of X as unique double limit, when $(k, q) \to (0, 0)$. As we had already proved a similar fact for the contour integrals, *this proves Stokes's theorem for our bounded function* $X(x, y, z)$, *having bounded partial derivates with respect to* y *and* z, *whose double integrals with respect to* (y, z) *are continuous functions of* (x, y, z).

* Part I, § 9. † Part III, § 34.

SUR LES FORMULES D'INVERSION DE FOURIER ET DE HANKEL

Par MICHEL PLANCHEREL.

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1. Dans une intéressante Note M. E. C. Titchmarsh* a montré comment on peut établir les propriétés de la transformation fonctionnelle de Hankel sans avoir recours à la théorie générale des transformations fonctionnelles linéaires bornées. La méthode de M. Titchmarsh est, pour ainsi dire, parallèle à celle que l'on donne généralement pour établir la formule

$$f(x) = \int_0^\infty d\xi\, \xi J_\nu(\xi x) \int_0^\infty f(\lambda)\, \lambda J_\nu(\lambda \xi)\, d\lambda$$

en partant du développement de $f(x)$ en série de Bessel.

Je voudrais montrer comment la théorie que j'ai donnée jadis† permet d'établir les propriétés de la transformation de Fourier et de celle de Hankel à l'aide de quelques formules bien connues. Tout revient pour cela à démontrer pour $\sqrt{(xy)}\, J_\nu(xy)$ la possibilité d'un développement

$$\int_0^x \int_0^y \sqrt{(\xi\eta)}\, J_\nu(\xi\eta)\, d\xi\, d\eta = \sum_{p=1}^{\infty} \int_0^x \phi_p(\xi)\, d\xi \int_0^y \psi_p(\eta)\, d\eta$$

où le système $[\phi_p(x)]$ et le système $[\psi_p(y)]$ sont des systèmes fermés de fonctions orthogonales et normées, relativement à l'intervalle $(0, \infty)$.

Le système orthogonal $[\phi_p]$ peut être pris arbitrairement; le système $[\psi_p]$ est entièrement déterminé par le choix du système $[\phi_p]$, sa simplicité dépend du choix plus ou moins habile que l'on fait du système $[\phi_p]$. Mais quoiqu'il en soit, la détermination explicite est longue et conduit

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* "On Hankel Transforms", *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473.

† "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (2e sem. 1910), 289–335.

à des formules compliquées.* Je me propose précisément de montrer comment, grâce à un théorème que j'établis au §5, cette détermination peut être évitée.

2. Je veux d'abord résumer l'essentiel de la théorie des transformations fonctionnelles, pour autant qu'elle intéresse la question traitée.

Considérons le champ $\Omega = \Omega(0, \infty)$ des fonctions réelles $f(x)$ de carré intégrable (au sens de Lebesgue) dans l'intervalle $(0, \infty)$. Nous dirons qu'une transformation T qui fait correspondre à toute fonction f de Ω une fonction $T(f)$ de ce même champ est *linéaire* et *bornée*, lorsque

(a) linéarité

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2)$$

(a_1, a_2 constantes arbitraires),

(b) limitation $$\int_0^\infty [T(f)]^2 dx \leqslant m^2 \int_0^\infty f^2 dx,$$

m étant une constante indépendante de f.

La condition (b) pourrait être remplacée par la suivante :

(b') Si $$\int_0^\infty (f - f_n)^2 dx \to 0 \quad \text{pour} \quad n \to \infty$$

on a $$\int_0^\infty [T(f) - T(f_n)]^2 dx \to 0.$$

Or (b') est une conséquence de (a) et (b), car

$$\int_0^\infty [T(f) - T(f_n)]^2 dx = \int_0^\infty [T(f - f_n)]^2 dx \leqslant m^2 \int_0^\infty (f - f_n)^2 dx.$$

Inversement (a) et (b') ont pour conséquence une limitation (b).†

Si T est une transformation linéaire bornée du champ Ω, il existe une fonction $\Phi(x, y)$, définie et continue dans le domaine $0 \leqslant x < \infty$,

* La détermination explicite a été donnée par M. H. Weyl: "Singuläre Integralgleichungen", *Math. Annalen*, 66 (1908), 273–324. Voir, en particulier, 317–324.

† "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (2e sem. 1910), 300–301.

$0 \leqslant y < \infty$, ayant presque partout des dérivées partielles du premier ordre et telle que

(i) $$\int_0^\infty \left(\frac{\partial \Phi}{\partial x}\right)^2 dx \quad \text{et} \quad \int_0^\infty \left(\frac{\partial \Phi}{\partial y}\right)^2 dy$$

sont des fonctions continues, la première de y, la seconde de x ;

(ii) $$T(f) = \frac{d}{dx}\int_0^\infty f(\xi)\,\frac{\partial \Phi(\xi, x)}{\partial \xi}\,d\xi ;$$

(iii) $$\Phi(0, y) = \Phi(x, 0) = 0.$$

Cette fonction Φ est univoquement déterminée. Nous l'appellerons la *génératrice* de la transformation T. Elle peut se construire de la manière suivante :

Soit $[\phi_p(x)]$ un système orthogonal normé et fermé pour l'intervalle $(0, \infty)$ et soit

$$\Phi_p = T(\phi_p).$$

On a $$\Phi(x, y) = \sum_{p=1}^{\infty} \int_0^x \phi_p(\xi)\,d\xi \int_0^y \Phi_p(\eta)\,d\eta.$$

En particulier, si $$\frac{\partial^2 \Phi}{\partial x\,\partial y} = \phi(x, y)$$

existe presque partout et si

$$\Phi(x, y) = \int_0^x \int_0^y \phi(\xi, \eta)\,d\xi\,d\eta,$$

on aura $$T(f) = \lim_{z \to \infty} \int_0^z f(\xi)\,\phi(\xi, x)\,d\xi,$$

chaque fois que le second membre existe presque partout dans

$$0 \leqslant x < \infty .$$

3. Seules nous intéressent ici les transformations T dont la génératrice Φ est telle que pour toute fonction f de Ω on ait, en notant

$$F(x) = T(f) = \frac{d}{dx}\int_0^\infty f(\xi)\,\frac{\partial \Phi(\xi, x)}{\partial \xi}\,d\xi, \tag{1}$$

la formule d'inversion

$$f(x) = \frac{d}{dx}\int_0^\infty F(\xi)\,\frac{\partial\Phi(x,\,\xi)}{\partial\xi}\,d\xi. \tag{2}$$

La condition nécessaire et suffisante à laquelle Φ doit alors satisfaire est que cette fonction puisse s'exprimer sous la forme

$$\Phi(x,\,y) = \sum_{p=1}^{\infty}\int_0^x \phi_p(\xi)\,d\xi\int_0^y \psi_p(\eta)\,d\eta, \tag{3}$$

où $[\phi_p]$ est un système orthogonal normé et fermé pour l'intervalle $(0,\,\infty)$ et où $[\psi_p]$ est un système orthogonal normé, mais non nécessairement fermé, pour ce même intervalle. Le choix du système $[\phi_p]$ est arbitraire.

La suffisance de la condition est aisée à démontrer;* c'est d'elle seule que nous aurons à faire usage pour établir les propriétés des transformations de Fourier et de Hankel.

Si f et g sont deux fonctions quelconques de Ω et si $F = T(f)$, $G = T(g)$, on a l'analogue de la formule de Parseval

$$\int_0^\infty FG\,dx = \int_0^\infty fg\,dx.$$

Si le système $[\psi_p]$ est lui-même fermé, toute fonction de Ω est la transformée par T d'une et d'une seule fonction de Ω et réciproquement. Dans ce cas, l'inverse T^{-1} de T est encore une transformation linéaire bornée du champ et sa génératrice est la fonction $\Phi(y, x)$. Le cas le plus intéressant est celui où Φ est une fonction symétrique de x, y. Dans ce cas, T et T^{-1} sont identiques, $T[T(f)] = f$ et T est une transformation involutive. Les formules de transformation deviennent alors, en notant

$$\Psi(x,\,y) = \frac{\partial\Phi(x,\,y)}{\partial x},$$

$$F = T(f) = \frac{d}{dx}\int_0^x f(\xi)\,\Psi(\xi,\,x)\,d\xi, \tag{5}$$

$$f = T(F) = \frac{d}{dx}\int_0^\infty F(\xi)\,\Psi(\xi,\,x)\,d\xi.$$

C'est précisément ce cas qui se présente dans les transformations de Fourier et de Hankel.

* *Loc. cit.*, § 5, 298-300.

4. Nous dirons qu'un système de fonctions $[\chi_p(x)]$ de $\Omega(a, b)$ forme une *base* pour le champ $\Omega(a, b)$ des fonctions de carré intégrable de l'intervalle (a, b) lorsque, étant donné un nombre positif ϵ arbitrairement petit, on peut déterminer pour toute fonction f de $\Omega(a, b)$ une combinaison linéaire $a_1\chi_1+\ldots+a_n\chi_n$ telle que

$$\int_a^b [f-(a_1\chi_1+a_2\chi_2+\ldots+a_n\chi_n)]^2 dx < \epsilon. \tag{6}$$

Pour que le système $[\chi_p]$ forme une base, il faut et il suffit que le système orthogonal qui s'en déduit par le procédé d'orthogonalisation de E. Schmidt soit fermé ou encore que $f \sim 0$ soit la seule fonction de $\Omega(a, b)$ telle que $\int_a^b f\chi_p dx = 0 \quad (p = 1, 2, 3, \ldots)$.

Si $x = \mu(y)$ établit une correspondance biunivoque, continue et dérivable entre les intervalles $a \leqslant x \leqslant b$ et $A \leqslant y \leqslant B$, le système

$$\left[\sqrt{\left(\left|\frac{d\mu}{dy}\right|\right)}\,\chi_p\{\mu(y)\}\right]$$

forme une base pour le champ $\Omega(A, B)$.

Pour démontrer qu'un système $[\chi_p]$ forme une base, il suffit de montrer que l'inégalité (6) est vérifiée par une fonction *continue* f arbitraire du champ Ω.

Le théorème de Weierstrass sur l'approximation uniforme d'une fonction continue par des polynômes montre que le système x^{p-1} $(p = 1, 2, 3, \ldots)$ forme une base pour $\Omega(0, 1)$. Plus généralement, d'après un théorème de Lerch, si une fonction f dont on ne suppose pas qu'elle appartienne à $\Omega(0, 1)$ est telle que $\int_0^1 fx^{p-1} dx = 0$ $(p = 1, 2, 3, \ldots)$, on a $f = 0$ presque partout. La substitution $x = e^{-y}$ montre ensuite que le système $e^{-(p-\frac{1}{2})x}$ $(p = 1, 2, 3, \ldots)$ forme une base pour le champ $\Omega(0, \infty)$. Il en est de même du système $x^{\nu+\frac{1}{2}}e^{-px^2}$ $(p = 1, 2, 3, \ldots)$; c'est ce que montre la substitution $e^{-x^2} = y$ et le théorème de Lerch.

5. THÉORÈME. — *Si la fonction* $\phi(x, y)$ *définie pour*

$$0 \leqslant x < \infty, \quad 0 \leqslant y < \infty$$

est telle que

$$\int_0^\infty \left(\int_0^y \phi(\xi, \eta)\, d\eta\right)^2 d\xi$$

soit une fonction finie de y et si pour une base convenable $[\chi_p(x)]$ du champ $\Omega = \Omega(0, \infty)$ les intégrales

$$\int_0^\infty |\phi(x, y)\,\chi_p(x)|\,dx$$

sont des fonctions intégrales de y dans tout intervalle fini; si, de plus, les fonctions

$$f_p(x) = \int_0^\infty \phi(\xi, x)\,\chi_p(\xi)\,d\xi$$

sont de carré intégrable dans l'intervalle $(0, \infty)$ et vérifient les relations

$$\int_0^\infty f_p f_q\,dx = \int_0^\infty \chi_p \chi_q\,dx \quad (p, q = 1, 2, 3, \ldots),$$

on a une décomposition

$$\int_0^x \int_0^y \phi(\xi, \eta)\,d\xi\,d\eta = \sum_{p=1}^{\infty} \int_0^x \phi_p(\xi)\,d\xi \int_0^y \psi_p(\eta)\,d\eta \tag{7}$$

dans laquelle $[\phi_p]$ est un système orthogonal fermé et normé pour l'intervalle $(0, \infty)$ et $[\psi_p]$ un système orthogonal normé, non nécessairement fermé, pour le même intervalle.

COROLLAIRE.—*Si $\phi(x, y)$ est une fonction symétrique de x, y, le système orthogonal $[\psi_p]$ est lui aussi fermé.*

Soit, en effet, $$\phi_p(x) = \sum_{\rho=1}^{p} a_{p\rho}\,\chi_\rho(x) \quad (p = 1, 2, 3, \ldots),$$

le système orthogonal normé et fermé obtenu par orthogonalisation de la base $[\chi_p]$. Par conséquent,

$$\int_0^\infty \phi_p\,\phi_q\,dx = \sum_{\rho=1}^{p} \sum_{\sigma=1}^{q} a_{p\rho} a_{q\sigma} \int_0^\infty \chi_\rho \chi_\sigma\,dx = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$$

Notons $$\psi_p(x) = \int_0^\infty \phi(\xi, x)\,\phi_p(\xi)\,d\xi = \sum_{\rho=1}^{p} a_{p\rho} f_\rho(x).$$

On aura

$$\int_0^\infty \psi_p \psi_q\,dx = \sum_{\rho=1}^{p} \sum_{\sigma=1}^{q} a_{p\rho} a_{q\sigma} \int_0^\infty f_\rho f_\sigma\,dx$$

$$= \sum_{\rho=1}^{\rho} \sum_{\sigma=1}^{q} a_{p\rho} a_{q\sigma} \int_0^\infty \chi_\rho \chi_\sigma\,dx = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}.$$

Le système $[\psi_p]$ est donc orthogonal et normé.

La légitimité de l'interversion des intégrations dans la formule

$$\int_0^\infty d\xi \left(\int_0^y \phi(\xi, \eta)\, d\eta \right) \phi_p(\xi) = \int_0^y d\eta \int_0^\infty \phi(\xi, \eta)\, \phi_p(\xi)\, d\xi$$
$$= \int_0^y \psi_p(\eta)\, d\eta$$

est assurée par l'existence de l'intégrale double $\int_0^y d\eta \int_0^\infty |\, \phi(\xi, \eta)\, \phi_p(\xi)\, |\, d\xi$ et elle montre que $\int_0^y \phi(\xi, \eta)\, d\eta$, que nous savons être, par hypothèse, de carré intégrable en ξ, possède relativement au système orthogonal fermé $[\phi_p]$ les coefficients de Fourier $\int_0^y \psi_p(\eta)\, d\eta$. On a donc, symboliquement,

$$\int_0^y \phi(\xi, \eta)\, d\eta \sim \sum_{p=1}^{\infty} \int_0^y \psi_p(\eta)\, d\eta\, \phi_p(\xi).$$

L'intégration terme à terme de cette équivalence est légitime, comme on sait ; elle donne la formule (7) où la série est uniformément convergente dans tout domaine borné.

Pour démontrer le corollaire, soit

$$\psi_p \sim \sum_{r=1}^{\infty} o_{pr}\, \phi_r \quad \text{ou} \quad \int_0^y \psi_p(\eta)\, d\eta = \sum_{r=1}^{\infty} o_{pr} \int_0^y \phi_p(\eta)\, d\eta.$$

La fermeture du système $[\phi_p]$ et l'orthogonalité des ψ_p entraînent que

$$\sum_{r=1}^{\infty} o_{pr}^2 = 1, \quad \sum_{r=1}^{\infty} o_{pr}\, o_{qr} = 0 \quad (p \neq q). \tag{8}$$

On déduit de (7)

$$\int_0^x d\xi \int_0^y \phi(\xi, \eta)\, d\eta = \sum_{p, r}^{1 \ldots \infty} o_{pr} \int_0^x \phi_p(\xi)\, d\xi \int_0^y \phi_r(\eta)\, d\eta.$$

On voit de même que

$$\int_0^y d\eta \int_0^x \phi(\eta, \xi)\, d\xi = \sum_{p, r}^{1 \ldots \infty} o_{rp} \int_0^x \phi_p(\xi)\, d\xi \int_0^y \phi_r(\eta)\, d\eta.$$

La symétrie de $\phi(x, y)$ entraîne l'égalité

$$\sum_{p, r}^{1 \ldots \infty} o_{pr} \int_0^x \phi_p(\xi)\, d\xi \int_0^y \phi_r(\eta)\, d\eta = \sum_{p, r}^{1 \ldots \infty} o_{rp} \int_0^x \phi_p(\xi)\, d\xi \int_0^y \phi_r(\eta)\, d\eta,$$

d'où découle, par suite de la fermeture du système $[\phi_p]$

$$o_{pr} = o_{rp},$$

c'est-à-dire $$\int_0^\infty \psi_p \phi_r \, dx = \int_0^\infty \psi_r \phi_p \, dx.$$

(8) montre ensuite que l'on a encore

$$\sum_{r=1}^{\infty} o_{rp}^2 = 1, \quad \sum_{r=1}^{\infty} o_{rp} o_{rq} = 0 \quad (p \neq q). \tag{9}$$

On déduit de (8) et (9) que le système $[\psi_p]$ est fermé, en remarquant que si une fonction f de Ω est telle que

$$\int_0^\infty f\psi_p \, dx = 0 \quad (p = 1, 2, 3, \ldots),$$

nécessairement $f = 0$ presque partout. Car,

$$\int_0^\infty f\psi_p \, dx = \sum_{r=1}^{\infty} o_{pr} \int_0^\infty f\phi_r \, dx = 0$$

entraîne

$$\begin{aligned} 0 &= \sum_{p=1}^{\infty} o_{ps} \int_0^\infty f\psi_p \, dx \\ &= \sum_{p=1}^{\infty} \sum_{r=1}^{\infty} o_{ps} o_{pr} \int_0^\infty f\phi_r \, dx = \sum_{r=1}^{\infty} \int_0^\infty f\phi_r \, dx \sum_{p=1}^{\infty} o_{ps} o_{pr} \\ &= \int_0^\infty f\phi_s \, dx. \end{aligned}$$

Le système $[\phi_s]$ étant fermé, f est nul presque partout.

6. Le théorème précédent trouve une application immédiate à la transformation de Fourier et à celle de Hankel. La fonction $\phi(x, y)$ est alors

$$\phi(x, y) = \sqrt{(2/\pi)} \cos xy$$

pour la première et

$$\phi(x, y) = \sqrt{(xy)} \, J_\nu(xy) \quad (\nu > -1)$$

pour la seconde.

Le fait que $\int_0^\infty \left(\int_0^y \phi(\xi, \eta) \, d\eta \right)^2 d\xi$ existe se vérifie immédiatement dans le cas de Fourier ; il s'établit aisément à l'aide de la valeur asymptotique connue de $J_\nu(x)$ $(x \to \infty)$ dans le cas de Hankel. Il suffit, en effet, de remarquer qu'en notant $\xi\eta = t$

$$\int_0^y \sqrt{(\xi\eta)} \, J_\nu(\xi\eta) \, d\eta = \frac{1}{\xi} \int_0^{\xi y} \sqrt{t} \, J_\nu(t) \, dt.$$

Tout revient donc à trouver une base convenable $[\chi_p]$ réalisant les conditions du théorème. Dans le cas de Fourier, on prendra, par exemple,

$$\chi_p(x) = e^{-(p-\frac{1}{2})x} \quad (p = 1, 2, 3, \ldots)$$

et on utilisera la formule bien connue

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\alpha\xi} \cos \xi x \, d\xi = \sqrt{\frac{2}{\pi}} \frac{\alpha}{x^2+\alpha^2} \quad (\alpha > 0).$$

On vérifie, en effet, que si $\alpha > 0$, $\beta > 0$,

$$\int_0^\infty e^{-\alpha\xi} e^{-\beta\xi} \, d\xi = \frac{2}{\pi} \alpha\beta \int_0^\infty \frac{dx}{(x^2+\alpha^2)(x^2+\beta^2)}.$$

Dans le cas de Hankel, le plus simple est de prendre

$$\chi_p(x) = x^{\nu+\frac{1}{2}} e^{-px^2} \quad (p = 1, 2, 3, \ldots).$$

On utilisera la formule

$$\int_0^\infty J_\nu(x\xi) \, e^{-\alpha^2\xi^2} \, \xi^{\nu+1} \, d\xi = \frac{x^\nu}{(2\alpha^2)^{\nu+1}} \, e^{-(x^2/4\alpha^2)} \quad (\nu > -1, \ \alpha \neq 0).$$

que l'on peut vérifier en remplaçant au premier membre $J_\nu(x\xi)$ par son développement en série et en intégrant terme à terme. Il suffira de vérifier que l'on a

$$\int_0^\infty (\xi^{\nu+\frac{1}{2}} e^{-\alpha^2\xi^2})(\xi^{\nu+\frac{1}{2}} e^{-\beta^2\xi^2}) \, d\xi = \frac{1}{(2\alpha^2 . 2\beta^2)^{\nu+1}} \int_0^\infty (\xi^\nu \, e^{-(\xi^2/4\alpha^2)})(\xi^\nu \, e^{-(\xi^2/4\beta^2)}) \, d\xi$$

pour pouvoir appliquer le théorème.

Les transformations de Fourier et de Hankel sont donc des transformations linéaires bornées involutives dans le champ $\Omega(0, \infty)$.

THE CONFORMAL REPRESENTATION OF CERTAIN ASSOCIATED FIGURES

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IN a previous paper I have discussed the relations of the functions which effect the conformal representations of the various triangles bounded by the arcs of three intersecting circles.* The present paper deals with the same question from a different, and perhaps more constructive, point of view. The methods now shown possess the advantage that, in dealing with those figures for which representation of the fundamental reduced triangle is effected by means of the polyhedral functions, it is possible to obtain explicit results for the associated figures without having to solve equations of order higher than the first.

The methods are based on a theorem of Klein, enunciated without proof, so far as I have traced, in fact treated by him as obvious. This theorem is to the effect that, if we have two curvilinear triangles of angles $\lambda\pi, \mu\pi, \nu\pi$ and $\lambda'\pi, \mu'\pi, \nu'\pi$ in the planes of the complex variables u, w respectively, and these triangles are both represented on the half-plane of a variable z, then w is a uniform function of u if $1/\lambda$, $1/\mu$, $1/\nu$, λ'/λ, μ'/μ, ν'/ν are all integers.†

The functions thus obtained are characterized by the property that, if the variable u is subjected to one of the homographic transformations

* *Proc. London Math. Soc.* (2), 15 (1915–16), 166–181.

† Klein, *Icosahedron* (English edition), 144.

of its appropriate group, w is subjected to a transformation of its group. If the two triangles are bounded by the same three circles, these groups are the same.

The property was remarked by Klein (*loc. cit.*, 149) in the special case for which his theorem was enunciated.

Similar results are obtained for figures associated with the curvilinear rectangle, but two points of divergence are to be noted. The reasoning applied to the consideration of triangular figures does not apply in its entirety to figures with a larger number of sides. In particular, it is impossible to make the points of the real axis in the z-plane which correspond to the angular points of the figures quite arbitrary when the number of such points is more than three. It follows that the rectangle of the u-plane is not of the same type as the rectangle of the w-plane with which the actual figure in that plane is associated. As a consequence it follows that the groups of the variables w, u are not the same.

For rectilinear figures it may be remarked that the associated figures are all of infinite area, and the ordinary Schwarz-Christoffel formula requires various modifications. Kluyver* has considered the most interesting case, viz. that of the area exterior to a closed rectilinear polygon, in considerable detail. The other necessary modifications can be found in similar fashion, and it will suffice here to say that the explicit results obtainable have the same general character as those of Kluyver.

Proof of the Theorem of Klein.

The representation of the u triangle is given by the Schwarzian formula

$$\{u, z\} = \tfrac{1}{2}\left\{\frac{1-\lambda^2}{z^2}+\frac{1-\nu^2}{(z-1)^2}+\frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\right\},$$

where $\{u, z\}$ is the Schwarzian Derivative. The points 0, ∞, 1 in the z-plane correspond to the angular points of the u triangle.

Suppose we take any initial value, z_0, of z, and describe a circuit enclosing one of the points 0, 1, ∞, returning to z_0. If we take a corresponding initial value, u_0, of u, the final value of u will not be u_0 but one of the set of transformations of u_0.† Conversely, if we take any initial

* Kluyver, *Zeitschrift der Math. u. Phys.*, 40 (1895), 129–150.

† For these properties, see Forsyth, *Theory of Functions*, Ch. 20.

value u_0 of u, and describe a path to one of its transformations, the corresponding z point will trace out a path from an initial value z_0, returning to the same point after encircling one or more of the critical points 0, 1, ∞.

The w triangle now being supposed to be represented on the same z-half-plane, it follows that, when w is expressed as a function of u, if u be subjected to one of the group of its transformations, w is also subjected to one of the group of its transformations.

Sufficient conditions for the uniformity of w are found by considering the solutions of the differential equations. Whatever path the variable z may describe, if it returns to its initial value z_0 without enclosing a critical point 0, 1, or ∞, the variables u, w describe corresponding paths, also returning to the initial values u_0, w_0. Consequently any departure from uniformity of w as a function of u can only take place in the neighbourhood of the points which correspond to 0, 1, ∞, *i.e.* the angular points of the triangles.

The solution of the differential equation for u is the ratio of two solutions of a hypergeometric equation. It follows that the solution may be written :

$$U = \frac{au+b}{cu+d} = z^{\lambda} S(z),$$

where $S(z)$ is an absolutely convergent series of positive powers of z, the expansion being valid in the neighbourhood of the z origin.

This series may be reversed for sufficiently small values of $|U|$ in the form

$$z = U^{1/\lambda} T(U^{1/\lambda}).$$

Similarly we have $$W = \frac{a'w+b'}{c'w+d'} = z^{\lambda'} V(z),$$

whence $$W = U^{\lambda'/\lambda} X(U^{1/\lambda}).$$

Uniformity is ensured in the neighbourhood of the critical point if $1/\lambda$ and λ'/λ are integers. By considering the values of u, w corresponding to the other critical z points 1, ∞, we finally obtain that w will be a uniform function of u if $1/\lambda$, $1/\mu$, $1/\nu$, λ'/λ, μ'/μ, ν'/ν are all integers.

When the u triangle is a reduced triangle such that λ, μ, ν are the reciprocals of integers, and the w triangle is a triangle associated with the replica in the w-plane of the u triangle (or of one of its transformations), these conditions are all satisfied. Further, the groups of the transformations of the variables u, w are the same.

Polyhedral Functions.

The calculation of the polyhedral functions is simplified by the fact that the whole plane contains a finite number of repetitions of the original triangle. Consequently any associated triangle in the w-plane contains a finite number of repetitions, say m, of the reduced triangle. It follows that, when the u triangle is subjected to the complete group of transformations so that its plane is completely covered once, the w triangle and its repetitions covers the w-plane m times, and thus the same w point will correspond to m different u points, *i.e.* the relation between u, w is of the m-th order in u, and, therefore, as w is a uniform function of u, w can be expressed as a rational fraction whose numerator and denominator are polynomials in u of order m.

(*a*) *Cyclic and Dihedral Functions.*—General formulae can be given for functions having the characteristic property in the case of these two simple groups.

The cyclic substitutions are given by

$$u' = e^{(2ir\pi)/n}u, \quad r = 0, 1, 2, \ldots, (n-1).$$

The general function of the required type is $u^m f(u^n)$, where $f(t)$ is a uniform function of t.

The dihedral substitutions are given by

$$u' = e^{(2ir\pi)/n}u, \quad u' = e^{(2ir\pi)/n}/u, \quad r = 0, 1, \ldots, (n-1),$$

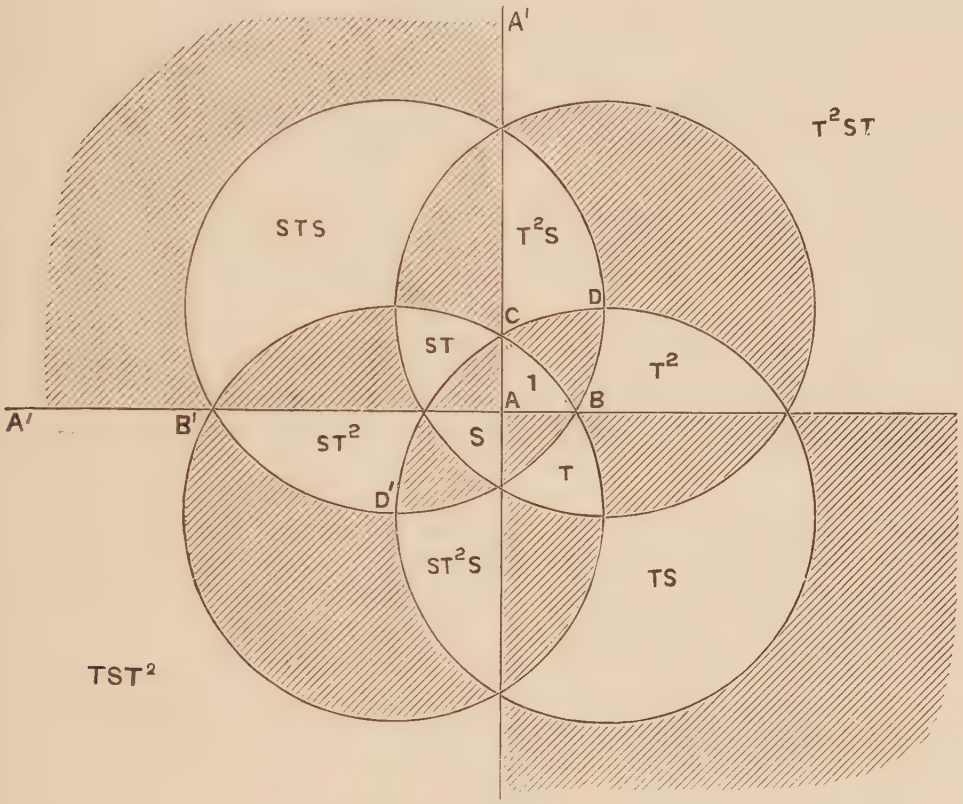
and the typical function is now

$$w = \frac{u^m(a_0u^{pn} + a_1u^{(p-1)n} + \ldots + a_p)}{a_pu^{pn} + a_{p-1}u^{(p-1)n} + \ldots + a_0}.$$

In either case if m is not prime to n, the function w only undergoes a sub-group of the transformations.

(*b*) *Tetrahedral Functions.*—The fundamental triangle has angles $\pi/2$, $\pi/3$, $\pi/3$. The tetrahedral group of transformations is generated by two substitutions $S(u)$, $T(u)$, which transform the fundamental triangle ABC marked 1 into those marked S, T respectively. The

symbols marked in each triangle show one way in which the original triangle can be transformed into that particular repetition. [The convention as to the meaning of ST not being fixed, I have written $ST(u)$ as being equivalent to $S\{T(u)\}$].



We consider the associated triangle whose angles are $\pi/2$, $2\pi/3$, $\pi/3$. We may choose any of its repetitions we please. I have selected that one which exhibits the group of its transformations in exactly the same order as the u triangle, viz. $A'\,B'\,CA'$ in the figure (which we may use, as convenient, to represent either the u-plane or the w-plane). This is contrived by taking a repetition of the w triangle bounded by the same three circles as $A\,B\,C$, and such that, when the boundary of each triangle is described positively, corresponding angular points occur in the same order.

Now $A'\,B'\,CA'$ contains three repetitions of the triangle ABC, so that the relation between w, u is of the third order in u. From the following

selection of the properties of w we can now determine this function :—

$$u \to 0, \quad w \to \infty ; \qquad u \to \infty , \quad w \to 0 ;$$

$$u' = S(u) = -u, \qquad w' = S(w) = -w ;$$

$$u' = T^2ST(u) = i/u, \qquad w' = T^2ST(w) = i/w ;$$

$$u = T^2(A) = e^{(i\pi)/4} \text{ (at } D), \qquad w = T^2(A') = -e^{(i\pi)/4} \text{ (at } D') ;$$

$$u = \frac{\sqrt{2}}{\sqrt{3}+1} \text{ (at } B), \qquad w = -\frac{\sqrt{2}}{\sqrt{3}-1} \text{ (at } B').*$$

The conditions determine w, and we get

$$w = \frac{\sqrt{3}\,u^2+1}{u(u^2-\sqrt{3})}.$$

In a similar way, if we take the triangle $A'BCA'$, whose angles are $\pi/2$, $4\pi/3$, $\pi/3$, we obtain a function which is determined by the following properties; the relation between u, w is of the seventh order in u :—

$$u \to 0, \quad w \to \infty ; \qquad u \to \infty , \quad w \to 0 ;$$

$$u' = S(u) = -u ; \qquad w' = S(w) = -w ;$$

$$u' = T^2ST(u) = i/u, \qquad w' = T^2ST(w) = i/w ;$$

$$u = T^2(A) = e^{(i\pi)/4} \text{ (at } D), \qquad w' = T^2(A') = -e^{(i\pi)/4} \text{ (at } D') ;$$

$u = i\alpha$ is a zero of the fourth order of $w - i\alpha$,

where $$\alpha = \frac{\sqrt{2}}{\sqrt{3}+1} \text{ (at } C).$$

This leads to $$w = \frac{7\sqrt{3}(u^6+u^2)+7u^4-1}{u\{(u^6-7u^2)+7\sqrt{3}(u^4+1)\}}.$$

(c) *Octahedral and Icosahedral Functions.*—In a precisely similar manner I have obtained functions for some of the associated triangles of the octahedral and icosahedral configurations. Thus for the triangles of angles $(\pi/2, \pi/4, 2\pi/3)$ and of angles $(\pi/2, 3\pi/4, 2\pi/3)$ which are

* For the numerical values assigned to the coordinates of points in the figure, see Forsyth, *Theory of Functions*, Ch. 20.

associated triangles of the octahedral configuration, I obtain the functions

$$-\frac{u(u^4+5)}{5u^4+1} \quad \text{and} \quad -\frac{11u^8-22u^4-1}{u^3(u^8+22u^4-11)} \quad \text{respectively,}$$

and for the triangles of angles $(\pi/2, 2\pi/3, \pi/5)$ and of angles $(\pi/2, \pi/3, 4\pi/5)$, which are associated triangles of the icosahedral configuration, I obtain the functions

$$-\frac{u(u^{10}+66u^5-11)}{11u^{10}+66u^5-1} \quad \text{and} \quad \frac{57u^{15}-247u^{10}-171u^5-1}{u^4(u^{15}-171u^{10}+247u^5+57)} \quad \text{respectively.}$$

(*d*) *The development of new functions of the type.*—The total number of associated triangles bounded by three circles is thirty-two, and for each such triangle can be found a function of the type we are considering. But these are not all.

Suppose $v_1(u)$, $v_2(u)$ are functions obtained in this manner from two associated triangles of one of the regular configurations. When u is subjected to the transformation of its group, so also are $v_1(u)$, $v_2(u)$, and therefore so also is $v_1\{v_2(u)\}$. Clearly there is no limit to the number of functions which can be thus generated.

Further, when we state that there are in all thirty-two associated triangles we are restricting ourselves to simple envelopment of the plane. If we allow multiple envelopment the number of associated triangles is also unlimited. Such an associated triangle, for example, can be obtained by making a slit along one side of the fundamental reduced triangle, and joining a repetition of the triangle to one edge of the slit, the associated triangle now consisting of the whole plane plus one repetition.

(*e*) *The Application of the Methods of the Theory of Binary Quantics.*—As in the case of the ordinary polyhedral functions, we can develop functions of the type we are now considering by the methods of the theory of binary quantics, and this very simply. The method possesses the disadvantage that it does not display the relation of the function to its triangle; it is wider because it develops functions which do not appear to be associated with any triangle at all.

Suppose we have a quantic $f(u_1, u_2)$ in two variables u_1, u_2 which we then subject to a linear transformation

$$u_1 = l_1 U_1 + m_1 U_2,$$

$$u_2 = l_2 U_1 + m_2 U_2,$$

where $$l_1 m_2 - l_2 m_1 = 1.$$

Let $f(u_1, u_2)$ be transformed into $F(U_1, U_2)$.

Then $$-\frac{\dfrac{\partial f}{\partial u_2}}{\dfrac{\partial f}{\partial u_1}} = \frac{l_1\left\{-\dfrac{\partial F}{\partial U_2}\Big/\dfrac{\partial F}{\partial U_1}\right\}+m_1}{l_2\left\{-\dfrac{\partial F}{\partial U_2}\Big/\dfrac{\partial F}{\partial U_1}\right\}+m_2}.$$

Suppose we take one of the finite groups of linear substitutions. If the quantic $f(u_1, u_2)$ be such that $F(U_1, U_2)$ differ from $f(U_1, U_2)$ at most by a constant factor, then on putting $u_1/u_2 = u$, $-\dfrac{\partial f}{\partial u_2}\Big/\dfrac{\partial f}{\partial u_1}$ is a function such that, when u is subject to one of its group of transformations, it also is subject to the same transformation.

Such quantics, $f(u_1, u_2)$ are known to exist, viz. the three fundamental ground forms associated with each polyhedron and covariants of these forms.* We can thus calculate again the six functions already determined. Thus the ground forms for the tetrahedron are

$$u_1 u_2(u_1^4+u_2^4), \quad u_1^4+2\sqrt{3}u_1^2u_2^2-u_2^4, \quad u_1^4-2\sqrt{3}u_1^2u_2^2-u_2^4$$

The covariants required in order to derive the examples given are the ground form $u_1^4-2\sqrt{3}u_1^2u_2^2-u_2^4$, and the fractional covariant

$$\frac{(u_1^4-2\sqrt{3}u_1^2u_2^2-u_2^4)^4}{(u_1^4+2\sqrt{3}u_1^2u_2^2-u_2^4)^3}.$$

The two examples of octahedral functions can be obtained from the two ground forms $u_1u_2(u_1^4+u_2^4)$ and $u_1^{12}+33u_1^8u_2^4-33u_1^4u_2^8-u_2^{12}$, and the two icosahedral functions from the ground forms

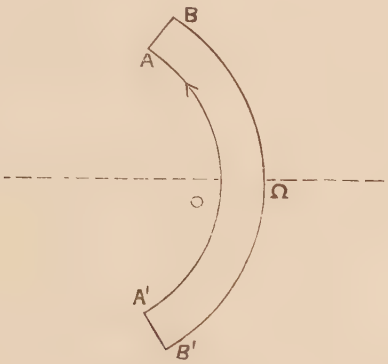
$$u_1u_2(u_1^{10}+11u_1^5u_2^5-u_2^{10}), \quad u_1^{20}-228u_1^{15}u_2^5+494u_1^{10}u_2^{10}+228u_1^5u_2^{15}+u_2^{20}.$$

Two Areas associated with a Curvilinear Rectangle.

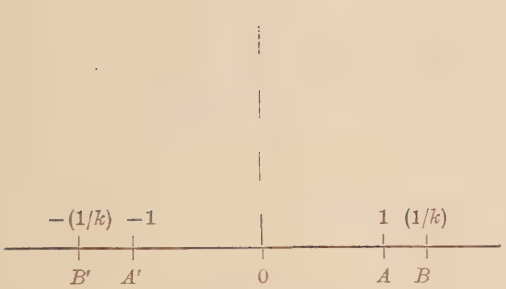
(*a*) *The Area exterior to a Curvilinear Rectangle.*—The circular arcs which bound a curvilinear rectangle must belong to two systems of coaxal circles. If the members of each system touch one another, we can transform the figure by inversion into a rectilinear figure. In all

* Klein, *Icosahedron* (English edition), 56.

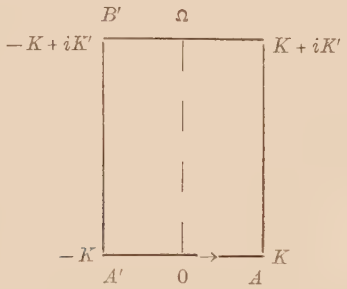
other cases we can invert the figure into one bounded by the arcs of concentric circles and two radii.



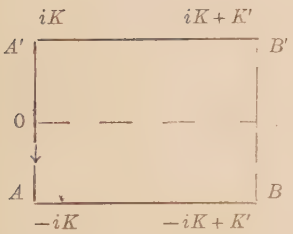
w-plane.



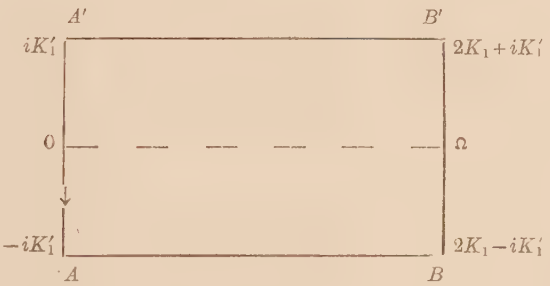
z-plane.



u-plane.



v-plane.



t-plane.

The angles of the w figure being $3\pi/2$, $3\pi/2$, $3\pi/2$, $3\pi/2$, the representa-

tion upon the half-plane of a variable z is given by

$$\{w, z\} = \tfrac{1}{2}\left[-\frac{5}{4(z-1)^2}-\frac{5}{4(z+1)^2}-\frac{5}{4\{z-(1/k)\}^2}-\frac{5}{4\{z+(1/k)\}^2}\right]$$

$$+\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{z-(1/k)}+\frac{D}{z+(1/k)}$$

where

$$A+B+C+D=0,$$

$$A-B+\frac{C-D}{k}=\frac{5}{2},$$

$$A+B+\frac{C+D}{k^2}=0,^*$$

the symmetry of the figure allowing the choice of ± 1, $\pm 1/k$ as the homologues of the angular points. We may write for convenience $A=-B=\frac{5}{8}-\lambda$, $C=-D=(\frac{5}{8}+\lambda)k$, and we get, after reduction,

$$\{w, z\} = -\tfrac{5}{2}\left\{\frac{1}{(1-z^2)^2}+\frac{k^2}{(1-k^2z^2)^2}\right\}+\frac{2\lambda(1-k^2)}{(1-z^2)(1-k^2z^2)}.$$

We transform this relation by means of the subsidiary variable u where $z=\operatorname{sn}(u, k)$ and the relation

$$\{w, u\} = \left(\frac{dz}{du}\right)^2[\{w, z\}-\{u, z\}],$$

But $$\{u, z\} = \tfrac{3}{2}\left[\frac{1}{(1-z^2)^2}+\frac{k^2}{(1-k^2z^2)^2}\right]-\tfrac{1}{2}\frac{(1-k^2)}{(1-z^2)(1-k^2z^2)},$$

whence

$$\{w, u\} = \left(\frac{dz}{du}\right)^2\left[-4\left\{\frac{1}{(1-z^2)^2}+\frac{k^2}{(1-k^2z^2)^2}\right\}+\frac{(2\lambda+\frac{1}{2})(1-k^2)}{(1-z^2)(1-k^2z^2)}\right]$$

$$= -\frac{4(1-k^2)^2\operatorname{sn}^2 u}{\operatorname{cn}^2 u\operatorname{dn}^2 u}+\text{const.}$$

We put $u=iv$, and apply Jacobi's imaginary transformation of the elliptic functions, and we get

$$\{w, v\} = -\frac{4(1-k^2)^2\operatorname{sn}^2 v\operatorname{cn}^2 v}{\operatorname{dn}^2 v}+\text{const.} \pmod{k'}.$$

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* See, *e.g.*, Forsyth, *Theory of Functions*, Ch. 20.

We again transform by applying Landen's transformation,

$$t = (1+k)v, \quad k_1 = \frac{1-k}{1+k}, \quad K_1 = \frac{(1+k)K'}{2}, \quad K_1' = (1+k)K,$$

whence $$\left\{w, \frac{t}{1+k}\right\} = -4\frac{(1-k^3)^2}{(1+k)^2}\operatorname{sn}^2(t, k_1)+\text{const.}$$

or $$\{w, t\} = -4k_1^2 \operatorname{sn}^2(t, k_1)+\text{const.}$$

We require then the ratio of two solutions of Lamé's equation of order 1,

$$\frac{d^2y}{dt^2} = (2k_1^2 \operatorname{sn}^2 t+\text{const.})\, y.$$

Two such solutions are, in general,

$$y = \frac{\mathrm{H}(t\pm\beta)}{\Theta(t)}\, e^{\mp tZ(\beta)}.$$

i.e we may take $$w = -\frac{\mathrm{H}(t+\beta)}{\mathrm{H}(t-\beta)}\, e^{-2tZ(\beta)},$$

while $$z = \operatorname{sn} u \pmod{k}$$

$$= \frac{i \operatorname{sn} v}{\operatorname{cn} v} \pmod{k'}$$

$$= \frac{i(\pm \operatorname{cn} t \pm \operatorname{dn} t)}{(1-k_1)\operatorname{sn} t} \pmod{k_1}.$$

The ambiguous signs are now determined by making $z \to 0$ as $t \to 0$ and $z \to -1$ as $t \to iK_1'$ and we get finally

$$z = \frac{i(\operatorname{cn} t - \operatorname{dn} t)}{(1-k_1)\operatorname{sn} t}.$$

On account of the symmetry of the original figure the value of β will be real and lie between 0 and K_1, and we get for the correspondence of the principal points of the figure

O, $t = 0$, $w = 1$;

A, $t = -iK_1'$, $w = -e^{(i\pi\beta/K_1)+2iK'Z(\beta)}$;

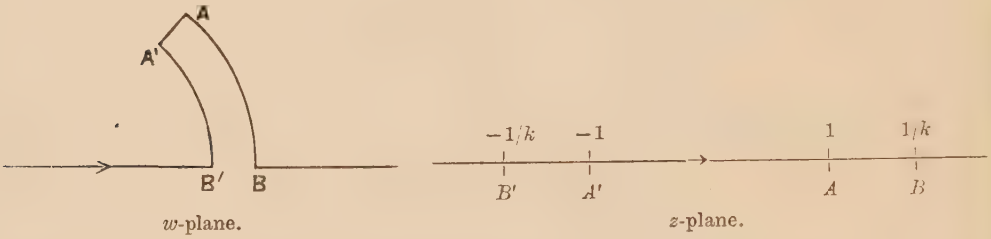
A', $t = -iK_1'$, $w = -e^{-(i\pi\beta/K_1)-2iK'Z(\beta)}$;

B, $t = 2K_1 - iK_1'$, $w = -e^{-4K_1Z(\beta)+(i\pi\beta/K_1)+2iK'Z(\beta)}$;

B', $t = 2K_1 + iK_1'$, $w = -e^{-4K_1Z(B)-(i\pi\beta/K_1)-2iK'Z(\beta)}$.

The radii of the circles are 1 and $e^{-4K_1'Z(\beta)}$ and the angle subtended at the centre is $2\pi-(2\pi\beta/K_1)-4K_1'Z(B)$.

(*b*) *A Second Area associated with a Curvilinear Rectangle.*—



The angles of the figure shown are $\pi/2$, $3\pi/2$, $3\pi/2$, $\pi/2$.

We can fix the z homologues of A, A' as ± 1, and have still one arbitrary element at our disposal. Let the homologues of B, B' be b, b'. Then, by a real homographic transformation of the z-plane with ± 1 as double points, we can arrange that $b=-b'$, *i.e.* we may write the homologues of B, B' as $\pm 1/k$.

We have then

$$\{w, z\} = \tfrac{1}{2}\left\{-\frac{5}{4(z-1)^2}-\frac{5}{4(z+1)^2}+\frac{3}{4\{z-(1/k)\}^2}+\frac{3}{4\{z+(1/k)\}^2}\right\}$$
$$+\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{z-(1/k)}+\frac{D}{z+(1/k)},$$

where we may write

$$A=-B=\tfrac{5}{8}-\lambda, \quad C=-D=k(\lambda-\tfrac{3}{8}).$$

After reduction we put $z=\mathrm{sn}\,(u,k)$, and obtain

$$\{w, u\} = -\frac{4\,\mathrm{dn}^2 u}{\mathrm{cn}^2 u}+\text{const.}$$
$$= -4k^2\,\mathrm{sn}^2\,(u+K-iK')+\text{const.}$$

whence we derive a solution

$$w=-\frac{\mathrm{H}(u+K-iK'-\beta)}{\mathrm{H}(u+K-iK'+\beta)}\,e^{2(u+K-iK')Z(\beta)},$$

where $0<\beta<K$.

The other details of the solution are obtained exactly as in the last section, and need not be included here.

ON THE RANK OF THE DOUBLE-BINARY FORM*

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THE point of view which is adopted in this paper towards the *double-binary* form is one which attempts to study it by means of *binary* forms. This is the method naturally suggested by geometry. In this point of view the rank-concept of the double-binary form plays a fundamental rôle.

The paper is mainly occupied with the properties and the geometrical interpretation of the *Rank Covariants*. The reciprocity of the rank covariants for an equiform and the identical binary apolarity of an equiform and its penultimate rank-covariant are the topics of Section (4). Section (5) studies an important method of representing a double-binary form as a correspondence of a special type between two rational norm curves. In Section (6) a useful canonical shape is obtained for double-binary forms which are of even order in each of the variables. It is seen from the last section that this canonical shape is more advantageous for the (2, 2) form, in point of symmetry, than the numerical canonical types which are in ordinary use.

1. *Definition of rank.*

Let a double-binary form $F(x, y)$, of orders m, n in the independent binary variables x, y, be written out in the shape :

$$\begin{aligned} F &= \Sigma a_{rs} x_1^{m-r} x_2^r y_1^{n-s} y_2^s \\ &= y_1^n F_0(x) + y_1^{n-1} y_2 F_1(x) + \ldots + y_2^n F_n(x) \\ &= x_1^m \Phi_0(y) + x_1^{m-1} x_2 \Phi_1(y) + \ldots + x_2^m \Phi_m(y) \end{aligned} \qquad (1)$$

* The author is greatly indebted to Prof. H. W. Turnbull for advice and encouragement during the preparation of the paper, and to the referee for several useful suggestions.

where $$F_\kappa(x) = \sum_r a_{r\kappa} x_1^{m-1} x_2^r, \quad \Phi_\kappa(y) = \sum_s a_{\kappa s} y_1^{n-s} y_2^s.$$

Let Δ denote the matrix

$$\begin{vmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & & & \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{vmatrix}.$$

We define the rank of the double-binary form F to be the rank of the matrix Δ.

The rank of Δ is a numerical characteristic of the matrix which persists after pre- and post-multiplications of Δ by undegenerate square matrices of orders $m+1$, $n+1$ respectively. It is easily seen to follow from this that the rank of $F(x, y)$ is a numerical invariant of the form for arbitrary linear transformations of non-zero determinant of x and y. For, by such linear transformations, the sets

$$(x_1^m, x_1^{m-1} x_2, \dots, x_2^m), \quad (y_1^n, y_1^{n-1} y_2, \dots, y_2^n)$$

are subjected to two linear transformations, also of non-zero determinant, represented by two matrices δ_1, δ_2 (say); and the matrix Δ is thereby transformed into $\delta_1' \Delta \delta_2$, δ_1' being the transposed matrix of δ_1. Since this transformation of Δ does not alter its rank, it follows that the rank of a form is an invariant for linear transformations of the variables.

If Δ has the rank r, then only r of its rows or columns are linearly independent, and therefore only r of the binary forms $F_\kappa(x)$ or $\Phi_\kappa(y)$ are linearly independent. Hence it must be possible to express the $n+1$ forms $F_\kappa(x)$ linearly in terms of r independent forms $f_1(x), f_2(x), \dots, f_r(x)$. Writing

$$F_\kappa(x) = l_{\kappa 1} f_1(x) + l_{\kappa 2} f_2(x) + \dots + l_{\kappa r} f_r(x) \quad (\kappa = 0, 1, \dots, n),$$

it is obvious that the matrix $|l|$ is undegenerate, of rank r, since r of the forms $F_\kappa(x)$ are linearly independent. Substituting these expressions in (1), it follows that F is expressible in the shape :

$$F(x, y) = f_1(x)\, \phi_1(y) + f_2(x)\, \phi_2(y) + \dots + f_r(x)\, \phi_r(y), \tag{2}$$

$$\phi_p(y) = l_{0p} y_1^n + l_{1p} y_1^{n-1} y_2 + \dots + l_{np} y_2^n \quad (p = 1, 2, \dots, r).$$

It is important to notice that *not only the forms f but also the forms ϕ in* (2) *are linearly independent.* This may be seen in different ways. From the actual expressions for $\phi_\rho(y)$ and from the fact that the matrix $|l|$ is of rank r, it follows immediately that there can be no linear relation between $\phi_1 \phi_2 \dots \phi_r$. Otherwise, on writing out the expression (2) according to powers of x, it would follow that all the forms $\Phi_\kappa(y)$ are linear combinations of $\phi_1 \phi_2 \dots \phi_r$; hence any linear relation between these latter would mean that, among the forms $\Phi_\kappa(y)$, there are not as many as r which are linearly independent; and this would make the rank of Δ less than r, contrary to hypothesis. That the linear independence of the f's implies the linear independence of the ϕ's is in any case a consequence of the fundamental property of the matrix, that *the rank of a matrix has the same value, whether defined as the number of its linearly independent rows, or as the number of its linearly independent columns.*

2. *The rank covariants.*

The form $F(x, y)$, which may be represented symbolically in the shape

$$F(x, y) = (a_{\kappa 1} x_1 + a_{\kappa 2} x_2)^m (b_{\kappa 1} y_1 + b_{\kappa 2} y_2)^n = a_{\kappa_x}^m b_{\kappa_y}^n \quad (\kappa = 0, 1, \dots),$$

is ordinarily of rank equal to the lesser of the numbers $m+1$, $n+1$. In order to find the condition that it may be of any given rank r, we observe that, when F is of rank r and is expressed in the form (1), any $r+1$ linear combinations of the forms $\Phi_\kappa(y)$ must be subject to a linear relation. In particular the $r+1$ r-th polars

$$\frac{\partial^r F(x, y)}{\partial x_1^\lambda \partial x_2^{r-\lambda}} \quad (\lambda = 0, 1, \dots, r),$$

regarded as forms in y, must be subject to a linear relation. Now the symbolic condition* that the forms

$$c_{0_y}^n, \quad c_{1_y}^n, \quad \dots, \quad c_{r_y}^n$$

be linearly related is the identical vanishing of their Jacobian, namely

* Compare Grace and Young, *Algebra of Invariants* (1903), 220, 221.

the covariant*

$$(c_0c_1)(c_0c_2)\dots(c_0c_r)(c_1c_2)\dots(c_{r-1}c_r)\,c_{0_y}^{n-r}c_{1_y}^{n-r}\dots c_{r_y}^{n-r}.$$

Hence the necessary and sufficient condition for a linear relation between the $r+1$ r-th polars,

$$a_{01}^r a_{0_x}^{m-r} b_{0_y}^n,\quad a_{11}^{r-1}a_{12}a_{1_x}^{m-r}b_{1_y}^n,\quad \dots,\quad a_{r2}^r a_{r_x}^{m-r}b_{r_y}^n,$$

regarded as forms in y, is the identical vanishing of

$$(b_0b_1)\dots(b_{r-1}b_r)\,b_{0_y}^{n-r}\dots b_{r_y}^{n-r}\,a_{0_x}^{m-r}\dots a_{r_x}^{m-r}\,a_{01}^r a_{11}^{r-1}a_{12}\dots a_{r2}^r.$$

By deranging the equivalent symbols in all possible ways, this expression is seen to be equal to

$$\frac{1}{(r+1)!}(b_0b_1)\dots(b_{r-1}b_r)\,b_{0_y}^{n-r}\dots b_{r_y}^{n-r}\,a_{0_x}^{m-r}\dots a_{r_x}^{m-r}\begin{vmatrix} a_{01}^r a_{01}^{r-1}a_{02}\dots a_{02}^r \\ a_{11}^r\ \dots\ \dots\ \dots a_{12}^r \\ \vdots \\ a_{r1}^r a_{r1}^{r-1}a_{r2}\dots a_{r2}^r \end{vmatrix}$$

$$=\frac{1}{(r+1)!}(a_0a_1)\dots(a_{r-1}a_r)\,(b_0b_1)\dots(b_{r-1}b_r)\,a_{0_x}^{m-r}\dots a_{r_x}^{m-r}\,b_{0_y}^{n-r}\dots b_{r_y}^{n-r}$$

$$=\frac{1}{(r+1)!}\,G_r\ (\text{say}).$$

Thus, if F is of rank r, the covariant G_r (which we call the r-th *rank covariant* of F) vanishes identically. Conversely, if G_r vanishes identically, but not G_{r-1}, then there is a linear relation between any $r+1$ linear combinations of the forms Φ_κ, but not between any r. Hence F is of rank r. We have thus :

THEOREM I.—*The necessary and sufficient condition that F may be of rank r is that G_r be the first of the rank covariants $G_1\,G_2\,G_3\dots$ which vanishes identically.*

* This covariant has been called by several names—the functional determinant, the functional invariant, and the Jacobian, of the $r+1$ forms and of the linear system determined by them. The last of these names appears to be the best. The Jacobian of a number of *binary* forms can in no way be confused with the functional determinant of p functions of p variables.

In particular a form of rank *one* is of the shape $f(x)\ \phi(y)$; that is, it reduces to a product of two binary forms. Since the first rank covariant G_1 is the transvectant $(F, F)^{1,1}$, we have the result:

THEOREM II.—*The necessary and sufficient condition that a double-binary form may be the product of two binary forms is the identical vanishing of its* (1, 1) *transvectant.*

When the given form F is expressed in the shape

$$F = f_1(x)\,\phi_1(y) + f_2(x)\,\phi_2(y) + \ldots + f_p(x)\,\phi_p(y),$$

the rank covariants assume a useful form. If we consider the equivalent symbols $(a_0 b_0)$, $(a_1 b_1)$, ... in the symbolic expression for G_r to refer to $r+1$ distinct forms $F_0, F_1, \ldots, F_r$, where

$$F_\kappa = f^{(\kappa)}(x)\,\phi^{(\kappa)}(y) \quad (\kappa = 0, 1, \ldots, r),$$

then it is obvious that the expression for G_r reduces to the product of the Jacobian of the $r+1$ forms $f^{(\kappa)}(x)$ and the Jacobian of the $r+1$ forms $\phi^{(\kappa)}(y)$. Hence it follows that the r-th rank covariant of $F = f_1(x)\phi_1(y) + \ldots$ is given by

$$G_r = (r+1)!\,\Sigma\, J_{1,2,\ldots,r+1}(x)\, J'_{1,2,\ldots,r+1}(y),$$

$J_{1,2,\ldots,r+1}$ and $J'_{1,2,\ldots,r+1}$ denoting the Jacobians of $f_1 f_2 \ldots f_{r+1}$ and $\phi_1 \phi_2 \ldots \phi_{r+1}$ respectively. If the rank of F is less than $r+1$, then these Jacobians all vanish identically, causing G_r to vanish.

The foregoing form for G_r shows that, when G_r is a product of binary forms, not more than $r+1$ of the forms $f_\kappa(x)$ can be linearly independent. Hence $G_{r+1} = 0$, and so from Theorem II G_{r+1} is a factor of the (1, 1) transvectant of G_r. Also, when $G_{r-1} = 0$, both G_r and its (1, 1) transvectant vanish. These facts, combined with the orders of the (1, 1) transvectant of G_r, lead one to suspect the result:

THEOREM III.—*The* (1, 1) *transvectant of* G_r *is a numerical multiple of* $G_{r-1} G_{r+1}$.

This theorem (a symbolic proof of which does not appear to be easy) is the *first* of our extensions of a familiar property of the (2, 2) form, and is proved later on in a simple geometrical manner. Accepting its truth for

the present; we see that

(1) $G_{r-1} = 0$ implies $G_r = 0$.

(2) $G_r = 0$ implies *either* $G_{r-1} = 0$ *or* G_{r-1} is a product of binary forms.

3. *Connexion with binary theory.*

If the binary form $a_{0_x}^{m+n} = a_{1_x}^{m+n} = \ldots$ possesses an apolar r-ic $f_x^r (r < m+n)$, then all its $(m+n-r)$-th polars are apolar to f_x^r, and therefore it is directly evident that the rank of the double-binary form $a_{0_x}^{m+n-r} a_{0_y}^r$ is less than $r+1$. The condition for this is the identical vanishing of the r-th rank covariant G_r, which for this form reduces to

$$(a_0 a_1)^2 (a_0 a_2)^2 \ldots (a_{r-1} a_r)^2 a_{0_x}^{m+n-2r} \ldots a_{r_x}^{m+n-2r}.$$

This is the known covariant,* the vanishing of which is the condition for the existence of an apolar r-ic of the binary form $a_{0_x}^{m+n} = a_{1_x}^{m+n} = \ldots$. Thus the rank theory of the double-binary form may be regarded from the symbolic point of view as an extension of the theory of binary forms apolar to a given binary form. On this view Theorem II would be the extension of the binary theorem, that the identical vanishing of the Hessian is the necessary and sufficient condition for a form to be a power of a linear form.

4. *The reciprocity of the rank covariants for equiform.*

The fact that the $(1, 1)$ transvectant of the $(1, 1)$ transvectant of a $(2, 2)$ form $a_x^2 b_y^2$ reduces to a multiple of $a_x^2 b_y^2$ has been generalized in one direction in Theorem III; it can also be generalized in a different direction leading to a certain reciprocity between the rank covariants of a general (n, n) form.

Let the (n, n) form F, supposed to be of rank $n+1$, be written in the shape :

$$F(x, y) = f_0(x)\,\phi_0(y) + f_1(x)\,\phi_1(y) + \ldots + f_n(x)\,\phi_n(y).$$

The r-th rank covariant G_r of F has been already shown to be the product-sum of the Jacobians, $r+1$ at a time, of the binary forms f and ϕ. In

* Grace and Young, *Algebra of Invariants*, 232.

particular, if we define the forms $F_\kappa(x)$, $\Phi_\kappa(y)$ by

$$F_\kappa(x) = (-1)^\kappa J_{01\ldots(\kappa-1)(\kappa+1)\ldots n}(x),$$

$$\Phi_\kappa(y) = (-1)^\kappa J'_{01\ldots(\kappa-1)(\kappa+1)\ldots n}(y)$$

(where J and J' with subscripts denote as usual the Jacobians of the corresponding binary forms f and ϕ respectively), then the $(n-1)$-th rank covariant of F is given by

$$\frac{1}{n!} G_{n-1} = F_0(x)\Phi_0(y) + F_1(x)\Phi_1(y) + \ldots + F_n(x)\Phi_n(y).$$

Apolar sets of binary forms.

The set of $n+1$ binary n-ics $(F_0, F_1, \ldots, F_n)$ derived as above from the set $(f_0, f_1, \ldots, f_n)$ (that is, F_κ is the Jacobian with the suitable sign of all the f's except f_κ) may be called the *apolar set* of the latter. These sets stand in very important mutual relationships. Any form F_κ is from its definition apolar to all the forms f except f_κ. Also, by using the symbolic expression for the forms of the apolar set, it becomes obvious that the apolar invariant

$$(f_\kappa, F_\kappa)^n = \text{the invariant } J_{01\ldots n}$$

$$= j \quad \text{(say)},$$

for any κ.

If we take $$X_r = f_r(x) \quad (r = 0, 1, \ldots, n)$$

as the point-equations of a rational norm curve R_n in space of n dimensions, then the coordinates* of the t-dimensional osculating region at the point x of the curve would be the Jacobians, $t+1$ at a time, of the forms $f_0(x), f_1(x), \ldots, f_n(x)$. In particular, the coordinates of the $(n-1)$-dimensional region, which osculates the curve at the point x, would be proportional to the forms $F_0(x), F_1(x), \ldots, F_n(x)$ which constitute the apolar set. Thus the *dual* (or "plane") equations of the norm curve are of the form :

$$U_r = F_r(x) \quad (r = 0, 1, \ldots, n).$$

From this connexion of apolar sets with geometrical duality we shall now deduce two important consequences (A) and (B).

As our first consequence, consider the t-dimensional osculating region at the point x of the norm curve. The *point*-coordinates of this region

* For properties of region-coordinates, reference may be made to Bertini, *Geometria Proiettiva degli iperspazi* (1923, 2 ed.), 44, *et seq.*

are, as already remarked, the Jacobians of the forms $f_0(x), f_1(x), \ldots, f_n(x)$ $(t+1)$ at a time; the *dual* coordinates of the same region are similarly the Jacobians of the forms $F_0(x), F_1(x), \ldots, F_n(x)$ $(n-t)$ at a time. From the known relation between *point* and *dual* region-coordinates, we have :

The Jacobians of the type $J_{01\ldots t}(x)$ *of the forms* $f_0, f_1, \ldots, f_n$, $t+1$ *at a time, differ only by a universal constant multiplier from the Jacobians of the type* $J'_{(t+1)(t+2)\ldots n}(x)$ *of the forms* $F_0, F_1, \ldots, F_n$, $n-t$ *at a time.* (A)

From consideration of dimensions it is seen that the constant multiplier is a numerical multiple of $1/j^{n-t-1}$; the actual form of the numerical factor is complicated and is quite irrelevant for our purpose.

As our second consequence, consider the equibinary form

$$P_\kappa(x, y) = \Sigma J_{01\ldots\kappa}(x)\, J_{(\kappa+1)(\kappa+2)\ldots n}(y),$$

which is of order $(n-\kappa)(\kappa+1)$ in each of the variables. It is clear that $P_\kappa(x, y)$ is the expression for the lineo-linear invariant of the κ-dimensional osculating region at the point x and the $(n-\kappa-1)$-dimensional osculating region at the point y of the curve; thus the vanishing of P_κ is the condition that these osculating regions may intersect. But it is a peculiar property of the norm curve that the κ-dimensional osculating region at the point x and the $(n-\kappa-1)$-dimensional osculating region at the point y *never* intersect, unless x and y coincide. Hence P_κ can only differ by a constant multiplier—easily identified as the invariant j—from $(xy)^{(n-\kappa)(\kappa+1)}$. We have thus established the identity

$$P_\kappa(x, y) = \Sigma J_{01\ldots\kappa}(x)\, J_{(\kappa+1)(\kappa+2)\ldots n}(y) \equiv j(xy)^{(n-\kappa)(\kappa+1)}. \qquad \text{(B)}$$

When $\kappa = 0$ this identity takes the form

$$P_0(x, y) = f_0(x)\, F_0(y) + f_1(x)\, F_1(y) + \ldots + f_n(x)\, F_n(y) \equiv j(xy)^n. \qquad (B_1)$$

The identity (B_1) is the fundamental identity of apolar sets. It is the only case of identity (B) which we shall require here.

We are now in a position to establish the reciprocity of the rank covariants of a general equibinary (n, n) form F, of rank $n+1$. Writing F in the form

$$F = f_0(x)\, \phi_0(y) + f_1(x)\, \phi_1(y) + \ldots + f_n(x)\, \phi_n(y),$$

we have seen that its penultimate rank covariant G_{n-1} is given by

$$\frac{1}{n!}\,G_{n-1} = F_0(x)\,\Phi_0(y)+F_1(x)\,\Phi_1(y)+\ldots+F_n(x)\,\Phi_n(y),$$

the sets of forms $(F_0 \ldots F_n)$, $(\Phi_0 \ldots \Phi_n)$ being the respective apolar sets of $(f_0, f_1, \ldots, f_n)$, $(\phi_0, \phi_1, \ldots, \phi_n)$. By taking account of (A) above, it follows that the κ-th rank covariant of F, which is $(\kappa+1)!$ times the product-sum of the Jacobians of the forms f and ϕ, $\kappa+1$ at a time, can only differ by a constant factor λ_κ from the $(n-\kappa-1)$-th rank covariant of G_{n-1}, which is $(n-\kappa)!\,(n!)^{n-\kappa}$ times the product-sum of the Jacobians of the forms of the apolar sets, $n-\kappa$ at a time. Here λ_κ is a numerical multiple of

$$\frac{1}{j^{n-\kappa-1}j'^{n-\kappa-1}} = \frac{1}{G_n^{n-\kappa-1}} \times \{(n+1)!\}^{n-\kappa-1},$$

j and j' being the Jacobians of all the forms f and ϕ respectively. We therefore have the result :

THEOREM IV.—*The $(n-\kappa-1)$-th rank covariant of G_{n-1} is a numerical multiple of $G_n^{n-\kappa-1}G_\kappa$.*

In particular, the $(n-1)$-th rank covariant of G_{n-1} is a multiple of F itself. The simplest case of this reciprocity occurs, of course, for the $(2, 2)$ form F, for which we have

$$G_1 = (F, F)^{1,1}; \qquad (G_1, G_1)^{1,1} = \tfrac{1}{3}G_2F.$$

A double-binary form $F(x, y)$, on being equated to zero, defines a correspondence between the binary fields (x), (y), which may also be denoted by the same symbol F. Calling G_κ *the κ-th rank correspondence* of the correspondence F, Theorem IV can be stated as follows :—

The κ-th rank correspondence of F is also the $(n-\kappa-1)$-th rank correspondence of G_{n-1}.

Another important relation between F and G_{n-1} may be deduced from the fundamental identity (B_1). Considering F and G_{n-1} to be binary forms in x, their apolar invariant is

$$n!\,\Sigma\,(f_\kappa, F_\kappa)^n\,\phi_\kappa(y)\,\Phi_\kappa(y) = n!\,j\,\Sigma\,\phi_\kappa(y)\,\Phi_\kappa(y),$$

which vanishes in virtue of identity (B_1). We have therefore :

THEOREM V.—*A double-binary equiform and its penultimate rank covariant are identically apolar when considered as binary forms in either of the variables.**

This theorem is the extension of the known identical binary apolarity of a (2, 2) form and its (1, 1) transvectant.†

5. *Geometrical representation of rank covariants.*

Let C'_m, C_n be two rational norm curves, the former of which is situated in a space S_m of m dimensions, and the latter in a subspace S_n of S_m; and let them be considered to be carriers of the respective binary variables x, y. Let now Γ be a general correlation in S_m, involving $(m+1)^2$ constants. The S_{m-1} which Γ makes correspond to any point x of C'_m cuts C_n in n points. Also, the locus of points whose corresponding S_{m-1}'s pass through an assigned point is itself an S_{m-1} which cuts C'_m in m points. Thus Γ institutes an (m, n) correspondence F between the curves C'_m, C_n. To show that F is a general (m, n) correspondence, let Γ' be the general correlation [involving $(m-n)(m+1)$ constants], in which the S_{m-1} corresponding to *every* point contains S_n. Then the correlations Γ and $\Gamma+\Gamma'$ are such that the two S_{m-1}'s which they make correspond to the same point cut S_n in the same S_{n-1}. It is therefore clear that $\Gamma+\Gamma'$ is the most general correlation which produces the same (m, n) correspondence F as the correlation Γ. Hence the number of arbitrary constants in F is

$$(m+1)^2-(m-n)(m+1) = (m+1)(n+1),$$

which shows that F is a general (m, n) correspondence.

Now as the point x traces C'_m, the S_{m-1} corresponding to it in Γ osculates another norm curve C_m at a point whose parameter on C_m may also be supposed to be x. Since C'_m and C_n are in no special relation towards one another, neither are C_m and C_n. It follows, therefore, that the general form $F = a_x^m b_y^n$ $(m \geqslant n)$, of rank $n+1$, can be represented as a correspondence F of a special type between two suitably chosen norm curves C_m, C_n, namely, that in which any point x on C_m corresponds to those points y in which the osculating S_{m-1} at x cuts C_n.

* The apolarity of this theorem is *binary* apolarity, and is not to be confused with *double-binary* apolarity. Thus F and G_{n-1} are *not* apolar as double-binary forms, their double-binary apolar invariant being in fact equal to G_n.

† Coolidge, *Treatise on the Circle and Sphere* (1916), 210; also Kasner, cited hereafter.

This representation of $F = a_x^m b_y^n$ has the advantage that the rank covariants of F also receive a simple interpretation therein. For, from the symbolic derivation of the covariant G_r of F, it is seen that G_r is the binary Jacobian of the following forms in y :—

$$a_x^m a_y^n,\ a_{x_1}^m b_y^n,\ \ldots,\ a_{x_r}^m b_y^n,$$

where $x_1, x_2, \ldots, x_r$ differ infinitesimally from x. Hence the points y, which correspond in G_r to a given point x on C_m, are the roots of the Jacobian of the $r+1$ y-groups, which correspond in F to $r+1$ points infinitely near x. From the correspondence-representation adopted for F, it follows that these y-groups are cut out of C_n by S_{m-1}'s passing through the osculating S_{m-1-r} at x; therefore the Jacobian of these y-groups corresponds to those points y on C_n, the osculating S_r's at which intersect the osculating S_{m-1-r} at x. Thus the r-th rank correspondence defined by G_r is that in which corresponding points x, y on C_m, C_n are such that the osculating S_{m-1-r} at x and the osculating S_r at y meet each other. It is important to notice that, as thus represented, the given correspondence F, as well as its successive rank correspondences G_r constitute the fundamental class of correspondences which can be defined for two curves C_m, C_n—namely, those defined by the incidence of dual osculating elements of C_m, C_n.

This representation also shows the relation of the rank covariants to the rank of the form F. For instance, if $G_n = 0$, the *correspondence* G_n must become indeterminate both ways, and therefore the osculating S_{m-1-n} at any point x of C_m must meet the osculating S_n at any point y of C_n. But this osculating S_n is simply the S_n containing C_n, and it cannot meet an infinite number of osculating S_{m-1-n}'s of C_m, unless C_n is contained in a space of $n-1$ dimensions. When this happens, it is easy to see that the form F suffers a reduction in rank from $n+1$ to n. A similar argument holds for the case $G_r = 0$.

Proof of Theorem III.

When the *point* is the fundamental space-element (so that the *line*, *plane*, &c., are secondary or *derived* space-elements), a curve in S_m can be regarded not only as the set of its ∞^1 points, but also as the set of its ∞^1 derived osculating elements—*tangent-lines* or *osculating planes*, &c., or osculating S_{m-2}'s (which we may call the *penultimate osculating*

elements), or lastly, the osculating S_{m-1}'s (the *ultimate osculating elements*).

If we are given two curves in S_m (the first of which is not contained in any space of lower dimensions), the correspondence between them, which is defined by the incidence of the ultimate osculating elements of the first curve and the points of the second, may be called *the fundamental incidence-correspondence of the curves.* Similarly the correspondence defined by the incidence of the penultimate osculating elements of the first curve and the tangent lines of the second may be called *the first derived incidence-correspondence;* and so on.

The real content of our geometrical representation may now be stated compactly : *a double-binary form and its successive rank covariants can be simultaneously represented as the fundamental and the successive derived incidence-correspondences of two rational norm curves* C_m, C_n.

The coordinates of the osculating region of κ dimensions of the rational norm curve C_m are polynomials of degree $(m-\kappa)(\kappa+1)$ in the parameter. Since the linear manifold which contains the manifold of the S_κ's in a given S_m, is of dimensions $\binom{m+1}{\kappa+1}-1 \geqslant (m-\kappa)(\kappa+1)$, it follows that *when we take the region of* κ *dimensions instead of the point, as our fundamental space-element, the norm curve* C_m *still continues to be a norm curve* $C_{(m-\kappa)(\kappa+1)}$, *of order* $(m-\kappa)(\kappa+1)$. Hence, *the ultimate osculating element* of the curve $C_{(m-\kappa)(\kappa+1)}(=C_m)$ is the aggregate of the S_κ's which are linearly related to $(m-\kappa)(\kappa+1)$ consecutive osculating S_κ's of the curve C_m. Now, as was already mentioned in the previous section, all the $(m-\kappa)(\kappa+1)$ osculating S_κ's of C_m which meet the osculating $S_{m-1-\kappa}$ at a point x of C_m coalesce with the osculating S_κ at x. Hence all S_κ's which are linearly related to the osculating S_κ's at $(m-\kappa)(\kappa+1)$ points infinitely near x must meet the osculating $S_{m-1-\kappa}$ at x. It is not, however, true conversely, that every S_κ which meets the osculating $S_{m-1-\kappa}$ at x is linearly dependent on $(m-\kappa)(\kappa+1)$ osculating S_κ's at points consecutive to x. But, on the other hand, it is seen that these $(m-\kappa)(\kappa+1)$ consecutive osculating S_κ's have no transversal $S_{m-1-\kappa}$ other than the osculating $S_{m-1-\kappa}$ at x. Thus, the family of S_κ's which meet the $S_{m-1-\kappa}$ is a linear family which is uniquely representative of all osculating S_κ's at points in the immediate neighbourhood of x. We can therefore appropriately consider that

> *the ultimate osculating elements of* $C_{(m-\kappa)(\kappa+1)}$ (*which is simply the norm curve* C_m *considered as the aggregate of its osculating* S_κ's) *are the families of* S_κ's *incident with the osculating* $S_{m-1-\kappa}$'s *of* C_m...(C).

We saw that when a form F was represented as the fundamental incidence-correspondence between C_m, C_n, the rank covariant G_κ was represented as the correspondence defined by the incidence of the osculating $S_{m-1-\kappa}$'s of C_m and the osculating S_κ's of C_n. Thus by agreeing to the above view (C) of the ultimate osculating elements of $C_{(m-\kappa)(\kappa+1)}$, we can state that: *just as F is the fundamental incidence-correspondence between C_m, C_n, so G_κ is the fundamental incidence-correspondence between $C_{(m-\kappa)(\kappa+1)}$ and $C_{(n-\kappa)(\kappa+1)}$ (which are simply the curves C_m, C_n regarded as the aggregate of their osculating S_κ's).* The importance of this result is, that by conceiving G_κ not as a *derived*, but as a *fundamental* incidence-correspondence, we are enabled to construct its first derived incidence-correspondence, or in other words to interpret the (1, 1) transvectant of the *form* G_κ.

The first derived incidence-correspondence of G_κ is defined by the incidence of the penultimate osculating elements of $C_{(m-\kappa)(\kappa+1)}$ and the "tangent-lines" (or rather what corresponds to them in S_κ-geometry) of $C_{(n-\kappa)(\kappa+1)}$. The penultimate osculating element of $C_{(m-\kappa)(\kappa+1)}$ being the intersection of two consecutive ultimate osculating elements, consists of all the S_κ's which meet two consecutive osculating $S_{m-1-\kappa}$'s of C_m—say at the points x and $x+dx$. Since these osculating $S_{m-1-\kappa}$'s intersect in the osculating $S_{m-2-\kappa}$ and are contained in the osculating $S_{m-\kappa}$ at x, the S_κ's which meet them must either meet the osculating $S_{m-2-\kappa}$ at x in a point or the osculating $S_{m-\kappa}$ at x in a line. Thus the penultimate osculating element of $C_{(m-\kappa)(\kappa+1)}$ corresponding to any point x of C_m splits into two families $(x)_1$ and $(x)_2$, the former consisting of the S_κ's meeting the osculating $S_{m-2-\kappa}$ at x, the latter of those which meet the osculating $S_{m-\kappa}$ in a line.* In the same manner what corresponds to the "tangent-line" of $C_{(n-\kappa)(\kappa+1)}$ is simply the set of S_κ's which contain the osculating $S_{\kappa-1}$ and are contained in the osculating $S_{\kappa+1}$, at a point y of C_n; let this set of S_κ's be denoted by (y).

Thus the first rank correspondence of G_κ reduces to the *sum* of two correspondences, defined respectively by the incidence of the families $\{(x)_1, (y)\}$ and $\{(x)_2, (y)\}$. Now it is evident that the families $(x)_1$ and (y) have a common member only if the osculating $S_{m-2-\kappa}$ at x and the osculating $S_{\kappa+1}$ at y meet each other—in other words, only if x and y correspond in $G_{\kappa+1}$. Similarly $(x)_2$ and (y) have a common member only if the osculating $S_{m-\kappa}$ at x meets the osculating $S_{\kappa-1}$ at y—in other words,

* This splitting up of the osculating element is made possible only by the fact that the manifold of the S_κ's is not a linear manifold. It will be noticed that this splitting up shows in advance that the (1, 1) transvectant of G_κ splits into factors.

only if x and y correspond in $G_{\kappa-1}$. We have thus finally the result *that the* (1, 1) *transvectant of* G_κ *can differ only by a numerical multiple from* $G_{\kappa+1}G_{\kappa-1}$.

The statement (C) is in no way essential to this proof; it has been introduced only for keeping up the analogy.

The Equiform. Reciprocity of the rank covariants.

The (n, n) form F would be represented in the above manner as the fundamental incidence-correspondence between two norm curves C_n, C'_n in S_n—that is, as the correspondence between a point x on C_n and the points y in which the osculating S_{n-1} at x cuts C'_n. The rank covariant G_{n-1} will then be represented as the correspondence between a point x on C_n and the points of contact y in which the osculating S_{n-1}'s from x to C'_n—that is, as the fundamental incidence-correspondence between C'_nC_n. This dual symmetry of F and G_{n-1} shows that, if we attempt to form the successive rank correspondences of G_{n-1}, we would only be retracing in reverse order the rank correspondences of F. The reciprocity of the rank covariants is thus geometrically established.

The identical binary apolarity of F *and* G_{n-1} is also put in evidence by our representation. For the y-group which corresponds to x in F is cut out of C'_n by an S_{n-1} through x, and the y-group corresponding to the same x in G_{n-1} is cut out of C'_n by the polar S_{n-1} of x with respect to C'_n. The apolarity of these y-groups is a well-known fundamental property of the norm curve C'_n.

A special case worthy of mention is that of the (2, 2) form F. This can be represented as a (2, 2) correspondence between two conics C_2, C'_2 in the same plane, in which any point x on C_2 corresponds to the two points y in which the tangent at x meets C'_2. The (1, 1) transvectant G_1 (which is the first rank covariant of F) is then represented by the correspondence between a point x on C_2 and the two points of contact y of tangents from x to C_2.*

6. *Canonical Form.*

The shape

$$F = a_x^m b_y^n = f_1(x)\,\phi_1(y) + f_2(x)\,\phi_2(y) + \ldots + f_r(x)\,\phi_r(y).$$

* For the usual geometrical representation of the (2, 2) form in *cogredient* variables, see Clifford, *Math. Papers* (1882), 205.

obtained for the form F of rank r, is not unique. If, in fact, we write

$$f_p(x) = \sum_q \lambda_{pq} f'_q(x),$$

$$\phi_p(y) = \sum_q \mu_{pq} \phi'_q(y) \quad (p, q = 1, 2, \ldots, r),$$

then we would have identically

$$F = \Sigma f_p(x)\, \phi_p(y) \equiv f'_p(x)\, \phi'_p(y),$$

provided the linear substitutions $|\lambda|$, $|\mu|$ are contragredient. Let us now write

$$(f_p, f_q)^m = d_{pq}; \quad (\phi_p, \phi_q)^n = \delta_{pq};$$

$$(f'_p, f'_q)^m = d'_{pq}; \quad (\phi'_p, \phi'_q)^n = \delta'_{pq};$$

If $|\lambda'|$, $|\mu'|$ denote the transposed matrices of $|\lambda|$ and $|\mu|$, we have the matrix equations

$$|d_{pq}| = |\lambda|\,|d'_{pq}|\,|\lambda'|, \tag{1}$$

$$|\delta_{pq}| = |\mu|\,|\delta'_{pq}|\,|\mu'|.$$

Since we suppose $|\lambda|$, $|\mu|$ to be contragredient, $|\lambda| = |\mu'|^{-1}$ and the second of these equations may be written

$$|\delta_{pq}|^{-1} = |\lambda|\,|\delta'_{pq}|^{-1}\,|\lambda'|. \tag{2}$$

If we now suppose that both the partial orders m, n are even, then the matrices $|d|$, $|\delta|$, $|d'|$, $|\delta'|$, as well as their inverses, are all symmetric. By comparing equations (1), (2) with the known theory of reduction of two quadratic forms, we conclude in this case that it is in general possible to choose $|\lambda|$ uniquely, in such wise that $|d'|$ as well as $|\delta'|^{-1}$ (and therefore also $|\delta'|$) are in normal shape (that is to say, have vanishing non-diagonal elements). If $|d'|$ and $|\delta'|$ are both in normal shape, then in the transformed shape of F the forms f' as well as the forms ϕ' will be mutually apolar. Hence we have :

THEOREM VI.—*A double-binary form F of rank r, whose partial orders m, n are both even, can in general be reduced to the shape :*

$$F = f_1(x)\, \phi_1(y) + f_2(x)\, \phi_2(y) + \ldots + f_r(x)\, \phi_r(y),$$

where the forms f are mutually apolar m-ics, and the forms ϕ mutually apolar n-ics; and this reduction is in general unique in the sense that the only other sets with this property are permutations of the set $(f'\ \phi')$ given by

$$f'_\kappa = \lambda_\kappa f_\kappa; \qquad \phi_\kappa = (1/\lambda_\kappa)\,\phi_\kappa \quad (\kappa = 1, 2, \ldots, r).$$

The *actual* possibility of this reduction in any particular case, as well as its uniqueness when possible, will, of course, depend on the invariant-factors of $|d|+\lambda|\delta|^{-1}$.

If one or both of the partial orders m, n are odd, then the reduction of F on the above lines will depend on the simultaneous reduction of two bilinear forms, which are either symmetric and skew-symmetric, or both skew-symmetric. There is, however, no simple type of general canonical reduction for these cases suitable to our purpose.

As a special instance, suppose $m = n = 3$. Then $|d|$ and $|\delta|^{-1}$ are skew-symmetric matrices of the fourth order, and may be considered to represent two linear complexes in space. These complexes have in general a unique pair of common polar lines; by choosing these as edges of the tetrahedron of reference, $|d|$ and $|\delta|$ can be reduced simultaneously to the shape in which

$$d_{13} = d_{14} = d_{23} = d_{24} = 0,$$

$$\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0.$$

Hence the (3, 3) form can be reduced (but not uniquely) to the shape

$$f_1(x)\,\phi_1(y)+f_2(x)\,\phi_2(y)+f_3(x)\,\phi_3(y)+f_4(x)\,\phi_4(y),$$

where each of f_1, f_2 is apolar to each of f_3, f_4, and each of ϕ_1, ϕ_2 apolar to each of ϕ_3, ϕ_4.

As a second instance, consider a $(3, 2n)$ form of rank $4(n > 1)$. The matrices $|d|$ and $|\delta|^{-1}$ will now be skew-symmetric and symmetric matrices of order 4, and therefore correspond to a linear complex and a quadric in space. The complex will contain two pairs of generators of the quadric, and by choosing these as edges of the tetrahedron of reference, $|d|$ and $|\delta|$ could be brought to the shape in which

$$d_{13} = d_{14} = d_{23} = d_{24} = 0,$$

$$\delta_{11} = \delta_{22} = \delta_{33} = \delta_{44} = \delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0.$$

Hence the $(3, 2n)$ form can be reduced *uniquely* to the shape

$$f_1(x)\,\phi_1(y)+f_2(x)\,\phi_2(y)+f_3(x)\,\phi_3(y)+f_4(x)\,\phi_4(y)$$

in which

(1) each of the cubics f_1, f_2 is apolar to each of f_3, f_4;

(2) each of the $2n$-ics ϕ_1, ϕ_2, ϕ_3, ϕ_4 is apolar to itself;

(3) in addition, each of ϕ_1, ϕ_2 is apolar to each of ϕ_3, ϕ_4 $(n > 1)$.

These reductions for the $(3, 3)$ and the $(3, 2n)$ forms are, of course, equally valid for the $(2m+1, 2n+1)$ form of rank 4, and the $(2m+1, 2n)$ form of rank 4, respectively.

7. *The canonical shape for the* $(2, 2)$ *form.*

From Theorem VI of the last section, the general $(2, 2)$ form of rank three can be put into the shape

$$F = a_x^2 b_y^2 = f_1'(x)\,\phi_1'(y)+f_2'(x)\,\phi_2'(y)+f_3'(x)\,\phi_3'(y),$$

where $(f_1' f_2' f_3')$ and $(\phi_1' \phi_2' \phi_3')$ are mutually apolar triads of quadratics. Now none of the quadratics f' (or ϕ') can be a perfect square; for $f_1' f_2' f_3'$ which form a mutually apolar triad would thereby obtain a common factor, which would reduce the rank of the form. Hence we can replace the quadratics f', ϕ' by numerical multiples of themselves with unit discriminants, and write:

$$F = \lambda_1 f_1(x)\,\phi_1(y)+\lambda_2 f_2(x)\,\phi_2(y)+\lambda_3 f_3(x)\,\phi_3(y) \qquad \text{(D)}$$

where

$$(f_r, f_s)^2 = (\phi_r, \phi_s)^2 = 0,$$

$$(f_r, f_r)^2 = (\phi_r, \phi_r)^2 = 1 \quad (r \neq s;\ r, s = 1, 2, 3).$$

From the fact that f, ϕ are mutually apolar triads of quadratics with *unit* discriminants, we have the further relations:

$$f_1^2+f_2^2+f_3^2 = \phi_1^2+\phi_2^2+\phi_3^2 \equiv 0,$$

$$(f_2, f_3)^1 = \delta f_1, \text{ etc.}, \qquad (\phi_2, \phi_3)^1 = \delta' \phi_1, \text{ etc.},$$

where δ, δ' are the determinants of the triads f, ϕ, and are given by :

$$2\delta^2 = 2\delta'^2 = 1.$$

We may clearly suppose that δ and δ' have the same sign (if they have not, we could change the sign of one of the quadratics f, ϕ), so that

$$\delta = \delta' = \pm \frac{1}{\sqrt{2}}.$$

The form (D) is very convenient for the study of (2, 2) form, and has the advantage of not being too specialized. The two discriminants of the form are :*

$$D_1(x) = (F, F)^{0,2} = \lambda_1^2 f_1^2 + \lambda_2^2 f_2^2 + \lambda_3^2 f_3^2,$$

$$D_2(y) = (F, F)^{2,0} = \lambda_1^2 \phi_1^2 + \lambda_2^2 \phi_2^2 + \lambda_3^2 \phi_3^2,$$

and their linear equivalence is apparent. The principal equibinary concomitants of F are :

$$G = (F, F)^{1,1} = 2\delta\delta' \Sigma \lambda_2 \lambda_3 f_1 \phi_1 = \Sigma \lambda_2 \lambda_3 f_1 \phi_1,$$

$$-2I = (F, F)^{2,2} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$

$$3J = (F, G)^{2,2} = 3\lambda_1 \lambda_2 \lambda_3,$$

$$H = -(I/2)F - (F, G)^{1,1} = \tfrac{1}{4} \Sigma \lambda_1 (\lambda_1^2 - \lambda_2^2 - \lambda_3^2) f_1 \phi_1,$$

$$4K = (F, H)^{2,2} = \tfrac{1}{4}(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\lambda_2^2\lambda_3^2 - 2\lambda_3^2\lambda_1^2 - 2\lambda_1^2\lambda_2^2).$$

$3J$ is the second rank covariant the vanishing of which is the condition that F may be of rank two. The (1, 1) transvectant of G is seen on inspection to be JF. The equi-transvectants of F, G, H may be easily found from the above expressions. The two discriminants or *branch quartics* of H are seen to be $KD_1(x)$ and $KD_2(y)$.

If F is interpreted geometrically as a cyclic, the condition that it is a pair of circles is that both its branch quartics should be perfect squares,†

* The nomenclature used for the concomitants is the same as in Kasner, "The invariant theory of the inversion group", *Trans. Amer. Math. Soc.*, 1 (1900), 430–498.

† Kasner, *loc. cit.*

that is,

$$(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)=0.$$

The condition that it may be a repeated circle is the identical vanishing of both the branch quartics $D_1(x)$, $D_2(y)$; that is,

$$\lambda_1^2=\lambda_2^2=\lambda_3^2.$$

The covariant system of cyclics $aF+bG+cH$ is thus seen to contain *four* repeated circles, namely the director circles of F, and also of every cyclic of the system. To study this system, and also the *confocal* system of cyclics, we may represent $F=\lambda_1 f_1\phi_1+\lambda_2 f_2\phi_2+\lambda_3 f_3\phi_3$ by the point F, whose trilinear coordinates with respect to a fundamental triangle ABC are $\lambda_1\lambda_2\lambda_3$. In this representation the points ABC will correspond to the forms of rank *one* (namely $f_1\phi_1$, $f_2\phi_2$, $f_3\phi_3$) in the co-director system, and points on the sides of ABC to forms of rank two. The in- and ex-centres of ABC, P, P_1, P_2, P_3 will correspond to the four repeated circles—namely, the common director circles—and the points on the internal and external bisectors of the angles A, B, C will correspond to pairs of circles. The points F, G are seen to form a pair of the Steinerian involution determined by the quadrangle $PP_1P_2P_3$; that is, they are *isogonal conjugates* with respect to the triangle ABC.

Since any line contains a unique pair of isogonal conjugates, we have :

*Any pencil of cyclics of a co-director system contains a unique pair of cyclics F, G, such that $G=\rho(F, F)^{1,1}$, $F=\rho'(G, G)^{1,1}$; it also contains six pairs of circles which belong to the coaxal systems determined by every two of the director circles.**

A confocal system of cyclics (that is, a system every member of which has the same branch quartics) corresponds to a conic (rectangular hyperbola) through $PP_1P_2P_3$; the confocal system of F corresponds to the conic $PP_1P_2P_3F$. Since F, G are isogonal conjugates, FG is the tangent at F to this conic. The point H may be constructed either as the point of contact of the *other* tangent from G to this conic, or as the point which corresponds to the third polar of F with respect to the quadruple $PP_1P_2P_3$ on the conic.

* That the pencil contains six pairs of circles appears to have been first noticed by Cayley. For this theorem and generally in regard to the contents of this section, reference may be made to Turnbull, "Geometrical Interpretation of the Complete System of the (2, 2) Form", *Proc. Royal Soc. Edinburgh*, 44 (1924), 23-50.

[*Added November* 1924.—The last non-vanishing rank covariant of a (m, n) form $F = f_1(x)\,\phi_1(y) + \ldots + f_r(x)\,\phi_r(y)$, of rank r, is in the usual notation $J_{1,2,\ldots,r}(x)J_{1,2,\ldots,r}(y)$. Thus this last rank covariant is a product of binary forms; it reduces, however, to a constant if $m = n = r-1$, and to a binary form if $m \neq n$, and the form F has its maximum rank.

In particular, if $m = n+1$, and $r = n+1$, the last rank covariant G_n is a multiple of the binary $(n+1)$-ic $J_{1,2,\ldots,n+1}(x)$. Since the Jacobian of $n+1$ binary $(n+1)$-ics is simply their apolar $(n+1)$-ic, it follows that, in this case, *F is identically apolar to G_n*. This fact has interesting geometrical applications when x, y are cogredient.

Among the double-binary forms, those of rank *two* (which, utilizing a term of Sylvester, may be called *sub-rational forms*) have many special features. It has been shown, for instance, in a recent paper, that the symmetric, and the alternating, double-binary forms of rank two, which factorize completely into bilinear factors, are related to the Polyhedral Groups.*

The incidence-correspondence of two twisted cubics in space, together with a symbolic expression for some of its invariants, occur in a different connexion in a recent paper of Coble.†]

* A. Narasinga Rao, "On the Homographic Break-up of Certain Derived Loci", *Journal of Ind. Math. Soc.*, 15 (1924), 104–112.

† Coble, "On the Geometric Aspects of Modular Functions of Genus 4", *Amer. J. of Math.*, 46 (1924), 143–192.

APPLICATIONS OF THE ABSOLUTE DIFFERENTIAL CALCULUS TO THE THEORY OF ELASTICITY

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Introduction.

Ricci and Levi-Civita* have given a suggestive outline of the application of the Absolute Calculus to the theory of elasticity. The objects of the present paper are :—

(1) To establish the foundations of a theory of elasticity in Riemannian space—we shall limit ourselves to three dimensions.

(2) To obtain a geometrical representation of the elastic properties at a point in the form of two infinitesimal surfaces and two invariants of position. A geometrical representation for Euclidean space has been given by Voigt,† but his method is not only less general, but also considerably more difficult than that given here.

(3) To apply this geometrical representation directly to determine the strain-energy function in the case of isotropy, and to show that the satisfaction of Cauchy's relations is a consequence of an additional scalar symmetry.

(4) To treat isotropy and transverse-isotropy directly by tensorial methods.

Although the generalization to Riemannian space is made in the present paper, the methods employed nevertheless provide the most symmetrical and suggestive mode of approach in the particular case of Euclidean space. For example, the relations consequent on isotropy have

* *Math. Annalen*, 54 (1900), 200.

† *Lehrbuch der Kristallphysik* (1910), 578.

hitherto only been reached by a consideration of rotations about a pair of axes.*

1. In the extension into Riemannian space of a theory already established in Euclidean space, two conditions must be satisfied : (*a*) the laws must be expressed in invariant form, and (*b*) the laws must reduce to their classical form when the space degenerates into Euclidean space. These conditions are clearly not sufficient to define a unique extension of the theory, as it would be possible arbitrarily to introduce the curvature-tensors which vanish in Euclidean space. We must therefore add the condition that the extended theory should be as simple as possible in form.

Let a system of particles occupying a Riemannian space with line-element $ds^2 = g_{ij}\,dx^i\,dx^j$ receive infinitesimal displacements $dx^i = u^i$. Forming from this contravariant vector a covariant vector u_i, and from this latter by covariant derivation the tensor u_{ij}, we shall define the *strain-tensor* to be

$$e_{ij} = \tfrac{1}{2}(u_{ij} + u_{ji}). \tag{1.1}$$

From this point the development is formally similar to that of Ricci and Levi-Civita. The *strain-energy* or *potential function* W is defined by

$$2W = c^{ijkl}\,e_{ij}\,e_{kl}, \tag{1.2}$$

where c^{ijkl} is a contravariant tensor as indicated, W being therefore an invariant. The tensor c^{ijkl} is the *elasticity-tensor*, and its components the *coefficients of elasticity*. There is obviously no loss of generality in writing

$$c^{ijkl} = c^{jikl} = c^{klij}. \tag{1.3}$$

We define

$$T^{ij} = \frac{\partial W}{\partial e_{ij}} = c^{ijkl}\,e_{kl}, \tag{1.4}$$

to be the *stress-tensor*. Solving (1.4) for the components of the strain-tensor in terms of those of the stress-tensor, we obtain relations of the form

$$e_{ij} = s_{ijkl}\,T^{kl}, \tag{1.5}$$

* Cf. Love, *Mathematical Theory of Elasticity* (1916), 152.

and define s_{ijkl} to be the *modulus-tensor*, and its components the *moduli of elasticity*. The equations of equilibrium have been given* in the form

$$g_{jk}T^{ijk}+\rho P^i = 0, \tag{1.6}$$

where ρ is the density, T^{ijk} the contravariant derivative of T^{ij}, and P^i the contravariant force-vector.

We may here further define the equations of motion to be

$$\frac{\partial^2 u^i}{\partial t^2} = \frac{1}{\rho} g_{jk}T^{ijk}+P^i, \tag{1.7}$$

if the existence of an invariant (Newtonian) time is to be assumed.

2. If $c_{ij} = g^{kl}c_{ijkl}$, the equations

$$c_{ijkl}dx^i dx^j dx^k dx^l = \text{constant}, \tag{2.1}$$

$$c_{ij}dx^i dx^j = \text{constant}, \tag{2.2}$$

define invariant infinitesimal surfaces, which we shall call the *elasticity-quartic* and the *elasticity-quadric* respectively. The quantities

$$I_1 = g^{ij}g^{kl}c_{ijkl}, \tag{2.3}$$

$$I_2 = g^{ik}g^{jl}c_{ijkl}, \tag{2.4}$$

are invariants; we shall call them the *first* and *second elasticity invariants*. The quartic contains fifteen coefficients, and thus a knowledge of its form provides us with fourteen linear equations in the coefficients of elasticity; similarly a knowledge of the form of the quadric provides us with five equations. Thus if we are given the forms of the two surfaces and the values of I_1 and I_2, we have twenty-one equations, sufficient to determine the coefficients of elasticity, of which there are only twenty-one independent by virtue of (1.3).

The law of elastic symmetry for crystals may be stated:

Every covering operation for the crystalline form must be a covering operation for the elasticity-quartic and for the elasticity-quadric.

From the above geometrical representation we would expect that, in the case of isotropy, there would be just two independent coefficients of elasticity corresponding to I_1 and I_2.

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* *Math. Annalen*, 54 (1900), 201.

While the quartic and quadric defined above are sufficient to determine the elastic properties, there is another invariant quadric

$$g^{ik}c_{ijkl}\,dx^j\,dx^l = \text{constant}. \tag{2.5}$$

3. In the case of isotropy the quartic and quadric must reduce to a pair of spheres. Therefore

$$c_{ijkl}\,dx^i\,dx^j\,dx^k\,dx^l \equiv \theta(g_{ij}\,dx^i\,dx^j)^2, \tag{3.1}$$

$$c_{ij}\,dx^i\,dx^j \equiv \phi g_{ij}\,dx^i\,dx^j, \tag{3.2}$$

where θ, ϕ are indeterminate invariants, and the identities hold for all values of dx^1, dx^2, dx^3. Let the coordinate system be so chosen that, at the point

$$ds^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2 \quad \text{(local Cartesian system)}; \tag{3.3}$$

for this system

$$c_{ijkl} = c^{ijkl}, \tag{3.4}$$

and therefore, by (1.3),

$$c_{ijkl} = c_{jikl} = c_{klij}. \tag{3.5}$$

From (3.1) we obtain

$$\begin{cases} c_{1111} = c_{2222} = c_{3333} = \theta, \\ c_{2223} = c_{3331} = c_{1112} = c_{3332} = c_{1113} = c_{2221} = 0, \\ c_{2233}+2c_{2323} = c_{3311}+2c_{3131} = c_{1122}+2c_{1212} = \theta, \\ c_{1123}+2c_{1213} = c_{2231}+2c_{2321} = c_{3312}+2c_{3132} = 0, \end{cases} \tag{3.6}$$

and from (3.2),

$$\begin{cases} c_{1111}+c_{1122}+c_{1133} = \phi, \quad c_{2311}+c_{2322}+c_{2333} = 0, \\ c_{2211}+c_{2222}+c_{2233} = \phi, \quad c_{3111}+c_{3122}+c_{3133} = 0, \\ c_{3311}+c_{3322}+c_{3333} = \phi, \quad c_{1211}+c_{1222}+c_{1233} = 0. \end{cases} \tag{3.7}$$

Solving (3.6), (3.7), we obtain

$$\begin{cases} c_{1111} = c_{2222} = c_{3333} = \theta, \\ c_{2233} = c_{3311} = c_{1122} = \tfrac{1}{2}(\phi-\theta), \\ c_{2323} = c_{3131} = c_{1212} = \tfrac{1}{4}(3\theta-\phi), \end{cases} \tag{3.8}$$

all the other coefficients vanishing. Hence

$$2W = \theta(e_{11}^2+e_{22}^2+e_{33}^2)+(\phi-\theta)(e_{22}e_{33}+e_{33}e_{11}+e_{11}e_{22}) + (3\theta-\phi)(e_{23}^2+e_{31}^2+e_{12}^2)$$

$$= \theta(e_{11}+e_{22}+e_{33})^2+(3\theta-\phi)(e_{23}^2+e_{31}^2+e_{12}^2-e_{22}e_{33}-e_{33}e_{11}-e_{11}e_{22}). \tag{3.9}$$

Comparing this result with that given by Love,* in whose notation $e_{xx} = e_{11}, \ldots, e_{yz} = 2e_{23}, \ldots$, we find

$$\begin{cases} \theta = \lambda+2\mu, \\ \phi = 3\lambda+2\mu = 3k, \end{cases} \tag{3.91}$$

where k is the modulus of compression. We find for the invariants

$$\begin{cases} I_1 = 3\phi = 3(3\lambda+2\mu) = 9k, \\ I_2 = \frac{3}{2}(5\theta-\phi) = 3(\lambda+4\mu). \end{cases} \tag{3.92}$$

A further elastic symmetry, a *scalar symmetry*, is expressed by the invariant relation $I_1 = I_2$. In that case we find

$$\lambda = \mu, \quad \tfrac{1}{2}(\phi-\theta) = \tfrac{1}{4}(3\theta-\phi), \tag{3.93}$$

$$c_{2233} = c_{3311} = c_{1122} = c_{2323} = c_{3131} = c_{1212}.$$

Thus in an isotropic body possessing scalar symmetry, Cauchy's relations are satisfied.

4. We shall now find the elastic coefficients in the case of isotropy by a tensorial method of extreme simplicity, and completely independent of the arguments given in the two preceding sections. From the isotropic character we see that the elasticity tensor must be expressible in terms of the fundamental tensor g_{ij}, and certain invariant functions of position. The most general relation of this type, having regard to (1.3), is

$$c^{ijkl} = K_1 g^{ij}g^{kl}+\tfrac{1}{2}K_2(g^{ik}g^{jl}+g^{jk}g^{il}), \tag{4.1}$$

where K_1, K_2 are invariant functions of position. Hence, writing

$$e_j^i = g^{ik}e_{jk}, \qquad e = g^{ij}e_{ij} \tag{4.2}$$

—the generalized cubical dilatation—we find

$$2W = K_1e^2+K_2e_j^ie_i^j. \tag{4.3}$$

* *Ibid.*, 100.

For the invariants we obtain

$$(4.4)\qquad \begin{cases} I_1 = 9K_1+3K_2, \\ I_2 = 3K_1+6K_2. \end{cases}$$

In the case of transverse isotropy the elasticity tensor must be expressible in terms of the fundamental tensor g_{ij}, a contravariant unit vector λ^i, $(g_{ij}\lambda^i\lambda^j = 1)$, defining the axis of symmetry, and certain invariant functions of position. The most general relation of this type, having regard to (1 . 3), is

$$(4.5)\quad c^{ijkl} = K_1 g^{ij}g^{kl}+\tfrac{1}{2}K_2(g^{ik}g^{jl}+g^{jk}g^{il})+\tfrac{1}{2}K_3(g^{ij}\lambda^k\lambda^l+g^{kl}\lambda^i\lambda^j) \\ +\tfrac{1}{4}K_4(g^{ik}\lambda^j\lambda^l+g^{jk}\lambda^i\lambda^l+g^{il}\lambda^j\lambda^k+g^{jl}\lambda^i\lambda^k)+K_5\lambda^i\lambda^j\lambda^k\lambda^l,$$

where K_1, K_2, K_3, K_4, K_5 are invariant functions of position.

Hence

$$(4.6)\quad 2W = K_1 e^2+K_2 e^i_j e^j_i+K_3 e(\lambda^i\lambda^j e_{ij})+K_4 e^i_j\lambda^j\lambda^k e_{ik}+K_5(\lambda^i\lambda^j e_{ij})^2.$$

For the invariants we find

$$(4.7)\qquad \begin{cases} I_1 = 9K_1+3K_2+3K_3+K_4+K_5, \\ I_2 = 3K_1+6K_2+K_3+2K_4+K_5. \end{cases}$$

The expressions for W in these two cases are easily seen to reduce to their classical forms on taking local Cartesian coordinates, taking, in the case of transverse isotropy, one of the parametric lines in the direction of the axis of symmetry.

CONJUGATE TRIGONOMETRICAL INTEGRALS

By E. C. TITCHMARSH.

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1. Two series of the form

$$\Sigma(a_n \cos nx + b_n \sin nx), \qquad \Sigma(b_n \cos nx - a_n \sin nx)$$

are said to be conjugate trigonometrical series, and if one is a Fourier series, the other is said to be its allied series. Considerable attention has been paid to such series in recent years, and their main properties may be said to be known.*

It is the first object of the present paper to develop a corresponding theory of conjugate trigonometrical integrals, viz. integrals of the form

$$\int_0^\infty \{\phi(t) \cos xt + \psi(t) \sin xt\}\, dt, \tag{1}$$

$$\int_0^\infty \{\psi(t) \cos xt - \phi(t) \sin xt\}\, dt. \tag{2}$$

These integrals do not seem to have been considered explicitly before, though many of their properties are, of course, very similar to those of the series.

If the integral (1) has the Fourier form

$$\frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty \cos u(t-x) f(t)\, dt, \tag{3}$$

* For this theory see W. H. Young, *Proc. London Math. Soc.* (2), 10 (1911), 254–272; A. Plessner, *Zur Theorie der Konjugierten Trigonometrische Reihen*, Giessen (1922); G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (Records for June 14, 1923).

then the conjugate integral takes the form

$$\frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty \sin u(t-x)\,f(t)\,dt, \tag{4}$$

and (4) is called the allied integral of (3). The value of (3) is, by Fourier's theorem, generally $f(x)$, and it will be shown that (4) has generally the value

$$\frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt. \tag{5}$$

Now it is plain that the allied integral of an allied integral is minus the original integral. Hence, if we write

$$g(x)=\frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt, \tag{6}$$

then we have the reciprocal relation

$$f(x)=-\frac{1}{\pi}\int_0^\infty \frac{g(x+t)-g(x-t)}{t}\,dt. \tag{7}$$

The functional reciprocity expressed by (6) and (7) will be the subject of the second part of the paper. Two functions so connected are called Hilbert transforms of each other. The reciprocity has already been considered by a number of writers. It was noticed by Hardy* in the theory of Cauchy's principal values, and, in the case where $f(x)$ and $g(x)$ are periodic, was deduced by Young† from the theory of trigonometrical series. Different forms of it have also been given by Kellogg‡ and Hilbert§ in their researches on the theory of integral equations. More recently Plessner, in his dissertation referred to above, has deduced an equivalent reciprocity, for functions of integrable square, from the theory of conjugate trigonometrical series. Finally Hardy, in two notes which are to be published in the *Messenger of Mathematics*, has developed the theory *ab initio*, without making any use of trigonometrical integrals or series.

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* G. H. Hardy, *Proc. London Math. Soc.* (2), 7 (1908), 181–208.

† W. H. Young, *Messenger of Math.*, 41 (1912), 161–166.

‡ O. D. Kellogg, *Math. Annalen*, 58 (1904), 441–456; 60 (1905), 424–433.

§ D. Hilbert, *Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen* (1912), 75–77.

|| "Notes on Some Points in the Integral Calculus", 58–59: "On Hilbert Transforms."

2. *The Allied Integral of a Fourier Integral.*

I proceed to prove a number of theorems concerning the integral (4) of § 1.

2.1. THEOREM I.—*If $f(t)$ is integrable over $(-\infty, \infty)$, then the integral* (4) *converges to the value* (5), *provided the latter integral is absolutely convergent.*

We have

$$\int_0^\infty du \int_{-\infty}^\infty \sin u(t-x) f(t)\, dt = \lim_{\lambda\to\infty} \int_{-\infty}^\infty \frac{1-\cos\lambda(t-x)}{t-x} f(t)\, dt$$

provided the limit on the right exists. Now

$$\int_{-\infty}^\infty \frac{1-\cos\lambda(t-x)}{t-x} f(t)\, dt = \int_0^\infty \frac{1-\cos\lambda t}{t} \{f(x+t)-f(x-t)\}\, dt,$$

and it follows from our hypothesis about the absolute convergence of (5) that

$$\lim_{\lambda\to\infty} \int_0^\infty \frac{f(x+t)-f(x-t)}{t} \cos\lambda t\, dt = 0.$$

This proves the theorem.

2.2. THEOREM II.—*If $f(t)$ is integrable over $(-\infty, \infty)$ then the integral* (4) *is summable* (C 1) *to the value* (5) *for all values of x for which*

$$\int_0^t |f(x+u)-f(x-u)|\, du = o(t)$$

and for which the integral (5) *can be defined as the limit*

$$\lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{f(x+t)-f(x-t)}{t}\, dt.$$

This is the analogue of the theorem of Young on the allied series.* We have to prove that under the conditions stated

$$\lim_{\lambda\to\infty} \int_0^\lambda \left(1-\frac{u}{\lambda}\right) du \int_{-\infty}^\infty \sin u(t-x) f(t)\, dt = \int_0^\infty \frac{f(x+t)-f(x-t)}{t}\, dt.$$

* W. H. Young, *Proc. London Math. Soc.* (2), 10 (1911), 254–272; 271.

Now

$$\int_0^\lambda \left(1-\frac{u}{\lambda}\right) du \int_{-\infty}^{\infty} \sin u(t-x) f(t)\, dt = \int_{-\infty}^{\infty} \frac{\lambda(t-x)-\sin\lambda(t-x)}{\lambda(t-x)^2} f(t)\, dt$$

$$= \int_0^\infty \frac{\lambda t-\sin\lambda t}{\lambda t^2} \{f(x+t)-f(x-t)\}\, dt.$$

Hence we have to prove that

$$\int_0^\infty \frac{\sin\lambda t}{t^2} \{f(x+t)-f(x-t)\}\, dt = o(\lambda).$$

Now $\sin\lambda t/\lambda t$ is positive decreasing for $0<t<1/\lambda$, so that the Second Mean Value Theorem gives

$$\int_0^{1/\lambda} \frac{\sin\lambda t}{\lambda t} \frac{f(x+t)-f(x-t)}{t}\, dt = \int_0^\xi \frac{f(x+t)-f(x-t)}{t}\, dt,$$

where $0<\xi<1/\lambda$; and hence

$$\int_0^{1/\lambda} \frac{\sin\lambda t}{t^2} \{f(x+t)-f(x-t)\}\, dt = o(\lambda).$$

Finally

$$\left| \int_{1/\lambda}^\infty \frac{\sin\lambda t}{t^2} \{f(x+t)-f(x-t)\}\, dt \right| \leqslant \int_{1/\lambda}^\infty \frac{|f(x+t)-f(x-t)|}{t^2}\, dt$$

$$= -\lambda^2 \int_0^{1/\lambda} |f(x+u)-f(x-u)|\, du + 2\int_{1/\lambda}^\infty \frac{dt}{t^3} \int_0^t |f(x+u)-f(x-u)|\, du$$

$$= \lambda^2 o(1/\lambda) + \int_{1/\lambda}^\infty o(1/t^2)\, dt = o(\lambda).$$

This completes the proof.

2.3. THEOREM III.—*If, at a point where*

$$\int_0^t |f(x+u)-f(x-u)|\, du = o(t),$$

the integral (4) *is summable by any Cesàro mean of positive order, then it is summable by all Cesàro means of positive order to the same sum.*

We have to prove that, if the integral

$$I_k = \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^k du \int_{-\infty}^{\infty} \sin u(t-x) f(t)\, dt$$

tends to a limit when $\lambda \to \infty$ for $k = \alpha > 0$, then it tends to the same limit for $k = \beta > 0$; *i.e.* that $I_\alpha - I_\beta$ tends to zero. Now it is easily seen that

$$I_k = \int_0^\infty \{f(x+t)-f(x-t)\}\, dt \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^k \sin ut\, du.$$

A partial integration gives

$$\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^k \sin ut\, du = \frac{1}{t} + \frac{k}{\lambda t}\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^{k-1} \cos ut\, du$$

$$= \frac{1}{t} + \frac{k}{\lambda^k t^{k+1}} \int_0^{\lambda t} \frac{\cos(\lambda t - v)}{v^{1-k}}\, dv.$$

If $k \leqslant 1$ the last integral is bounded, and if $k > 1$ it is less than a constant multiple of $\lambda^{k-1} t^{k-1}$. Hence

$$\left| \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha \sin ut\, du - \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\beta \sin ut\, du \right| \leqslant \frac{A}{\lambda^a t^{a+1}} + \frac{B}{\lambda^b t^{b+1}},$$

where $a = \mathrm{Min}(\alpha, 1)$, $b = \mathrm{Min}(\beta, 1)$. Hence

$$\left| \int_{1/\lambda}^\infty \{f(x+t)-f(x-t)\}\, dt \int_0^\lambda \left\{ \left(1-\frac{u}{\lambda}\right)^\alpha - \left(1-\frac{u}{\lambda}\right)^\beta \right\} \sin ut\, du \right|$$

$$\leqslant \frac{A}{\lambda^a} \int_{1/\lambda}^\infty \frac{|f(x+t)-f(x-t)|}{t^{a+1}}\, dt + \frac{B}{\lambda^b} \int_{1/\lambda}^\infty \frac{|f(x+t)-f(x-t)|}{t^{b+1}}\, dt.$$

Now, integrating by parts,

$$\int_{1/\lambda}^\infty \frac{|f(x+t)-f(x-t)|}{t^{a+1}}\, dt = -\lambda^{a+1} \int_0^{1/\lambda} |f(x+u)-f(x-u)|\, du$$

$$+ (a+1) \int_{1/\lambda}^\infty \frac{dt}{t^{a+2}} \int_0^t |f(x+u)-f(x-u)|\, du$$

$$= -\lambda^{a+1} o(1/\lambda) + \int_{1/\lambda}^\infty o(t^{-a-1})\, dt = o(\lambda^a),$$

and similarly for the integral with b. Hence

$$\lim_{\lambda \to \infty} \int_{1/\lambda}^\infty \{f(x+t)-f(x-t)\}\, dt \int_0^\lambda \left\{ \left(1-\frac{u}{\lambda}\right)^\alpha - \left(1-\frac{u}{\lambda}\right)^\beta \right\} \sin ut\, du = 0.$$

Also the integral $\int_0^\lambda \left(1-\frac{u}{\lambda}\right)^k \sin ut\, du$

is a positive increasing function of t for $0 < t < 1/\lambda$, its greatest value being a constant multiple of λ. Hence

$$\int_0^{1/\lambda} \{f(x+t)-f(x-t)\} \int_0^{\lambda} \left(1-\frac{u}{\lambda}\right)^k \sin ut\, du = A\lambda \int_{\xi}^{1/\lambda} \{f(x+t)-f(x-t)\}\, dt$$

$$= o(1).$$

Combining this result for $k = \alpha$, $k = \beta$ with the previous result, it follows that

$$\lim_{\lambda \to \infty} (I_\alpha - I_\beta) = 0,$$

which gives the result stated.

2.4. THEOREM IV.—*The integral* (4) *is, for almost all values of* x, *summable* $(C\, k)$ *for every positive value of* k *to the sum* (5).*

This follows at once from the two previous theorems, and the fact that each of the conditions assumed in Theorem II holds for almost all values of x.

2.5. THEOREM V.—*If* $f(x)$ *has its square integrable over* $(-\infty, \infty)$, *then the integral* (4) *has the value* (5) *for almost all values of* x, *provided each integration of the repeated integral* (4) *is taken in the* $(C\, 1)$ *sense.*

The proof of this theorem depends on the theory of Fourier transforms, due to Plancherel.†

Suppose in the first place that $f(x)$ is an even function. Then the repeated integral (4) is

$$-\frac{2}{\pi} \int_0^{\infty} \sin ux\, du \int_0^{\infty} \cos ut\, f(t)\, dt.$$

Now Plancherel proved that, if $\phi(t)$ has its square integrable over $(0, \infty)$, then the integral

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \begin{matrix}\cos \\ \sin\end{matrix} xt\, \phi(t)\, dt$$

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* Cf. G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (Records for June 14, 1923).

† M. Plancherel, *Math. Annalen*, 76 (1914), 315–326. See also E. C. Titchmarsh, *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473.

is, for almost all values of x, summable (C 1) to the Fourier cosine or sine transform of $\phi(t)$, which itself is a function of integrable square. Hence, on performing the first (C 1) integration, we obtain almost everywhere the cosine transform of $f(x)$, viz.,

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{\sin xt}{t} f(t)\, dt.$$

Similarly the second (C 1) integration gives almost everywhere the sine transform of $F(x)$, viz.,

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{1-\cos xt}{t} F(t)\, dt.$$

Hence
$$g(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^\infty \frac{1-\cos xu}{u} \left\{\frac{d}{du} \int_0^\infty \frac{\sin ut}{t} f(t)\, dt\right\} du$$

$$= \frac{2}{\pi} \frac{d}{dx} \int_0^\infty \frac{1-\cos xu - xu \sin xu}{u^2}\, du \int_0^\infty \frac{\sin ut}{t} f(t)\, dt$$

on integrating by parts. Hence, if we can invert the order of integration, we obtain

$$g(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^\infty \frac{f(t)}{t}\, dt \int_0^\infty \frac{1-\cos xu - xu \sin xu}{u^2} \sin ut\, du.$$

Now, integrating by parts,

$$\int_0^\infty \frac{1-\cos xu - xu \sin xu}{u^2} \sin ut\, du = t \int_0^\infty \frac{1-\cos xu}{u} \cos ut\, du$$

$$= t \lim_{\delta \to 0} \int_\delta^\infty \left\{\cos ut - \tfrac{1}{2} \cos (x+t)u - \tfrac{1}{2} \cos (x-t)u\right\} \frac{du}{u}$$

$$= t \lim_{\delta \to 0} \left\{\int_{\delta t}^\infty \frac{\cos v}{v}\, dv - \tfrac{1}{2} \int_{\delta |x+t|}^\infty \frac{\cos v}{v}\, dv - \tfrac{1}{2} \int_{\delta |x-t|}^\infty \frac{\cos v}{v}\, dv\right\}$$

$$= \tfrac{1}{2} t \lim_{\delta \to 0} \left\{\int_{\delta t}^{\delta |x+t|} \frac{\cos v}{v}\, dv + \int_{\delta t}^{\delta |x-t|} \frac{\cos v}{v}\, dv\right\}$$

$$= \tfrac{1}{2} t \left\{\log \left|1+\frac{x}{t}\right| + \log \left|1-\frac{x}{t}\right|\right\}.$$

Hence
$$g(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^\infty f(t) \left\{\log \left|1+\frac{x}{t}\right| + \log \left|1-\frac{x}{t}\right|\right\} dt$$

$$= \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^\infty f(t) \log \left|1-\frac{x}{t}\right| dt.$$

We have still to justify the inversion of the repeated integral. Since the integral

$$\int_0^\infty \frac{\sin ut}{t} f(t)\, dt$$

is uniformly convergent over any finite u-interval, we have

$$\int_0^N \frac{1-\cos xu - xu \sin xu}{u^2}\, du \int_0^\infty \frac{\sin ut}{t} f(t)\, dt$$

$$= \int_0^\infty \frac{f(t)}{t}\, dt \int_0^N \frac{1-\cos xu - xu \sin xu}{u^2} \sin ut\, du,$$

and hence it is sufficient to prove that

$$\lim_{N\to\infty} \int_0^\infty \frac{f(t)}{t}\, dt \int_N^\infty \frac{1-\cos xu - xu \sin xu}{u^2} \sin ut\, du = 0,$$

In the first place

$$\left| \int_0^\infty \frac{f(t)}{t}\, dt \int_N^\infty \frac{1-\cos xu}{u^2} \sin ut\, du \right|$$

$$\leqslant \int_0^\delta \frac{|f(t)|}{t}\, dt \int_N^\infty \frac{(ut)^\alpha}{u^2}\, du + \int_\delta^\infty \frac{|f(t)|}{t}\, dt \int_N^\infty \frac{du}{u^2},$$

which tends to zero if $\frac{1}{2} < \alpha < 1$. Hence we have only to prove that

$$\lim_{N\to\infty} \int_0^\infty \frac{f(t)}{t}\, dt \int_N^\infty \frac{\sin xu \sin tu}{u}\, du = 0.$$

Now

$$\left| \int_N^\infty \frac{\sin xu \sin tu}{u}\, du \right| = \tfrac{1}{2} \left| \int_{N|x-t|}^{N|x+t|} \frac{\cos v}{v}\, dv \right| < \tfrac{1}{2} \log \left| \frac{x+t}{x-t} \right|,$$

and since the integral $\displaystyle\int_0^\infty \frac{|f(t)|}{t} \log \left| \frac{x+t}{x-t} \right| dt$

exists, the result follows from the convergence theorem of Lebesgue.

It has thus been proved that the value of the integral (4) taken in the sense stated is almost everywhere

$$-\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^\infty f(t) \log \left| 1 - \frac{x}{t} \right| dt.$$

Now it was shown by Hardy, in his paper on Hilbert transforms already referred to, that we may differentiate under the integral sign for almost all values of x; and it is easily seen that we obtain the value (5).

This proves the theorem for an even function. It may be proved for an odd function in the same way. In this case $g(x)$ is the cosine transform of the sine transform of $f(x)$,

$$g(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}\left\{\frac{d}{du}\int_0^\infty \frac{1-\cos ut}{t} f(t)\,dt\right\} du$$

$$= \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{xu\cos xu - \sin xu}{u^2}\,du \int_0^\infty \frac{1-\cos ut}{t} f(t)\,dt$$

$$= \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{f(t)}{t}\,dt \int_0^\infty \frac{xu\cos xu - \sin xu}{u^2}(1-\cos ut)\,du,$$

the inversion being justified as in the previous case. Also

$$\int_0^\infty \frac{xu\cos xu - \sin xu}{u^2}(1-\cos ut)\,du$$

$$= -t\int_0^\infty \frac{\sin xu}{u}\sin ut\,du = -\tfrac{1}{2}t\log\left|\frac{x-t}{x+t}\right|.$$

Hence

$$g(x) = -\frac{1}{\pi}\frac{d}{dx}\int_0^\infty f(t)\log\left|\frac{x-t}{x+t}\right| dt = -\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^\infty f(t)\log\left|1-\frac{x}{t}\right| dt$$

since $f(t)$ is odd. The result now follows as before.

3. *The Theory of Hilbert Transforms.*

The theory of conjugate integrals will now be applied to the theory of Hilbert transforms, concerning which I proceed to prove a number of theorems.

3.1. THEOREM VI.—*If the Fourier integral of a function $f(x)$ is absolutely convergent, then the reciprocal formulae* (6) (7) *of* § 1 *hold, if each integral is defined as*

$$\lim_{a\to 0,\, b\to\infty}\int_a^b;$$

and the Fourier integral of $g(x)$ is absolutely convergent.

Let $$f(x) = \int_0^\infty \{\phi(t) \cos xt + \psi(t) \sin xt\}\, dt,$$

$\phi(t)$ and $\psi(t)$ being integrable over $(0, \infty)$. Then

$$\int_a^b \frac{f(x+u)-f(x-u)}{u}\, du = \int_a^b \frac{du}{u} \int_0^\infty 2 \sin ut \{\psi(t) \cos xt - \phi(t) \sin xt\}\, dt$$

$$= 2\int_0^\infty \{\psi(t) \cos xt - \phi(t) \sin xt\}\, dt \int_a^b \frac{\sin ut}{u}\, du,$$

since the integral with respect to t converges uniformly for all values of u. Now the modulus of

$$\int_a^b \frac{\sin ut}{u}\, du$$

is less than a constant for all values of a, b, and t; hence the t-integrand is less than a constant multiple of the integrable function

$$|\psi(t)| + |\phi(t)|.$$

Hence, by Lebesgue's convergence theorem, we can make $a \to 0$, $b \to \infty$, under the integral sign, which gives

$$\int_0^\infty \frac{f(x+u)-f(x-u)}{u}\, du = \pi \int_0^\infty \{\psi(t) \cos xt - \phi(t) \sin xt\}\, dt.$$

Hence $$g(x) = \int_0^\infty \{\psi(t) \cos xt - \phi(t) \sin xt\}\, dt,$$

so that $g(x)$ is of the form stated.

If we perform the same operation on $g(x)$, we clearly obtain $-f(x)$. Hence the reciprocity holds.

3.2. THEOREM VII.—*If $f(x)$ is a function whose square is integrable over $(-\infty, \infty)$, then there is a function $g(x)$ with the same property such that for almost all values of x*

(1) $$g(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^\infty f(t) \log \left|1 - \frac{x}{t}\right| dt,$$

(2) $$f(x) = \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^\infty g(t) \log \left|1 - \frac{x}{t}\right| dt.$$

This form of the reciprocity is due to Hardy, who proved it directly; but it is most easily deduced from the theory of Fourier transforms. Suppose first that $f(x)$ is an even function. Then, as was proved in § 2.5, the equation (1) shows that $g(x)$ is almost everywhere equal to the sine transform of the cosine transform of $f(x)$, and the right-hand side of (2) is almost everywhere equal to the cosine transform of the sine transform of $g(x)$, viz. $f(x)$. Hence the reciprocity holds, and $g(x)$ has its square integrable. Similarly if $f(x)$ is an odd function, and so in the general case. This proves the theorem.

Hardy also proved the further theorem that *in the formulae of Theorem* VII *the differentiations may be performed under the integral sign for almost all values of* x, so that we are led again to formulae (6) and (7) of § 1. This, however, does not seem to follow at all easily from the theory of Fourier transforms. It depends on the fact that, if $f(x)$ has its square integrable, the limit

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\delta} \frac{f(x+t)-f(x-t)}{t}\, dt$$

exists for almost all values of x. A direct proof of this has been given by Besikovitch.*

3.3. THEOREM VIII.—*Suppose that* $f(x)$ *has its square integrable over* $(-\infty, \infty)$ *and that it satisfies a Lipschitz condition of order* α *uniformly in* x, *i.e.*

$$|f(x+h)-f(x)| < A|h|^{\alpha} \quad (0 < \alpha < 1),$$

for $|h| < k$. *Then* $g(x)$ *satisfies the same conditions, the reciprocity holds, and*

$$\int_{-\infty}^{\infty} \{f(x)\}^2 dx = \int_{-\infty}^{\infty} \{g(x)\}^2 dx.$$

The corresponding theorem for a periodic function $f(x)$ was deduced by Priwaloff† from the theory of conjugate trignometrical series.

The proof depends on a number of Lemmas.

3.31. LEMMA *a*.—*If*

$$F(\lambda, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+(t-x)^2} f(t)\, dt \quad (\lambda > 0),$$

* A. Besikovitch, *Fundamenta Mathematicae*, 4 (1923), 172–195.

† J. Priwaloff, *Bull. Soc. Math. de France*, 44 (1916), 100–103.

then $$|F(\lambda, x)-f(x)| < A\lambda^{\alpha}$$

where A is independent of x.

We have

$$F(\lambda, x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+t^2} f(t+x)\,dt, \quad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+t^2} f(x)\,dt.$$

Hence we have to prove that

$$\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+t^2} \{f(t+x)-f(x)\}\,dt = O(\lambda^{\alpha}).$$

It is easily seen that the parts of the integral over $(-\infty, -\delta)$ and (δ, ∞) are $O(\lambda)$; and if $\delta < k$ the remainder is

$$O\left(\int_{-\delta}^{\delta} \frac{\lambda}{\lambda^2+t^2} |t|^{\alpha}\,dt\right) = O\left(\int_0^{\lambda\delta} \frac{\lambda^{\alpha}u^{\alpha}\,du}{1+u^2}\right) = O(\lambda^{\alpha}).$$

3.32. LEMMA β.—*Let*

$$G(\lambda, x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{\lambda^2+(t-x)^2} f(t)\,dt \quad (\lambda > 0),$$

and $$g(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt.$$

Then $$|G(\lambda, x)-g(x)| < A\lambda^{\alpha}$$

where A is independent of x.

We have

$$\begin{aligned}
\pi\{G(\lambda, x)-g(x)\} &= P\int_{-\infty}^{\infty} \left(\frac{t}{\lambda^2+t^2}-\frac{1}{t}\right) f(x+t)\,dt \\
&= P\int_{-\infty}^{\infty} \frac{-\lambda^2}{t(\lambda^2+t^2)} f(x+t)\,dt \\
&= -\lambda^2\left\{\int_{-\infty}^{-\delta} + P\int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right\} \frac{f(x+t)}{t(\lambda^2+t^2)}\,dt \\
&= -\lambda^2(J_1+J_2+J_3).
\end{aligned}$$

If $\delta < k$,

$$|J_2| = \left| \int_{-\delta}^{\delta} \frac{f(x+t)-f(x)}{t(\lambda^2+t^2)}\, dt \right|$$

$$< 2A \int_0^{\delta} \frac{t^a\, dt}{t(\lambda^2+t^2)}$$

$$= 2A\lambda^{a-2} \int_0^{\delta/\lambda} \frac{du}{u^{1-a}(1+u^2)}$$

$$= O(\lambda^{a-2}).$$

Also J_1 and J_2 are clearly bounded for all values of λ. The result then follows at once.

3 . 33. We are now in a position to prove our theorem, except as regards the Lipschitz condition satisfied by $g(x)$. Suppose, for convenience, that $f(x)$ is even. It will be seen that the theorem can be proved in the same way if $f(x)$ is odd, and so in the general case.

Let $\phi(x)$ be the cosine transform of $f(x)$. Then if

$$\phi_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a f(t) \cos xt\, dt$$

we have $$\lim_{a\to\infty} \int_0^{\infty} e^{-\lambda u} \sin xu \{\phi(u)-\phi_a(u)\}\, du = 0.$$

Hence

$$\int_0^{\infty} e^{-\lambda u} \sin xu\, \phi(u) = \lim_{a\to\infty} \int_0^{\infty} e^{-\lambda u} \sin xu\, \phi_a(u)\, du$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \lim_{a\to\infty} \int_0^a f(t)\, dt \int_0^{\infty} e^{-\lambda u} \sin xu \cos tu\, du$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_0^{\infty} f(t) \left\{ \frac{x+t}{\lambda^2+(x+t)^2} + \frac{x-t}{\lambda^2+(x-t)^2} \right\} dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\, \frac{x-t}{\lambda^2+(x-t)^2}\, dt$$

$$= -\sqrt{\left(\frac{\pi}{2}\right)} G(\lambda, x).$$

Integrating with respect to x, we have

$$\int_0^x G(\lambda, t)\, dt = -\sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \frac{1-\cos xu}{u}\, e^{-\lambda u}\, \phi(u)\, du.$$

Making $\lambda \to 0$, by Lemma β

$$\int_0^x g(t)\,dt = -\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{1-\cos xu}{u}\,\phi(u)\,du,$$

$$g(x) = -\sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{1-\cos xu}{u}\,\phi(u)\,du.$$

Hence $g(x)$ is the sine transform of $-\phi(x)$, its square is integrable, and

$$\int_0^\infty \{g(x)\}^2\,dx = \int_0^\infty \{\phi(x)\}^2\,dx = \int_0^\infty \{f(x)\}^2\,dx.$$

The existence of the reciprocity follows from the theory of Fourier transforms exactly as in § 3 . 2.

To prove that $g(x)$ satisfies a Lipschitz condition we require further lemmas.

3 . 34. LEMMA γ.—*If $f(x)$ satisfies the conditions of Theorem* VIII, *then it tends to zero at infinity.**

If δ is any positive number, the points where $|f(x)| \geqslant \delta$ form a set of intervals, since $|f(x)|$ is continuous. Let $\mu(x_0, \delta)$ be the greatest of these intervals lying entirely to the right of $x = x_0$. Then for every value of δ

$$\lim_{x\to\infty} \mu(x, \delta) = 0.$$

For otherwise there would be an infinity of such intervals each of length greater than some positive number l; and this is inconsistent with the hypothesis that the square of $f(x)$ is integrable over $(0, \infty)$.

Having fixed δ, choose x_0 so that $\mu(x, \delta) < \epsilon$ for $x \geqslant x_0$ $(\epsilon < k)$. Let (x_1, x_2) be one of the intervals in which $|f(x)| \geqslant \delta$. Then if $x_1 < x < x_2$

$$|f(x)-f(x_1)| < A\,|x-x_1|^\alpha.$$

But $$|f(x_1)| = \delta, \quad |x-x_1| < |x_1-x_2| < \epsilon.$$

Hence $$|f(x)| < \delta + A\epsilon^\alpha$$

for all values of x greater than x_0. The right-hand side may be made as small as we please by choosing first δ and then x_0. This proves the lemma.

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* We shall require only the fact that it is bounded.

3.35. LEMMA δ.—*If* $h < A\lambda$, *then*

$$|G(\lambda, x+h)-G(\lambda, x-h)| < Ah\lambda^{\alpha-1}.$$

For

$$\pi\{G(\lambda, x+h)-G(\lambda, x-h)\}$$

$$= \int_{-x}^{\infty} \frac{t-x-h}{\lambda^2+(t-x-h)^2} f(t)\,dt - \int_{-\infty}^{\infty} \frac{t-x+h}{\lambda^2+(t-x+h)^2} f(t)\,dt$$

$$= 2h \int_{-\infty}^{x} \frac{t^2-\lambda^2-h^2}{\{\lambda^2+(t-h)^2\}\{\lambda^2+(t+h)^2\}} f(t+x)\,dt$$

$$= 2h \left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right)$$

$$= 2h(J_1+J_2+J_3).$$

If δ is fixed, J_1 and J_3 are clearly bounded for all small values of λ and h, and all values of x. Also

$$J_2 = f(x)\int_{-\delta}^{\delta} \frac{t^2-\lambda^2-h^2}{\{\lambda^2+(t-h)^2\}\{\lambda^2+(t+h)^2\}}\,dt$$

$$+\int_{-\delta}^{\delta} \frac{t^2-\lambda^2-h^2}{\{\lambda^2+(t-h)^2\}\{\lambda^2+(t+h)^2\}} \{f(t+x)-f(x)\}\,dt$$

$$= J_{2.1}+J_{2.2},$$

say. Now $f(x)$ is bounded, by Lemma γ, so that

$$|J_{2.1}| < \frac{A}{2h}\left|\int_{-\delta}^{\delta} \left\{\frac{t-h}{\lambda^2+(t-h)^2} - \frac{t+h}{\lambda^2+(t+h)^2}\right\} dt\right|$$

$$= \frac{A}{h}\left|\int_{-\delta}^{\delta} \frac{t-h}{\lambda^2+(t-h)^2}\,dt\right|$$

$$= \frac{A}{2h} \log\left\{1-\frac{4\delta h}{\lambda^2+(\delta-h)^2}\right\},$$

which is bounded for all small values of λ and h. Also

$$|J_{2.2}| < A\int_0^{\delta} \frac{t^2-h^2-\lambda^2}{\{\lambda^2+(t-h)^2\}\{\lambda^2+(t+h)^2\}}\, t^{\alpha}\,dt$$

$$= \frac{A}{\lambda^{1-\alpha}}\int_0^{\delta/\lambda} \frac{t^2-(h^2/\lambda^2)-1}{[1+\{t-(h/\lambda)\}^2][1+\{t+(h/\lambda)\}^2]}\, t^{\alpha}\,dt,$$

and the last integral is bounded. Thus $J_2 = O(\lambda^{\alpha-1})$, and the lemma follows.

If $\alpha = 1$, the last integral is $O\{\log(1/\lambda)\}$, so that in this case

$$|G(\lambda, x+h)-G(\lambda, x-h)| < Ah\log(1/\lambda).$$

3.36. It follows at once from the last lemma that $g(x)$ satisfies a Lipschitz condition of order α. For

$$\begin{aligned}|g(x+h)-g(x-h)| &\leqslant |g(x+h)-G(\lambda, x+h)| \\ &\quad +|G(\lambda, x+h)-G(\lambda, x-h)|+|G(\lambda, x-h)-g(x-h)| \\ &< A\lambda^{\alpha}+Ah\lambda^{\alpha-1}+A\lambda^{\alpha}\end{aligned}$$

by Lemmas β and δ. Taking $h = \lambda$ the result stated follows.

If $\alpha = 1$, we have similarly

$$|g(x+h)-g(x-h)| < A\lambda+Ah\log(1/\lambda)+A\lambda = O\big(h\log(1/h)\big).$$

3.4. The proof of Theorem VIII is now complete. Before leaving it reference must be made to a remarkable extension of the theory of Hilbert transforms which has recently been made by M. Riesz. He has proved that, if $f(x)$ has any power greater than the first, say the p-th, integrable over $(-\infty, \infty)$, then the reciprocal formulae (6) (7) of § 1 hold for almost all values of x, and $g(x)$ also has its p-th power integrable over $(-\infty, \infty)$.

Now it may easily be verified that Lemmas β, γ, and δ are equally true if $f(x)$ has any power greater than the first integrable over $(-\infty, \infty)$. Hence, combining the work of the present paper with that of Riesz, we are led to the result that *if $f(x)$ has its p-th power $(p>1)$ integrable over $(-\infty, \infty)$, and satisfies a Lipschitz condition of order α $(0<\alpha<1)$ uniformly in x, then the reciprocal formulae* (6) (7) *hold for all values of x, and $g(x)$ satisfies the same conditions as $f(x)$.*

3.5. A final extension of the theory of Hilbert transforms will now be made to non-absolutely convergent integrals. This is contained in the following theorem :—

THEOREM IX.—*Suppose that $f(x)$ is bounded and satisfies a Lipschitz condition of order α $(0<\alpha<1)$ uniformly in x, and that the integrals*

$$(1) \qquad \int^{\infty} \frac{f(t)\log t}{t}\,dt, \qquad \int_{-\infty} \frac{f(t)\log(-t)}{t}\,dt$$

are convergent (not necessarily absolutely). Then the reciprocal formulae (6) (7) *of* § 1 *hold*; $g(x) = O(\log x)$ *as* $x \to \infty$; $g(x)$ *satisfies the same Lipschitz condition as* $f(x)$; *and the integrals*

$$\int^{\infty} \frac{g(t)}{t}\,dt, \qquad \int_{-\infty} \frac{g(t)}{t}\,dt \tag{2}$$

*are convergent (non-absolutely).**

In the first place it may be verified without difficulty that Lemmas α, β and δ of the last theorem are still true under our present hypotheses. The only alteration needed in the proofs is the use of the Second Mean Value Theorem in certain places where simple considerations were previously sufficient; and the use of the datum that $f(x)$ is bounded, instead of Lemma γ.

The fact that $g(x)$ satifies the same Lipschitz condition as $f(x)$ follows from these lemmas, and is therefore proved. To establish the reciprocity we require two more lemmas.

3 . 51. LEMMA ϵ.—*The integral*

$$\int_A^{\infty} G(\lambda, x+h)\,\frac{dh}{h}$$

is uniformly convergent for $0 < \lambda < \lambda_0$.

The argument to be used is an elaboration of that of Hardy.† As we are only considering a particular value of x, we may put $x = 0$ without loss of generality. Then

$$\int_A^B G(\lambda, h)\,\frac{dh}{h} = \frac{1}{\pi}\int_A^B \frac{dh}{h}\int_{-\infty}^{\infty} \frac{t-h}{\lambda^2+(t-h)^2}\,f(t)\,dt \tag{1}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_A^B \frac{t-h}{\lambda^2+(t-h)^2}\,\frac{dh}{h}$$

* This theorem includes that of Hardy, *Proc. London Math. Soc.* (2), 7 (1908), 203. Hardy assumed that $f(x)$ was bounded and had two continuous derivatives, and merely deduced the existence of the reciprocity.

† *Loc. cit.*, 202–203.

since the t-integral converges uniformly over any finite h-interval. Hence it is sufficient to show that

$$\left| \int_{-\infty}^{\infty} f(t)\,dt \int_{B}^{\infty} \frac{t-h}{\lambda^2+(t-h)^2} \frac{dh}{h} \right| < \epsilon \tag{2}$$

for $B > B_0(\epsilon)$, $0 < \lambda < \lambda_0$. Now

(3)

$$\int_{B}^{\infty} \frac{t-h}{\lambda^2+(t-h)^2} \frac{dh}{h} = \tfrac{1}{2} \frac{t}{\lambda^2+t^2} \log \frac{\lambda^2+(t-B)^2}{B^2} - \frac{\lambda}{\lambda^2+t^2} \left(\frac{\pi}{2} + \arctan \frac{t-B}{\lambda} \right).$$

The last term on the right gives the integral

$$\int_{-\infty}^{\infty} \left(\frac{\pi}{2} + \arctan \frac{t-B}{\lambda} \right) \frac{\lambda f(t)}{\lambda^2+t^2}\,dt = \int_{-\infty}^{-M} + \int_{-M}^{N} + \int_{N}^{\infty} = J_1 + J_2 + J_3, \tag{4}$$

say. Since the function $\frac{1}{2}\pi + \arctan\{(t-B)/\lambda\}$ is positive, increasing and less than π, it is easily seen from the Second Mean Value Theorem that J_1 and J_2 may be made as small as we please by choosing M and N sufficiently large, for all small values of λ; and having fixed M and N, we have

$$J_2 = \left(\frac{\pi}{2} + \arctan \frac{N-B}{\lambda} \right) \int_{M'}^{N} \frac{\lambda f(t)}{\lambda^2+t^2}\,dt.$$

Here the integral is bounded, and the factor in brackets tends to zero when $B \to \infty$, uniformly in λ.

The integral (4) thus behaves in the required manner. The remaining integral is

$$\int_{-\infty}^{\infty} \frac{t}{\lambda^2+t^2} \log \left\{ \frac{\lambda^2}{B^2} + \left(1-\frac{t}{B}\right)^2 \right\} f(t)\,dt = \int_{-\infty}^{-M} + \int_{-M}^{N} + \int_{N}^{\infty}.$$

In the interval $(-M, N)$ the logarithm can be made as small as we please for all values of λ, by choice of B, and the integral

$$\int_{-M}^{N} \frac{t}{\lambda^2+t^2} |f(t)|\,dt$$

is bounded—in fact it tends to a limit as $\lambda \to 0$, by Lemma β. Hence the integral over $(-M, N)$ tends uniformly to zero.

Now let $$\int_N^\infty = \int_N^A + \int_A^{(1-\theta)B} + \int_{(1-\theta)B}^{(1+\theta)B} + \int_{(1+\theta)B}^\infty \quad (0<\theta<1)$$

$$= J_1 + J_2 + J_3 + J_4,$$

and choose A so that $$\left|\int_{A'}^{A''} \frac{f(t)\log t}{t}\,dt\right| < \sigma$$

for $A'' > A' \geqslant A$. When A has been chosen we can choose B_1 so that

$$|J_1| < \sigma \quad (B \geqslant B_1, \quad \lambda < \lambda_0).$$

In J_2 the logarithm is negative decreasing, so that

$$J_2 = \log\left(\frac{\lambda^2}{B_2}+\theta^2\right)\int_\xi^{(1-\theta)B} \frac{tf(t)}{\lambda^2+t^2}\,dt,$$

and we can choose B_2 so that

$$|J_2| < \sigma \log(1/\theta) \quad (B \geqslant B_2, \quad \lambda < \lambda_0).$$

Again

$$|J_3| = \left|\int_{1-\theta}^{1+\theta} \frac{u}{\lambda^2/B^2+u^2} \log\left\{\frac{\lambda^2}{B^2}+(1-u)^2\right\} f(Bu)\,du\right|$$

$$< \frac{K}{1-\theta}\int_{1-\theta}^{1+\theta} \log\left(\frac{1}{1-u}\right)^2 du$$

$$= \frac{2K\theta}{1-\theta}\left(1+\log\frac{1}{\theta}\right).$$

Finally $$J_4 = J_{4.1} + J_{4.2} + J_{4.3},$$

where $$J_{4.1} = \int_{(1+\theta)B}^\infty \frac{f(t)\log t}{t^2+\lambda^2}\,t\,dt,$$

$$J_{4.2} = -\log B \int_{(1+\theta)B}^\infty \frac{tf(t)}{t^2+\lambda^2}\,dt,$$

$$J_{4.3} = \int_{(1+\theta)B}^\infty \frac{f(t)}{\lambda^2+t^2} \log\left\{\frac{\lambda^2}{t^2}+\left(\frac{B}{t}-1\right)^2\right\} t\,dt.$$

Hence we have

$$|J_{4.1}| < \sigma \quad (B \geqslant B_{4.1}, \quad \lambda < \lambda_0), \qquad |J_{4.2}| < \sigma \quad (B \geqslant B_{4.2}, \quad \lambda < \lambda_0)$$

and
$$|J_{4.3}| = \left| \log \left\{ \frac{\lambda^2}{(1+\theta)^2 B^2} + \left(\frac{\theta}{1+\theta}\right)^2 \right\} \int_{(1+\theta)B}^{\xi} \frac{f(t)}{t^2+\lambda^2}\, dt \right|$$

$$< \sigma \log \frac{1}{\theta} \quad (B \geqslant B_{4.3}, \quad \lambda < \lambda_0).$$

Combining the above results we have, for $B > B_1$, $\lambda < \lambda_0$,

$$\left| \int_N^{\infty} \right| < \sigma + \sigma \log \frac{1}{\theta} + \frac{2K\theta}{1-\theta}\left(1 + \log \frac{1}{\theta}\right) + \sigma + \sigma + \sigma \log \frac{1}{\theta}$$

which can be made as small as we please by choosing first θ and then σ. Hence this integral tends uniformly to zero.

The remaining integral over $(-\infty, -M)$ can be dealt with in a similar way. It will, however, be simpler, as there is no region where t is nearly equal to B.

3.52. LEMMA ξ.—*We have the relation*

$$F(\lambda, x) = -\frac{1}{\pi} \int_0^{\infty} \frac{G(\lambda, x+h) - G(\lambda, x-h)}{h}\, dh.$$

It is clear from the argument used in the previous lemma that we may substitute the integral defining each G, and invert the order of integration in the resulting repeated integral. The result of this step is easily found to be

$$\int_0^{\infty} \frac{G(\lambda, x+h) - G(\lambda, x-h)}{h}\, dh$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} f(t+x)\, dt \int_0^{\infty} \frac{t^2 - h^2 - \lambda^2}{\{\lambda^2 + (t-h)^2\}\{\lambda^2 + (t+h)^2\}}\, dh.$$

An elementary integration gives

$$\int_0^{\infty} \frac{t^2 - h^2 - \lambda^2}{\{\lambda^2 + (t-h)^2\}\{\lambda^2 + (t+h)^2\}}\, dh = -\frac{\pi\lambda}{2(t^2+\lambda^2)}$$

and the result then follows from the definition of $F(\lambda, x)$.

3 . 53. The reciprocal formula § 1 (7) may now be obtained. It follows from Lemmas α and ξ that

$$\lim_{\lambda\to 0}\left\{-\frac{1}{\pi}\int_0^\infty \frac{G(\lambda, x+h)-G(\lambda, x-h)}{h}\,dh\right\} = \lim_{\lambda\to 0} F(\lambda, x) = f(x).$$

Hence we have only to prove that

$$\lim_{\lambda\to 0}\int_0^\infty \frac{G(\lambda, x+h)-G(\lambda, x-h)}{h}\,dh = \int_0^\infty \frac{g(x+h)-g(x-h)}{h}\,dh.$$

If the integrals were taken over (δ, ∞), where $\delta > 0$, this would follow at once from Lemmas β and ϵ. Hence we have only to prove that

$$\lim_{\lambda\to 0}\int_0^\delta \frac{G(\lambda, x+h)-G(\lambda, x-h)}{h}\,dh = \int_0^\delta \frac{g(x+h)-g(x-h)}{h}\,dh.$$

Now, by Lemma β,

$$|\,G(\lambda, x+h)-G(\lambda, x-h)-g(x+h)+g(x-h)\,| < A\lambda^\alpha,$$

and by Lemma δ and the Lipschitz condition for $g(x)$, the same expression is less than $Ah\lambda^{\alpha-1}+Ah^\alpha$. Hence it is also less than

$$(Ah\lambda^{\alpha-1}+Ah^\alpha)^\beta (A\lambda^\alpha)^{1-\beta} < Ah^\beta\lambda^{\alpha-\beta}+Ah^{\alpha\beta}\lambda^{\alpha-\alpha\beta}.$$

Taking $0 < \beta < \alpha$, it follows that

$$\left|\int_0^\delta \frac{G(\lambda, x+h)-G(\lambda, x-h)}{h}\,dh - \int_0^\delta \frac{g(x+h)-g(x-h)}{h}\,dh\right|$$

$$< A\lambda^{\alpha-\beta}\int_0^\delta h^{\beta-1}\,dh + A\lambda^{\alpha-\alpha\beta}\int_0^\delta h^{\alpha\beta-1}\,dh$$

which tends to zero with λ. This proves the theorem.

3 . 54. It only remains to prove that $g(x) = O(\log x)$. We have, if $x > 0$,

$$\pi g(x) = P\int_{-\infty}^{\infty} \frac{f(t+x)}{t}\,dt$$

$$= \left(\int_{-\infty}^{-2x} + \int_{-2x}^{-\delta} + P\int_{-\delta}^{\delta} + \int_{\delta}^{x} + \int_{x}^{\infty}\right)\frac{f(t+x)}{t}\,dt$$

$$= J_1+J_2+J_3+J_4+J_5.$$

Now $$J_1 = \int_{-\infty}^{-x} \frac{f(u)}{u-x}\,du = -\int_x^{\infty} \frac{u}{u+x}\,\frac{f(-u)}{u}\,du = o(1)$$

by the Second Mean Value Theorem. Similarly

$$J_5 = \int_{2x}^{\infty} \frac{f(u)}{u-x}\,du = \int_{2x}^{\infty} \frac{u}{u-x}\,\frac{f(u)}{u}\,du = o(1).$$

Also

$$J_4 = O\left(\int_{\delta}^{x} \frac{dt}{t}\right) = O(\log x),$$

$$J_2 = O\left(\int_{\delta}^{2x} \frac{dt}{t}\right) = O(\log x),$$

$$|J_3| = \left|\int_0^{\delta} \frac{f(x+t)-f(x-t)}{t}\,dt\right| < A\int_0^{\delta} t^{a-1}\,dt = O(1).$$

Combining the results we have the theorem stated.

I have also constructed an example which shows that it does not necessarily follow from our hypotheses that $g(x) = o(\log x)$; but it is too long to be inserted here.

ON A CLASS OF EXPANSIONS.

By J. Proudman.

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1. The expansions here considered were suggested in an investigation of applied mathematics which is recorded in another paper.*

The subjects of expansion are a null function and any function admitting a certain Sturm-Liouville expansion. In the latter case the general term of the new expansion tends asymptotically with increasing rank to that of the associated Sturm-Liouville expansion.

2. *Characteristic Functions.*

Suppose that $\eta(x)$ denotes a real function of x, given in the range $(0, 1)$, differentiable, and always greater than a definite positive number.

Consider the function $v(x)$, which is defined in the range of x $(0, 1)$ by the differential equation

$$\frac{d}{dx}\left\{\eta(x)\frac{d}{dx}v(x)\right\}+\lambda v(x)=-1, \tag{1}$$

λ denoting a parameter independent of x and possibly complex, together with the boundary conditions

$$v'(0)=0, \qquad v(1)=0, \tag{2}$$

where here and throughout the paper $f'(x)$ denotes $d/dx\,.\,f(x)$.

For any value of x in $(0, 1)$, $v(x)$ will be a meromorphic function of λ, and its mean value with respect to x, $M(\lambda)$, say, will also be a meromorphic function of λ.

* J. Proudman and A. T. Doodson, "Time-relations in Meteorological Effects on the Sea"; next paper in these *Proceedings*.

We have $$\int_0^1 v(x)\,dx = M(\lambda), \tag{3}$$

and if $v(x)$ becomes $\bar{v}(x)$ when λ becomes $\bar{\lambda}$, it is easily proved that

$$(\lambda-\bar{\lambda})\int_0^1 v(x)\bar{v}(x)\,dx = M(\lambda)-M(\bar{\lambda}) \tag{4}$$

and $$(\lambda-\bar{\lambda})\int_0^1 \eta(x)\,v'(x)\,\bar{v}'(x)\,dx = \lambda M(\lambda)-\bar{\lambda}M(\bar{\lambda}), \tag{5}$$

from which we deduce

$$\int_0^1 \{v(x)\}^2 dx = M'(\lambda) \tag{6}$$

and $$\int_0^1 \eta(x)\{v'(x)\}^2 dx = M(\lambda)+\lambda M'(\lambda). \tag{7}$$

It is easy to see that when $\lambda = 0$,

$$v(x) = \int_x^1 \frac{y\,dy}{\eta(y)}, \tag{8}$$

so that $$M(0) = \int_0^1 \frac{x^2\,dx}{\eta(x)}. \tag{9}$$

Next suppose that a denotes a prescribed real number, and consider those values of λ which satisfy the equation

$$M(\lambda) = a\lambda. \tag{10}$$

These values of λ we shall call the characteristic constants, and denote by (λ_s), while the corresponding functions $v(x)$ we shall denote by $\{v_s(x)\}$. These functions are the characteristic functions of the expansions to be considered.

We deduce $$\int_0^1 v_r(x)\,v_s(x)\,dx = a \tag{11}$$

and $$\int_0^1 \eta(x)\,v_r'(x)\,v_s'(x)\,dx = a(\lambda_r+\lambda_s), \tag{12}$$

when $\lambda_r \neq \lambda_s$, while

$$\int_0^1 \{v_s(x)\}^2\,dx = M'(\lambda_s) \tag{13}$$

and $$\int_0^1 \eta(x)\{v_s'(x)\}^2\,dx = M(\lambda_s)+\lambda_s M'(\lambda_s). \tag{14}$$

3. *The Expansions.*

Suppose for the present that it is possible to find a sequence of numbers (A_s) depending only on $\eta(x)$ and a so that

$$\left.\begin{aligned} \sum_s A_s v_s(x) &= 0, \\ \sum_s A_s &= 1, \end{aligned}\right\} \tag{15}$$

for all values of x in the range $(0, 1)$. Also, having given an arbitrary function $f(x)$, defined in the range $(0, 1)$, suppose that it is possible to find a sequence of numbers (B_s), so that

$$\left.\begin{aligned} \sum_s B_s v_s(x) &= f(x), \\ \sum_s B_s &= 0, \end{aligned}\right\} \tag{16}$$

for all values of x in the range.

On multiplying the first equation in both (15) and (16) by $v_r(x)$, integrating and using (11), (13), and the second equations in (15), (16), we immediately deduce

$$(1-A_r)a+A_rM'(\lambda_r) = 0,$$

$$-B_ra+B_rM'(\lambda_r) = \int_0^1 f(x)v_r(x)\,dx,$$

so that

$$A_s = \frac{a}{a-M'(\lambda_s)}, \tag{17}$$

$$B_s = \frac{-1}{a-M'(\lambda_s)} \int_0^1 f(x)v_s(x)\,dx. \tag{18}$$

We have to prove that the values (17), (18) make valid the expansions (15), (16).

4. *Auxiliary Functions.*

Let $V(x)$ denote the function defined in the range of x $(0, 1)$ by the differential equation

$$\frac{d}{dx}\{\eta(x)V'(x)\}+\lambda V(x) = 0, \tag{19}$$

with the conditions

$$V(0) = 1, \qquad V'(0) = 0. \tag{20}$$

For any value of x $V(x)$ and $V'(x)$ will be integral functions of λ.

We easily obtain

$$v(x) = \frac{1}{\lambda}\left\{\frac{V(x)}{V(1)} - 1\right\}, \tag{21}$$

and

$$M(\lambda) = -\frac{1}{\lambda} - \frac{1}{\lambda^2}\frac{\eta(1)V'(1)}{V(1)}. \tag{22}$$

We thus see that the infinities of $v(x)$ and $M(\lambda)$ will be the zeros of $V(1)$, and these values of λ, which are well known to be positive, distinct and infinite in number, we shall denote by

$$\nu_1, \nu_2, \ldots, \nu_s, \ldots,$$

when arranged in increasing order of magnitude. The corresponding functions $V(x)$ we shall denote by

$$V_1(x), V_2(x), \ldots, V_s(x), \ldots.$$

Also from (1)

$$\frac{d}{dx}\left\{\eta(x)\frac{d}{d\lambda}v'(x)\right\} + \lambda\frac{d}{d\lambda}v(x) = -v(x), \tag{23}$$

and on solving this equation by the ordinary method we obtain

$$\frac{d}{d\lambda}v(x) = V(x)\int_{y=x}^{1}\int_{z=0}^{y}\frac{v(z)V(z)}{\eta(y)V^2(y)}\,dy\,dz, \tag{24}$$

providing that $V(x)$ does not vanish anywhere for the value of λ taken. Since from (6)

$$M''(\lambda) = 2\int_0^1 v(x)\frac{d}{d\lambda}v(x)\,dx,$$

we deduce

$$M''(\lambda) = 2\int_{x=0}^{1}\frac{1}{\eta(x)V^2(x)}\left\{\int_{y=0}^{x}v(y)V(y)\,dy\right\}^2 dx. \tag{25}$$

For negative values of λ, $V(x)$ and $V'(x)$ are easily seen to be positive. For very large values of λ we have approximately

$$V(x) = \left\{\frac{\eta(0)}{\eta(x)}\right\}^{\frac{1}{4}}\cos\left(\lambda^{\frac{1}{2}}\int_0^x\frac{dy}{\{\eta(y)\}^{\frac{1}{2}}}\right), \tag{26}$$

and

$$M(\lambda) = -\frac{1}{\lambda} + \left\{\frac{\eta(1)}{\lambda^3}\right\}^{\frac{1}{2}}\tan\left(\lambda^{\frac{1}{2}}\int_0^1\frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\right), \tag{27}$$

the expressions (26) and (27) becoming accurate for all values of λ when $\eta(x)$ is constant.

The relations of this section are independent of α.

5. *Characteristic Constants.*

Let us first study the function $M(\lambda)$ for *real* values of λ.

For negative values of λ we know that $V(x)$ and $V'(x)$ are positive; it follows from (21) that $\eta(x)$ is positive and less than $-1/\lambda$. We deduce that

$$M(\lambda) \to 0 \quad \text{as} \quad \lambda \to -\infty . \tag{28}$$

Now (6) shows that $M'(\lambda)$ is positive for all real values of λ, and (22) shows that $M(\lambda)$ changes from $+\infty$ to $-\infty$ at

$$\nu_1, \nu_2, \ldots, \nu_s, \ldots .$$

Also (25) shows that $M''(\lambda)$ is positive for all values of $\lambda < \nu_1$.

Next let us consider the real roots of the equation

$$M(\lambda) = a\lambda. \tag{10}$$

For negative values of a there will be one negative root, one root between each consecutive pair of values of ν_s, and no other root.

For positive values of a there will be no negative root, not more than two roots between 0 and ν_1, and at least one root between each consecutive pair of values of ν_s. For simplicity we shall restrict our consideration to the case in which there can only be one root lying between a consecutive pair of values of ν_s. The necessary and sufficient condition for this is that the tangent to the graph of $M(\lambda)$ shall cut the λ-axis to the right of the origin, or that

$$\lambda M'(\lambda) > M(\lambda), \quad (\lambda > \nu_1). \tag{29}$$

We may note that this condition is satisfied when $\eta(x)$ is constant, for which special case with $\eta = 1$ the graph of Figure 1 is drawn.

As a increases from $-\infty$ to $+\infty$, the root of (10) lying between ν_s and ν_{s+1} will increase steadily from ν_s to ν_{s+1}.

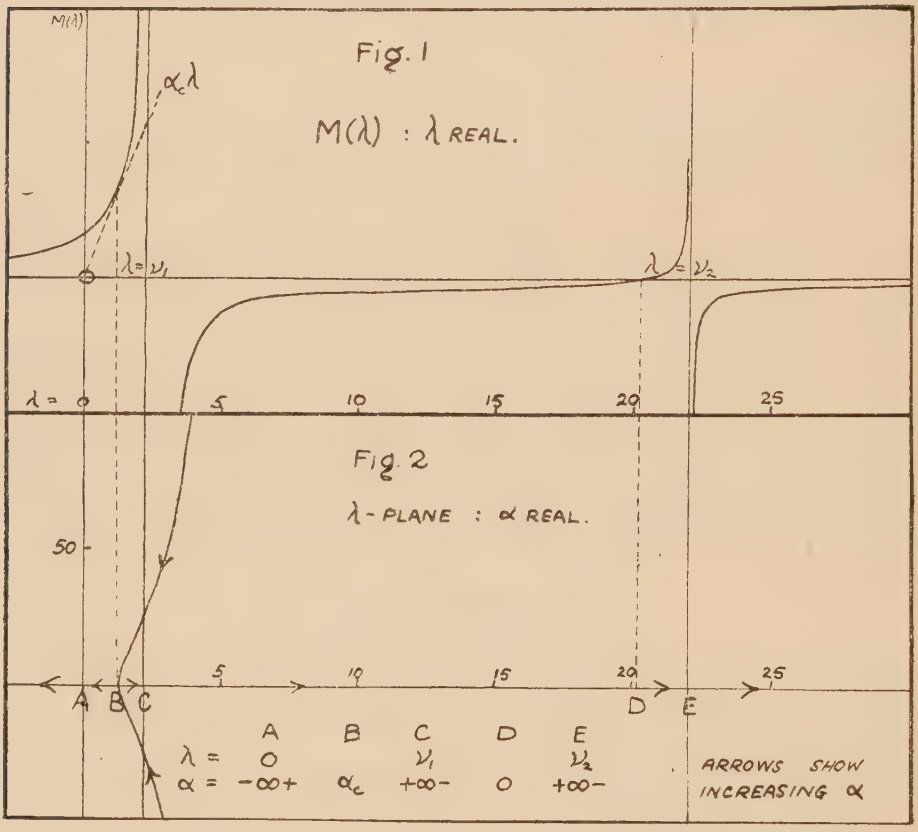
For values of a less than a certain critical value, a_c say, there will be no root lying between 0 and ν_1, while for values of a greater than a_c there will be two roots lying between 0 and ν_1, and as $a \to \infty$ one of these roots will tend to zero and the other to ν_1.

Next let us consider *complex* characteristic constants. These will occur in conjugate pairs, and the corresponding characteristic functions will also be conjugate. We see from (11), by supposing λ_r and λ_s to be conjugate, that such complex constants can only arise when a is positive, and then from (12) that the real part of a complex constant must be positive.

On writing (10) in the form

$$a = \frac{M(\lambda)}{\lambda}, \tag{30}$$

let us regard a for a moment as a function (meromorphic) of the complex variable λ. We require those curves on the λ-plane which make a real. The real axis is one of these curves, and Figure 2 shows the corresponding values of a. Also, we see from (27) that the point at infinity on the



imaginary axis makes $a = 0$. We next examine how other curves may cut the real axis.

We have approximately

$$\delta a = \frac{da}{d\lambda}\,\delta\lambda + \tfrac{1}{2}\,\frac{d^2a}{d\lambda^2}\,(\delta\lambda)^2,$$

and since $da/d\lambda$ and $d^2a/d\lambda^2$ are real on the real axis, we see that the intersections required can only occur where $da/d\lambda = 0$, as a little examination

shows that none can occur at the infinities of α. Now α_c is a minimum of α on the real axis, and consequently $d\alpha/d\lambda = 0$ there, so that corresponding to this value of α a curve of real values of α will cut the real axis of λ at right angles. This curve will be symmetrical about the real axis, and presumably will have an infinite branch along which α will vary from α_c to 0. This has been verified for $\eta =$ constant, and the curve is shown in Figure 2 for $\eta = 1$.

For values of α between 0 and α_c there will thus be two conjugate complex characteristic constants.

6. *Convergence.*

For very large values of λ we have approximately

$$\left\{\frac{\eta(1)}{\lambda_s^3}\right\}^{\frac{1}{2}} \tan\left(\lambda_s^{\frac{1}{2}}\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\right) = \alpha\lambda_s, \tag{31}$$

so that

$$\lambda_s = (s-\tfrac{1}{2})^2\pi^2 \Big/ \left(\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\right)^2. \tag{32}$$

Also

$$M'(\lambda_s) = \tfrac{1}{2}\,\frac{\{\eta(1)\}^{\frac{1}{2}}}{\lambda_s^2}\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\,\tan^2\left(\lambda_s^{\frac{1}{2}}\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\right)$$

$$= \tfrac{1}{2}\,\frac{\alpha^2}{\{\eta(1)\}^{\frac{1}{2}}}\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\,.\,\lambda_s^3 \tag{33}$$

approximately, while when $\lambda = \lambda_s$

$$\lambda\,|\,V(1)\,| = \lambda\left|\left\{\frac{\eta(0)}{\eta(1)}\right\}^{\frac{1}{4}}\cos\left(\lambda^{\frac{1}{2}}\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}\right)\right|$$

$$= \left|\frac{\{\eta(0)\,\eta(1)\}^{\frac{1}{4}}}{\alpha\lambda_s^{3/2}}\right| \tag{34}$$

approximately.

From (32), (33), and (34) we see that the series (15) and the second series in (16) are absolutely convergent. The general term of the first series in (16) tends asymptotically to the form

$$\frac{2}{\{\eta(1)\}^{\frac{1}{2}}\displaystyle\int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}}\int_0^1 f(x)\,V_s(x)\,dx\,.\,V_s(x). \tag{35}$$

We shall see in § 8 that (35) is ultimately the general term of a Sturm-Liouville expansion of $f(x)$, and we shall suppose $f(x)$ such that this expansion is convergent.

7. *Contour Integrals.*

Consider the integrals

$$I_1 = \int \frac{a d\lambda}{a\lambda - M(\lambda)}, \tag{36}$$

$$I_2 = \int \frac{a v(x)\, d\lambda}{a\lambda - M(\lambda)}, \tag{37}$$

$$I_3 = \int \left\{ - \int_0^1 \frac{f(x)\, v(x)\, dx}{a\lambda - M(\lambda)} \right\} d\lambda, \tag{38}$$

$$I_4 = \int \left\{ - \int_0^1 \frac{f(x)\, v(x)\, dx}{a\lambda - M(\lambda)} \right\} v(x)\, d\lambda, \tag{39}$$

each taken round the circle

$$|\lambda|^{\frac{1}{2}} = n\pi \Big/ \int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}, \tag{40}$$

n being a large integer. We easily see that as $n \to \infty$

$$I_1 \to 2\pi i, \tag{41}$$

$$I_2,\ I_3,\ I_4 \to 0. \tag{42}$$

Now the integrand of I_1 has for its residues the terms of the second series in (15) with (17), and therefore Cauchy's theorem proves the validity of this formula.

In I_2 and I_3 it is easily seen that the poles of $v(x)$, viz. (ν_s), do not give poles of the integrand, as they occur also in $M(\lambda)$. It therefore follows that these integrands have for their residues respectively the terms of the first series in (15) with (17), and the terms of the second series in (16) with (18). Cauchy's theorem therefore establishes these formulae.

The integrand of I_4, however, has two sets of poles, viz. (λ_s) and (ν_s). The residues corresponding to (λ_s) are the terms of the first series in (16), while that corresponding to ν_s is

$$\left[\int_0^1 f(x) V_s(x)\, dx \Big/ \eta(1) \left\{ V'(1) \frac{d}{d\lambda} V(1) \right\}_{\lambda = \nu_s} \right] . V_s(x) \tag{43}$$

with opposite sign. Cauchy's theorem therefore proves that the first series in (16) with (18) is equivalent to the sum of the terms (43) for all values of s. We shall show in the next section that this series has for its sum $f(x)$.

8. *Associated Sturm-Liouville Expansions.*

The functions $\{V_s(x)\}$ form a system of Sturm-Liouville functions for the range $(0, 1)$.

It is known that, with certain restrictions on $f(x)$, which we shall suppose satisfied, this function may be expanded as a series of the functions $V_s(x)$, and in fact we have

$$f(x) = \sum_s \left[\int_0^1 f(x) V_s(x)\, dx \Big/ \int_0^1 \{V_s(x)\}^2 dx \right] V_s(x). \tag{44}$$

Now it is easily shown that

$$\int_0^1 \{V_s(x)\}^2 dx = \eta(1) \left\{ V'(1) \frac{d}{d\lambda} V(1) \right\}_{\lambda=\nu_s}, \tag{45}$$

and therefore the general term of (44) is equivalent to (43). This completes the proof of the validity of the expansions.

To justify the remark made at the end of § 6, we note that as $\nu_s \to \infty$, (45) tends to

$$\tfrac{1}{2} \{\eta(1)\}^{\frac{1}{2}} \int_0^1 \frac{dx}{\{\eta(x)\}^{\frac{1}{2}}}, \tag{46}$$

the reciprocal of the coefficient in (35).

TIME-RELATIONS IN METEOROLOGICAL EFFECTS ON THE SEA

By J. PROUDMAN *and* A. T. DOODSON.

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1. *Introduction.*

Researches by one of the authors (A.T.D.) into the correlation between the unpredicted sea-level at Liverpool and the neighbouring barometric distribution led to a large measure of success when this distribution did not change too rapidly. But a preliminary study of the effects of storms seemed to show that, when the distribution changed suddenly, the effects on the sea-level followed very rapidly and reached magnitudes greater than if the change had taken place slowly, oscillations of considerable amplitude being often generated.

The only theoretical investigations into the response of the sea to atmospheric conditions which appear to be generally known relate primarily to a steady state,* though H. Jeffreys has estimated the order of magnitude of the time necessary to establish a steady state.

The problem of seeking the time-relations involved in a changing state was therefore suggested, and led to the present paper.

These time-relations take their purest form in the decay of an initial disturbance of the sea, and in this form their investigation does not explicitly involve the atmospheric conditions at all. The problems of decay are considered in § 5, and then their solution is utilised in giving that of the establishment of a steady state in § 6, and that of the general variable state in § 7.

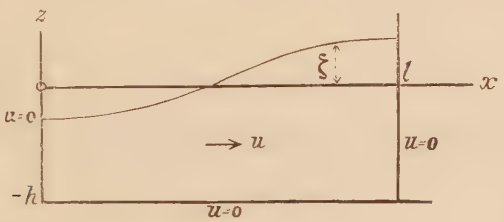
* V. W. Ekman "On the Influence of the Earth's Rotation on Ocean Currents", *Arkiv för Mat.*, 2 (1905). H. Jeffreys "On Turbulence in the Ocean", *Phil. Mag.*, 39 (1920), 578–586; "The Effect of a Steady Wind on the Sea-Level near a Straight Shore", *Phil. Mag.*, 46 (1923), 114–125.

The basin and the type of motion considered are the simplest conceivable for the purpose in hand, and, as the problem is essentially one of turbulent motion, the mathematical work is intended to represent qualitatively the chief features of the case, rather than to produce formulae which can be interpreted numerically in actual circumstances.

The order of magnitude of the establishment-time obtained is probably of less value than the estimate of Jeffreys, as, owing to the more complicated mathematics of the present investigation, an inferior form of the condition to be satisfied at the sea-bottom has been used. But the point of the present paper is the investigation of the *law* of change.

2. *Fundamental Equations.*

Using Cartesian coordinates, the basin is taken to be a rectangular one bounded by $x = 0$, $x = l$, $z = -h$, with the mean free surface at $z = 0$, and the motion is supposed to be entirely in the plane of x, z. Time



is denoted by t, and the elevation of the free surface by ξ, which will be a function of x and t. The current, assumed to be sensibly parallel to the x-axis, is denoted by u, which will be a function of x, z, and t.

The equation of continuity takes the form

$$\frac{\partial}{\partial x}\int_{-h}^{0} u\,dz + \frac{\partial \xi}{\partial t} = 0, \tag{1}$$

and the dynamical equation

$$\frac{\partial u}{\partial t} = -g\frac{\partial}{\partial x}(\xi - \bar{\xi}) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right). \tag{2}$$

Here $-g\rho\bar{\xi}$ denotes the variable part of the atmospheric pressure, g being the acceleration of gravity and ρ the density of the water, and $\bar{\xi}$ is supposed to be prescribed. Also $\rho\mu\partial u/\partial z$ denotes the frictional force

per horizontal area exerted by the water above the area on that below. We shall suppose μ to be constant, but the modifications necessary when μ is a function of z are indicated in the paper* giving the mathematical results quoted below, which paper we shall refer to as (E).

The boundary conditions are taken to be

$$u = 0, \text{ for } x = 0, \quad x = l, \quad z = -h, \tag{3}$$

together with a prescription of $\partial u/\partial z$ at $z = 0$, as representing the effect of the wind.

We shall assume that x enters ζ only through the factor $\cos \kappa x$ and u only through the factor $\sin \kappa x$. These assumptions are consistent with (1) and (2), and satisfy two of the conditions (3) if $\kappa l/\pi$ is taken to be an integer. The general case can be built up by the superposition of these particular cases with different values of κ. In what follows we shall use the constant κ instead of l.

We shall also group symbols so as to render each group of zero dimensions.

3. *Steady State.*

Suppose that ζ and u are independent of t, and at $z = 0$,

$$\left.\begin{aligned} \kappa\bar{\zeta} &= -P \cos \kappa x \\ \frac{\mu}{gh}\frac{\partial u}{\partial z} &= W \sin \kappa x \end{aligned}\right\} \tag{4}$$

P and W being constants; then it is easily verified that the solution of the problem of the preceding section is given by

$$\left.\begin{aligned} \kappa\zeta &= -(P + \tfrac{3}{2}W) \cos \kappa x \\ \frac{\mu}{gh^2}u &= \tfrac{3}{4}\left(\frac{z}{h} + 1\right)\left(\frac{z}{h} + \tfrac{1}{3}\right) W \sin \kappa x \end{aligned}\right\} \tag{5}$$

for all values of x, z, t; the solution satisfies

$$\int_{-h}^{0} u\,dz = 0.$$

* J. Proudman "On a Class of Expansions"; preceding paper in these *Proceedings*.

4. *Free Variable State.*

Now suppose that

$$\bar{\zeta} = 0, \quad \frac{\partial u}{\partial z} = 0 \tag{6}$$

at $z = 0$, so that there is no atmospheric disturbance.

Consider the elemental solution of the equations of § 2 of the form

$$\left.\begin{aligned} \kappa\zeta &= \cos \kappa x \,.\, e^{-\lambda\mu t/h^2} \\ \frac{\mu}{gh^2}\, u &= \sin \kappa x \,.\, v(z) \,.\, e^{-\lambda\mu t/h^2} \end{aligned}\right\} \tag{7}$$

λ denoting a numerical parameter.

The equation of continuity (1) gives

$$\frac{1}{h}\int_{-h}^{0} v(z)\,dz = a\lambda, \tag{8}$$

where

$$a = \frac{\mu^2}{g\kappa^2 h^5}, \tag{9}$$

while the dynamical equation (2) gives

$$h^2 \frac{d^2}{dz^2} v(z) + \lambda v(z) = -1. \tag{10}$$

We then see that

$$v(z) = \frac{1}{\lambda}\left\{\frac{\cos(\lambda^{\frac{1}{2}} z/h)}{\cos \lambda^{\frac{1}{2}}} - 1\right\}, \tag{11}$$

which yields

$$\frac{1}{h}\int_{-h}^{0} v(z)\,dz = \frac{1}{\lambda}\left(\frac{\tan \lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} - 1\right), \tag{12}$$

so that (8) becomes

$$\tan \lambda^{\frac{1}{2}} = \lambda^{\frac{1}{2}} + a\lambda^{\frac{3}{2}}. \tag{13}$$

The solutions of this equation are investigated in (E): they are infinite in number, and two only may be complex. Denoting the solutions by (λ_s), with $v_s(z)$ as the corresponding values of $v(z)$, we proceed to examine the solutions for free motion given by

$$\left.\begin{aligned} \kappa\zeta &= \cos \kappa x \sum_s C_s e^{-\lambda_s \mu t/h^2} \\ \frac{\mu}{gh^2}\, u &= \sin \kappa x \sum_s C_s v_s(z)\, e^{-\lambda_s \mu t/h^2} \end{aligned}\right\} \tag{14}$$

(C_s) being a sequence of constants.

5. Free Decay of Initial Disturbance.

Atmospheric disturbances being absent we again have the conditions (6). Let us suppose that, at $t = 0$,

$$\left.\begin{aligned} \kappa\zeta &= A\cos\kappa x \\ \frac{\mu}{gh^2}u &= \sin\kappa x\,.f(z) \end{aligned}\right\} \qquad (15)$$

where $f(z)$ is a function of z defined from $z = -h$ to 0, and subject only to such restrictions as are placed upon $f(x)$ in (E).

The general solution is given by (14), provided that we can satisfy

$$\left.\begin{aligned} \sum_s C_s &= A \\ \sum_s C_s v_s(z) &= f(z) \end{aligned}\right\} \qquad (16)$$

for all values of z between $-h$ and 0.

It is readily verified that

$$\left.\begin{aligned} \frac{1}{h}\int_{-h}^{0} v_r v_s\,dz &= a \\ \frac{1}{h}\int_{-h}^{0} v_s^2\,dz &= \tfrac{1}{2}\left(\frac{\tan^2\lambda^{\frac{1}{2}}}{\lambda^2} - 3a\right) \end{aligned}\right\} \qquad (17)$$

If, therefore, we multiply both sides of the second equation of (16) by v_s, and take the depth-means, we obtain

$$C_s = \left\{-2aA + \frac{2}{h}\int_{-h}^{0} f(z)\,v_s(z)\,dz\right\} \Big/ \left(\frac{\tan\lambda^{\frac{1}{2}}}{\lambda}\right)^2 - 5a \qquad (18)$$

The validity of the solution is vindicated in (E), where it is also shown that the solution can be stated in terms of the contour-integrals

$$\kappa\zeta = \frac{\cos\kappa x}{2\pi i}\int\left[\left\{-Aa + \frac{1}{h}\int_{-h}^{0} f(z)\,v(z)\,dz\right\}\frac{\lambda^2 e^{-\lambda\mu t/h^2}}{\lambda^{\frac{1}{2}}\tan\lambda^{\frac{1}{2}} - \lambda - a\lambda^3}\right]d\lambda,$$

$$\frac{\mu}{gh^2}u = \frac{\sin\kappa x}{2\pi i}\int\left\{-Aa + \frac{1}{h}\int_{-h}^{0} f(z)\,v(z)\,dz\right\}$$

$$\times\left\{\frac{1}{\lambda^{\frac{1}{2}}\tan\lambda^{\frac{1}{2}} - \lambda - a\lambda^3} - \frac{\sin\lambda^{\frac{1}{2}}\cos\lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right\}v(z)\,\lambda^2 e^{-\lambda\mu t/h^2}\,d\lambda,$$

taken round contours in the λ-plane which tend to enclose all the values λ_s. All the values of λ_s have their real parts positive, so that a portion of the contour may be taken as the imaginary axis. When this is done the solution is expressed in Fourier-integrals.

6. *Establishment of Steady State.*

We now proceed to consider the law of establishment of the ultimate state produced by atmospheric disturbance, including wind, suddenly created at $t = 0$ and then maintained constant. Accordingly we take, as in § 3,

$$\left.\begin{aligned} \kappa\bar{\zeta} &= -P\cos\kappa x \\ \frac{\mu}{gh}\frac{\partial u}{\partial z} &= W\sin\kappa x \end{aligned}\right\} \tag{4}$$

where P and W are constants. The conditions of the problems are satisfied by

$$\left.\begin{aligned} \kappa\zeta &= -\cos\kappa x\left\{P+\tfrac{3}{2}W+\sum_s C_s e^{-\lambda_s\mu t/h^2}\right\} \\ \frac{\mu}{gh^2}u &= \sin\kappa x\left\{\tfrac{3}{4}\left(\frac{z}{h}+1\right)\left(\frac{z}{h}+\tfrac{1}{3}\right)W-\sum_s C_s v_s(z)\,e^{-\lambda_s\mu t/h^2}\right\} \end{aligned}\right\} \tag{19}$$

if we write in (18)

$$\left.\begin{aligned} A &= -P-\tfrac{3}{2}W \\ f(z) &= \tfrac{3}{4}\left(\frac{z}{h}+1\right)\left(\frac{z}{h}+\tfrac{1}{3}\right)W \end{aligned}\right\} \tag{20}$$

and

$$\frac{1}{h}\int_{-h}^{0} f(z)\,v_s(z)\,dz = \left(-\tfrac{3}{2}a+\frac{\sec\lambda_s^{\frac{1}{2}}-1}{\lambda_s^2}\right)W$$

7. *General Variable State.*

The atmospheric disturbance (4) suddenly created at time $t = \tau$ and then maintained constant will contribute to $-\kappa\zeta\sec\kappa x$ the amount

$$P+\tfrac{3}{2}W+\sum_s C_s e^{-\lambda_s\mu(t-\tau)/h^2}$$

for $t > \tau$, while an equal and opposite disturbance suddenly created at

time $\tau+\delta\tau$, will contribute

$$-P-\tfrac{3}{2}W-\sum_s C_s e^{-\lambda_s\mu(t-\tau-\delta\tau)/h^2}$$

for $t>\tau+\delta\tau$. The joint contribution at time $t(>\tau+\delta\tau)$ is

$$-\sum_s C_s\frac{d}{d\tau}e^{-\lambda_s\mu(t-\tau)/h^2}\delta\tau. \tag{21}$$

The coefficients C_s, being given by (18) and (20), are linear functions of P and W, and if these are now regarded as functions of τ, the complete solution is given by integrating (21) from $\tau=-\infty$ to t. If we write

$$C_s'=\frac{d}{d\tau}C_s, \tag{22}$$

and carry out by parts the integration indicated, we obtain the first of the following expressions. But that these expressions give the complete solution is easily verified by direct substitution into the equations of § 2.

$$\left.\begin{aligned}\kappa\zeta &= -\cos\kappa x\left\{P(t)+\tfrac{3}{2}W(t)+\sum_s\int_{-\infty}^{t}C_s'(\tau)\,e^{-\lambda_s\mu(t-\tau)/h^2}d\tau\right\}\\ \frac{\mu}{gh^2}u &= \sin\kappa x\left\{\tfrac{3}{4}\left(\frac{z}{h}+1\right)\left(\frac{z}{h}+\tfrac{1}{3}\right)W(t)\right.\\ &\qquad\left.-\sum_s\int_{-\infty}^{t}C_s'(\tau)\,v_s(z)\,e^{-\lambda_s\mu(t-\tau)/h^2}d\tau\right\}\end{aligned}\right\} \tag{23}$$

8. *Numerical Solution of the Equation between* λ *and* α.

The fundamental relation between λ and α is given by

$$\tan\lambda^{\frac{1}{2}}=\lambda^{\frac{1}{2}}+a\lambda^{\frac{3}{2}}, \tag{13}$$

and if α is regarded as a function of λ it is a minimum when

$$\tan^2\lambda^{\frac{1}{2}}=5a\lambda^2,$$

whence the corresponding value of λ is given by

$$\tfrac{1}{5}\tan^2\lambda^{\frac{1}{2}}-\frac{\tan\lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}+1=0.$$

By trial we find $\lambda^{\frac{1}{2}}=1{\cdot}1122$ and $a=0{\cdot}5370.$ (24)

Consequently if α is less than 0·537, there are no real roots of the equation for $\lambda^{\frac{1}{2}}$ less than $\pi/2$, but there are then, however, two conjugate complex roots. For all values of α real roots exist which are approximately equal to $r\pi/2$ ($r = 3, 5, 7, \ldots$), and unless α is very small these roots are obtained very approximately by taking

$$\left.\begin{aligned} \lambda^{\frac{1}{2}} &= r\frac{\pi}{2} - \theta \\ \frac{1}{\theta} &= r\frac{\pi}{2} + \alpha\left(r\frac{\pi}{2}\right)^5 \end{aligned}\right\} \quad (r = 3, 5, 7, \ldots),$$

The smallest real root ($\lambda^{\frac{1}{2}} < \pi/2$) for $\alpha > 0{\cdot}537$ must be found by trial.

For the complex roots ($\alpha < 0{\cdot}537$) we take

$$\lambda^{\frac{1}{2}} = \xi \pm i\eta, \quad \alpha = a \pm ib,$$

where ξ, η, a, b are real, and with the upper sign

$$a = \frac{(t - \xi - \eta tT) + i(T - \eta + \xi tT)}{(1 - itT)(\xi + 1\eta)^5}, \tag{25}$$

where $t = \tan \xi$, $T = \tanh \eta$. We require a numerical relation between ξ, η, and α such that $b = 0$. The method of calculation of ξ and η for $b = 0$ was to fix η say, and to evaluate α for a sequence of values of ξ, ultimately taken at small intervals so that linear interpolation for $b = 0$ could be resorted to. The results, including the cases $\xi = \frac{1}{2}\pi$ and $\xi = \pi$, are given in the following table :—

η	ξ	α	$\xi^2 - \eta^2$	$2\xi\eta$
0·0	1·1122	·5370	1·237	0·000
0·2	1·1307	·4732	1·239	0·452
0·4	1·1837	·3376	1·241	0·947
0·6	1·2670	·2132	1·245	1·520
0·8	1·3757	·1284	1·253	2·201
1·0935	1·5708	·0615	1·272	3·435
1·5	1·8916	·02440	1·328	5·675
2·0	2·3362	·00936	1·458	9·345
2·8368	3·1416	·00268	1·822	17·824

It can be shown that the asymptotes of the ξ, η curve are given by

$$\xi = \pm\eta + \tfrac{1}{4}.$$

9. *Numerical Illustrations.*

Corresponding to $\lambda_s = (\xi^2-\eta^2) \pm 2i\xi\eta$, we have, say, $C_s = p \pm iq$, and the resulting contribution to $\kappa\zeta$ is

$$-\cos \kappa x \, e^{-(\xi^2-\eta^2)\mu t/h^2} R \cos (2\xi\eta\mu t/h^2 - \epsilon), \tag{26}$$

where $R = 2\sqrt{(p^2+q^2)}$ and $\tan \epsilon = q/p$.

The cases we shall illustrate are the establishment of a steady state with $P = 1$, $W = 0$, and $P = 0$, $W = 1$ (Figure 1), and two examples of variable states with $P = 0$ (Figure 2).

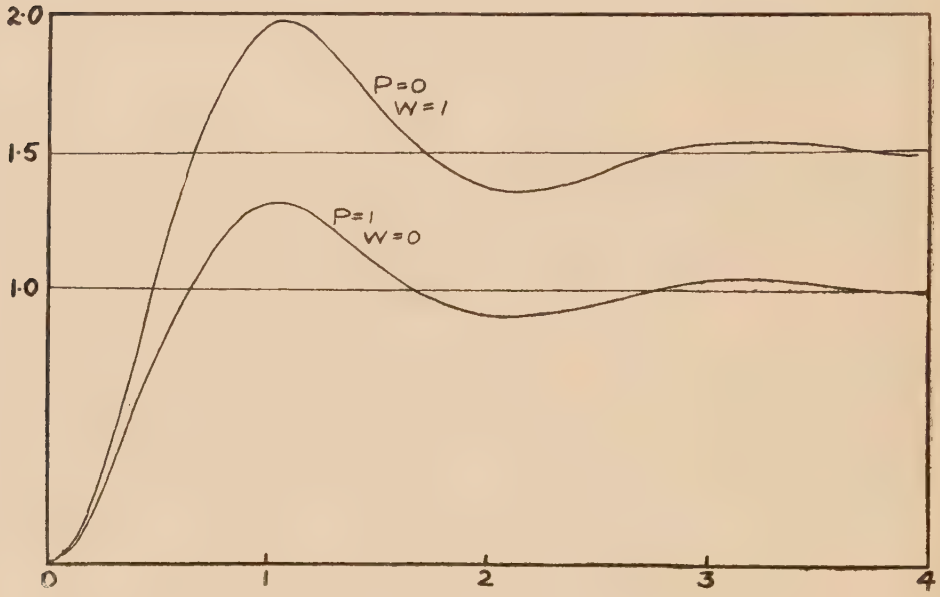


Fig. 1.

Examples of the approach of the elevation at any point to that of a steady state. The ordinates give $-\kappa\zeta \sec \kappa x$. The abscissal unit is the time $\cdot 864 h^2/\mu$.

Taking $\xi = 1{\cdot}5708$, $\eta = 1{\cdot}0935$, $\alpha = {\cdot}0615$, we obtain, from § 6,

$$-\kappa\zeta \sec \kappa x$$

$$= 1 - 1{\cdot}062 \cos (3{\cdot}435\mu t/h^2 - 19^\circ{\cdot}2) e^{-1{\cdot}272\mu t/h^2} + {\cdot}003 e^{-22{\cdot}14\mu t/h^2} + \ldots \tag{27}$$

for $P = 1$, $W = 0$, and

$$-\kappa\zeta \sec \kappa x$$

$$= \tfrac{3}{2} - 1{\cdot}613 \cos (3{\cdot}435\mu t/h^2 - 22^\circ{\cdot}8) e^{-1{\cdot}272\mu t/h^2} - {\cdot}014 e^{-22{\cdot}14\mu t/h^2} + \ldots \tag{28}$$

for $P = 0$, $W = 1$.

Possible basins with corresponding time-units are given by

h	π/κ	μ	time-unit
metres	kilometres	cm.^{2}/sec.	hours
20	100	436	2·2
20	200	218	4·4
40	200	1234	3·1
40	400	617	6·2

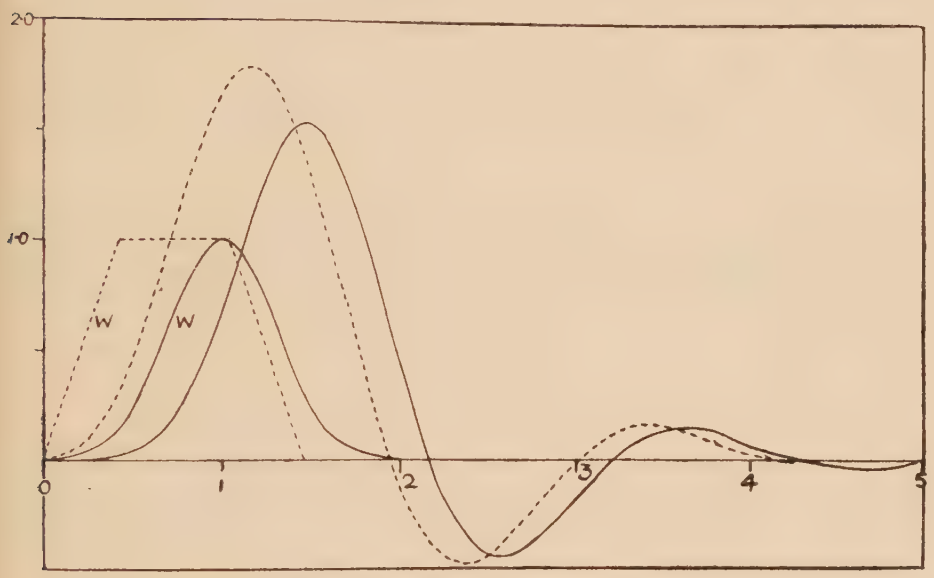


Fig. 2: Examples of variable state.

The curves marked W give surface values of $\frac{\mu}{gh}\frac{\partial u}{\partial z}\operatorname{cosec}\kappa x$. The corresponding full or dotted line gives the corresponding values of $-\kappa\zeta\sec\kappa x$. The abscissal unit is the time $\cdot 864h^2/\mu$.

FUNCTIONS DISCONTINUOUS AT RATIONAL POINTS

By J. L. BURCHNALL *and* T. W. CHAUNDY.

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1. Examples of functions which are continuous at all irrational points and discontinuous at all rational points are well known. Such, for instance, is the "denominator" function defined by the relations :

(1) x rational (p/q) $\qquad f(x) = 1/q,$

x irrational $\qquad f(x) = 0.$

The method described in this paper, which leads almost invariably to functions of this type, depends essentially on the expression of the argument x, now assumed for convenience to belong to the interval $(0, 1)$, in terms of a sequence

$$(a_1, a_2, \ldots, a_n, \ldots)$$

composed of positive integers or zeros.

For illustration we consider in especial four well-recognized modes of representation of x of this character :

Decimal fractions

(2) $$x = 0{\cdot}a_1 a_2 \ldots a_n \ldots \quad \text{where} \quad 0 \leqslant a_n \leqslant 9,$$

Continued fractions

(3) $$x = \frac{1}{a_1+}\frac{1}{a_2+}\cdots+\frac{1}{a_n+}\cdots \quad \text{where} \quad 1 \leqslant a_n,$$

(4) or $$x = 1-\frac{1}{a_1-}\frac{1}{a_2-}\cdots-\frac{1}{a_n-}\cdots \quad \text{where} \quad 2 \leqslant a_n,$$

Cantor's quasi-exponential series

$$x = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_n}{n!} + \ldots \quad \text{where} \quad 0 \leqslant a_n < n, \tag{5}$$

Cantor's product

$$x = -1 + \left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right)\ldots\left(1 + \frac{1}{a_n}\right)\ldots \quad \text{where} \quad 1 < a_1, \quad a_n^2 \leqslant a_{n+1}. \tag{6}$$

Now in the first place a *finite* ordered set of integers $(a_1, a_2, \ldots, a_n)$ defines, by way of the continued fraction

$$\frac{1}{a_1 +}\,\frac{1}{a_2 +}\cdots + \frac{1}{a_n},$$

a rational number, and the aggregate of such finite ordered sets is therefore countable. Schemes of representation can be devised to yield only a finite number or even none of these finite sets. If, for instance, $x = 0{\cdot}a_1a_2a_3a_4\ldots$, we may also write x as

$$(1+a_1)\,10^{-1} - (10-a_2)\,10^{-2} + (1+a_3)\,10^{-3} - (10-a_4)\,10^{-4} + \ldots$$

and define it by the sequence

$$(1+a_1,\ 10-a_2,\ 1+a_3,\ 10-a_4,\ \ldots) \tag{7}$$

in which none of the defining integers is zero, and the sequence is non-terminating.

We now order the aggregate of x's in $(0, 1)$ on "alphabetical" principles. That is to say, we first group together all those x's having the same a_1, and then order these groups, as wholes, monotonically among themselves according to the relative values of a_1. We then order the members of each group in similar fashion within the group according to their second integers a_2, and so on endlessly. This procedure in the examples considered suffices to achieve the monotonic ordering of the x's.

It follows that all those x's, for which $a_1 = b$, compose a complete interval (α, β), and all those, for which $a_1 = b+1$, compose a complete interval (α', β'), consecutive to the former. Since no x in $(0, 1)$ is omitted from the survey, these consecutive intervals are necessarily contiguous, *i.e.* $\beta = \alpha'$ (or $\alpha = \beta'$). Such a number β, α' has therefore two expressions of the form considered, namely $(b, \ldots)$ and $(b+1, \ldots)$.

A like argument applies to any n-th integer a_n. The "last" number for which $a_n = b$ is also the "first" number for which $a_n = b+1$, and consequently possesses alternative expressions. The existence of such "switch-points" is fundamental in our theory. When, at a switch-point, a_n suddenly changes from b to $b+1$, all the subsequent integers a_{n+r} change also, but the earlier integers are not affected. The aggregate of all such switch-points is thus equally numerous with the aggregate of terminating sequences of positive integers, and is therefore countable.

For the chosen forms (2) to (6) we may indicate by simple examples how equivalent representations can arise :

for (2) $$0{\cdot}a999\ldots = 0{\cdot}\overline{a+1}000\ldots,$$

for (3) $$\frac{1}{a+}\,\frac{1}{1} = \frac{1}{a+1},$$

for (4) $$\frac{1}{a} = \frac{1}{a+1-}\,\frac{1}{1},$$

for (5) $$\frac{a}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots \text{ to } \infty = \frac{a+1}{2!},$$

for (6) $$(1+a^{-1})(1+a^{-2})(1+a^{-4})\ldots \text{ to } \infty = 1+(a-1)^{-1}.$$

In each of the above examples it happens that the switch-points occur at rational numbers. This is adventitious, due to the particular form of representation chosen, as is seen by considering the representation which may be devised by use of an arbitrary conditionally-convergent series. Such a series is, we know, the convergent difference of two divergent positive series $\{s_n\}$, $\{t_n\}$, and the series, we also know, can be rearranged to converge to an arbitrary x by the following procedure :—

Choose n such that

$$s_{n-1} < x < s_n.$$

Then similarly choose m such that

$$t_{m-1} < s_n - x < t_m.$$

Again choose n' such that

$$s_{n+n'-1} < x + t_m < s_{n+n},$$

and so on.

The sequence of positive integers $(n, m, n', \ldots)$ and x correspond uniquely, except when x is such that any one of the above inequalities could be replaced by an equality. These values of x are the switch-points, and are equal to certain partial sums of the rearranged series. The series is arbitrary, and can certainly be chosen so that every partial sum is irrational.

It is also adventitious that the switch-points in the examples given correspond in general to terminating sequences, as appears from the contrary example (7) above.

A further characteristic of sequences of the type we are considering is that two x's, which agree in the first n integers $a_1, a_2, \ldots, a_n$ of their representations, differ at most by some ϵ_n, where $\epsilon_n \to 0$ as $n \to \infty$.

Now define $$f(x) \equiv (b_1, b_2, \ldots, b_n, \ldots),$$

where (α) $$b_n = \phi_n(a_n),$$

or (β) $$b_n = \phi_n(a_1, a_2, \ldots, a_n),$$

or (γ) more generally $$b_n = \phi_n(a_1, a_2, \ldots, a_m),$$

where m is bounded when n is bounded.

Suppose first that x is not a switch-point. Given ϵ, we can choose $n(\epsilon)$ so big that all the numbers $(b_1, b_2, \ldots, b_n, b'_{n+1}, b'_{n+2}, \ldots)$ for given $b_1, \ldots, b_n$, and *any* $b'_{n+1}, b'_{n+2}, \ldots$ differ by less than ϵ. By (γ) we can choose $m(n)$ so large that $b_1, b_2, \ldots, b_n$ are known, when $a_1, a_2, \ldots, a_m$ are known. Since x is not a switch-point, we can choose $\eta(m)$ so small that every number lying between $x \pm \eta$ begins with the same $a_1, a_2, \ldots, a_m$. Thus the inequality $|\delta x| < \eta$ secures the inequality $|\delta f(x)| < \epsilon$ and the continuity of $f(x)$ is established at all points not switch-points.

At a switch-point x has two representations which are, in general, the limiting forms of the representations of x on the right and on the left respectively.

These two representations may in exceptional cases lead to the same b-sequence. Still more exceptionally they may lead to distinct b-sequences representing the same switch-point of $f(x)$. Ordinarily there will be defined two b-sequences representing two distinct values of $f(x)$, namely the limits $f(x \pm 0)$. For precision we must agree to prefer one of these two values (or some other number) as the actual value of $f(x)$ at the switch-point. But such a preference can do nothing to bridge

the essential gap between $f(x \pm 0)$, and the discontinuity remains. Hence, in general, the functions $f(x)$ defined in this way are continuous everywhere save at a countable set of points, the switch-points, where they have gap-discontinuities.

The general theory may be illustrated by examples adapted to the five modes of representation (2) to (6). The b's in each case are deemed to be employed in the same mode as the a's.

Decimal fractions (2).*

(7) $$b_n = a_n + (-1)^{a_n},$$

(8) b_n is the greatest of $a_1, a_2, \ldots, a_n$.

The discontinuities of (7) are at the special rational points, $x = p\,10^{-q}$, expressible as terminating decimals. Those of (8) are at those rational points expressible as terminating decimals, in which the last figure is not the greatest figure. But if we write

(9) $$b_n = 9 - a_n,$$

then $f(x) = 1 - x$, continuous everywhere.

Continued fractions (3), (4).

(10) $$b_n = 1 + a_n.$$

(11) $$b_n = a_1 + a_2 + \ldots + a_n.$$

The discontinuities are at all the rational points. But, if in (3) we write

(12) $$b_n = a_{n-1}, \quad b_1 = 1,$$

then $f(x) = (1+x)^{-1}$, continuous throughout $(0, 1)$.

Cantor's series (5).

(13) $$b_{2n-1} = b_{2n} = a_n,$$

(14) b_n is the greatest integer in $2(a_2 + \ldots + a_n)/n$.

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* Peano, *Riv. di Mat.* 1 [quoted by Hobson, *Functions of a Real Variable* (1907), 248], has given examples coming under this head.

The discontinuities are again at the rational points. But, if we write

$$b_n = n-1-a_n, \tag{15}$$

then $f(x) = 1-x$, continuous everywhere.

Cantor's product (6).

$$b_n = a_n^2. \tag{16}$$

$$b_n = a_1 a_2 \dots a_n. \tag{17}$$

The discontinuities are again at the rational points.

We may, of course, combine two modes of representation and, for instance, write

$$x = 0{\cdot}a_1 a_2 \dots a_n \dots,$$

$$f(x) = \frac{1}{1+a_1+}\,\frac{1}{1+a_2+}\cdots\frac{1}{1+a_n+}\cdots.$$

The discontinuities are at the special rationals $p\,10^{-q}$ and $f(x)$ never assumes a rational value. If x is rational, $f(x)$ is a quadratic surd.

If x is expressed as a continued fraction, we may define a discontinuous $f(x)$ somewhat differently by first fixing some n and then writing

$$f(x) \equiv \phi(x, p_n, q_n) \quad \text{or} \quad \phi(x, p_m, q_m)$$

where p_n/q_n is the n-th convergent to x or, if x terminate before the n-th partial quotient, p_m/q_m is the last convergent. The discontinuities are at those rational values of x which have, in one of their alternative expressions as a continued fraction, just n partial quotients. We must, of course, choose ϕ so that there is continuity at other points. Thus

$$f(x) = 1/q_n, \quad f(x) = p_n/q_n, \quad f(x) = xq_n - p_n \tag{18}$$

(where, as explained above, we must read m for n at certain values of x) are each possible forms. Evidently the form (1) quoted at the head of the paper is akin to the first of these.

We may without difficulty extend the method to functions of several variables. Evidently, if $\phi(x)$, $\psi(x)$ are two functions of the type considered above, then $f(x, y) \equiv \phi(x)\,\psi(y)$ or $\phi(x)+\psi(y)$ is discontinuous,

when *either* argument is a switch-point. This will likewise be the case, if we give an *ad hoc* definition of $f(x, y)$, *e.g.*

$$x = 0{\cdot}a_1 a_2 \ldots a_n \ldots, \quad y = 0{\cdot}b_1 b_2 \ldots b_n \ldots, \tag{19}$$

$$f(x, y) = 0{\cdot}a_1 b_1 a_2 b_2 \ldots a_n b_n \ldots,$$

$$x = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots+\frac{1}{a_n+}\cdots, \quad y = \frac{1}{b_1+}\,\frac{1}{b_2+}\cdots+\frac{1}{b_n+}\cdots, \tag{20}$$

$$f(x, y) = \frac{1}{a_1+b_1+}\,\frac{1}{a_2+b_2+}\cdots+\frac{1}{a_n+b_n+}\cdots.$$

We conclude with the definition of a function $f(x)$, analogous to the function (1), continuous when x is transcendental, discontinuous when x is algebraic.

If x is an algebraic number, it satisfies an algebraic equation of degree n,

$$q_0 x^n + q_1 x^{n-1} + \ldots + q_n = 0 \tag{21}$$

with integral coefficients. For precision we suppose that x satisfies no such equation of lower degree, and that any factor common to all the q's has been removed. The equation is then unique for a given algebraic x. We define

$$f(x) = (n+|q_0|+|q_1|+\ldots+|q_n|)^{-1} \quad (x \text{ algebraic}), \tag{22}$$

$$f(x) = 0 \quad (x \text{ transcendental}).$$

Now if x is some algebraic ξ, we can find transcendentals, *e.g.* $\xi+\pi 2^{-m}$, for unbounded positive m, as near to ξ as we please. For these transcendentals we have, by (22),

$$|f(\xi+\pi 2^{-m})-f(\xi)| = |f(\xi)| \neq 0.$$

Hence $f(x)$ is discontinuous at $x = \xi$.

Again, if

$$n+|q_0|+|q_1|+\ldots+|q_n| < N, \tag{23}$$

then certainly $n < N$, and certainly every $|q| < N$. Of given degree n there are therefore not more than $(2N-1)^{n+1}$ equations (21), since each

q is here limited to the range of values

$$-(N-1),\ -(N-2),\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ N-2,\ N-1.$$

There are therefore in all not more than

$$\sum_{p=1}^{N-1} (2N-1)^{p+1}$$

distinct algebraic numbers which satisfy the inequality (23). This number is bounded when N is bounded.

If now ξ is transcendental, choose any ϵ. Then, as above, the condition

$$n+\Sigma|q| \leqslant \epsilon^{-1}$$

determines a finite set of algebraic numbers. If $\xi \pm \eta$ be the nearest to ξ of this set, then the inequality $|\delta x| < \eta$ defines an interval at ξ containing only transcendentals, and algebraic numbers for which

$$n+\Sigma|q| > \epsilon^{-1},$$

i.e. for which

$$|f(x)| < \epsilon,$$

Hence, at $x = \xi$, $|\delta x| < \eta$ secures $|\delta f(x)| < \epsilon$, and the continuity at all transcendental points is established.

A HYPERSPATIAL ANALOGUE OF FEUERBACH'S THEOREM

By J. P. GABBATT.

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IT is stated in Feuerbach's theorem that the inscribed and escribed circles of a Euclidean plane triangle all touch the nine-point circle of that triangle. T. C. Lewis has shown* that there is in general no corresponding theorem for the eight spheres which touch all the faces of a Euclidean tetrahedron.

The present writer has pointed out, however,† that it is essentially as "pure" poloconics q.‡ a certain plane cubic that the inscribed and escribed circles possess Feuerbach's property. Moreover, M. Malgouzou has proved§ an extension of Feuerbach's theorem, which may be stated as follows :—There are four circles which are "pure" poloconics q. any Euclidean plane cubic; and these four circles all touch a fifth circle. Now the poloconic is the two-dimensional representative of an n-ic $(n-1)$-fold in space of n dimensions. It seemed not unlikely, therefore, that Feuerbach's theorem might be the two-dimensional representative of a theorem on such n-ics; and quite simple analysis suffices to show that this is the case.

The analysis for n dimensions differs only slightly from that for three dimensions, and the geometrical interpretation is, of course, simpler in the latter case. Part I (consisting of §§ 1–3) of the present paper is therefore devoted to the extension of Feuerbach's and some related theorems to three-dimensional space, the corresponding theorems for n-dimensional space being dealt with as briefly as possible in Part II (§§ 4, 5).

In § 1 certain theorems on the general cubic surface and a related

* *Messenger of Math.*, 49 (1920), 187–192. The writer is indebted to Mr. Lewis for this reference.

† *Proc. Camb. Phil. Soc.*, 21 (1923), 342–348.

‡ The abbreviation q. signifies *with respect to*.

§ *Nouv. Ann.* (4), 19 (1919), 210–213.

transformation are introduced. The projective form of the three-dimensional analogue of Malgouzou's theorem is then given (2 . 12), and the analogue of Feuerbach's theorem is deduced from it, first in projective, and then in Euclidean form (3 . 19, 3 . 32). The latter form of the theorem is as follows:* Let [A], [B] denote the two tetrahedra inscribed in any given parallelepiped. Then there are eight cubic surfaces, each of which touches all the edges of [A], is circumscribed to [B], and contains the (cubic) curve at infinity which is anallagmatic for the isogonal transformation q. either tetrahedron:* and each of the eight cubic surfaces touches the pedal cubic surface of [B] along one of eight twisted cubics. Each of the eight cubic surfaces is associated with one of the eight in- or ex-centres of the tetrahedron [A]; and each in- or ex-centre has quasi-central properties q. the associated surface (3 . 33, cf. 5 . 14).

The extension of Malgouzou's theorem to projective space of n dimensions is given in § 4, and that of Feuerbach's theorem to Euclidean space of n dimensions in § 5.

The analogues of certain plane theorems subsidiary to that of Feuerbach occur naturally in the course of the work. The theorems in question are the following:—The point of contact of the inscribed circle with any side of the base-triangle is the projection from the opposite vertex of a certain point (Gergonne's point); and similarly for the escribed circles. The points common to the nine-point circle and any side of the base-triangle are the projections from the opposite vertex of two points (the centroid G and the orthocentre H). The centre N of the nine-point circle is that point of the line GH which is such that $NH : GH = \frac{3}{4}$. The middle point of any side of the base-triangle, and the middle point of the join of the opposite vertex and the orthocentre, are diametrically opposite points of the nine-point circle. The tangents to the nine-point circle at its points of contact with the in- and ex-circles all touch the ellipse (Steiner's in-ellipse) which touches the sides of the base-triangle at their middle points. The pedal line, q. the medial triangle, of the point of contact of the in- (or an ex-) circle with the nine-point circle is perpendicular to the polar line, q. the base-triangle, of the in- (or corresponding ex-) centre. Most of these theorems are extended to projective space of three dimensions (3 . 21–4), and all of them to Euclidean space of n dimensions (5 . 21–7).

* W. H. Salmon, *Arch. d. Math. u. Phys.* (3), 21 (1913), 309, has shown that there are cubic surfaces analogous to Tucker's circles. These surfaces also contain the above anallagmatic cubic.

I. *Three dimensions.*

1. *Preliminary.*—1.1. In space of three dimensions, four homogeneous coordinates (x_1, x_2, x_3, x_4) are sufficient to determine a point. The planes expressed by the equations $x_i = 0$ are the faces of the *base-tetrahedron*, which is denoted by [A]; that vertex of the tetrahedron which is not contained by the face $x_i = 0$ being denoted by A_i $(i = 1, 2, 3, 4)$.

1.2. Now let us consider the transformation determined by the equations

$$a_1x_1x_1' = a_2x_2x_2' = a_3x_3x_3' = a_4x_4x_4', \tag{1.21}$$

where the four numbers a_i are fixed. This transformation (which will be referred to throughout Part I as *the transformation*) is cubic, (1–1), reciprocal and bilinear. The points

$$X \equiv (x_1, x_2, x_3, x_4) \quad \text{and} \quad X' \equiv (x_1', x_2', x_3', x_4')$$

are *conjugate points* for the transformation.

Given any homogeneous form

$$f \equiv f(x_1, x_2, x_3, x_4),$$

then the form

$$\phi \equiv f\left(\frac{1}{a_1x_1}, \frac{1}{a_2x_2}, \frac{1}{a_3x_3}, \frac{1}{a_4x_4}\right)$$

is the *transform* of f; and so for the surfaces expressed by the equations $f = 0$, $\phi = 0$.* Thus, if a given surface be the locus of a variable point, then the transform of that surface is the locus of the conjugate point.

In particular, the transform of the plane

$$u \equiv a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

is (generally) a four-nodal cubic surface

$$T(u) \equiv \frac{a_1}{a_1x_1} + \frac{a_2}{a_2x_2} + \frac{a_3}{a_3x_3} + \frac{a_4}{a_4x_4} = 0, \tag{1.22}$$

of which the nodes are at the four points A_i. The surface $T(u)$ will be termed the *T-cubic* of the plane u.

Now the complete intersection of two four-nodal cubic surfaces, of which the (common) nodes are at the vertices of a given tetrahedron, consists of the edges of that tetrahedron and a twisted cubic containing

* The surface expressed by the equation $f = 0$ will be termed *the surface f*.

its vertices. In particular, the complete intersection of the T-cubics of any two planes, u, v, consists of the edges of the base-tetrahedron [A] and a twisted cubic containing the four points A_i. Moreover, the T-cubics of any two planes which both contain a given line l have the same intersection. The twisted cubic which forms part of this intersection will be termed the *transform* of the line l. (1.23)

1.3. It is sometimes convenient to introduce a fifth coordinate, x_0, connected with x_1, x_2, x_3, x_4 by a linear equation which, without loss of generality, may be supposed to be

$$x_0+x_1+x_2+x_3+x_4 = 0.$$

The faces of the base-tetrahedron, and the plane expressed by $x_0 = 0$, are the faces of the *base-pentahedron*.

For example, by proper choice of the base-pentahedron, the equation of the general cubic surface may be expressed* in Sylvester's form

$$U \equiv a_0x_0^3+a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3 = 0. \qquad (1.31)$$

Again, writing $x_0^2 = -x_0(x_1+x_2+x_3+x_4)$,

&c., the equation of any quadric surface may be written

$$Q \equiv \Sigma c_{ij}x_ix_j = 0,$$

where the sum (of ten terms) is taken for all unequal values of i, j from 0 to 4 inclusive. The surface

$$P(Q) \equiv -\Sigma \frac{c_{ij}}{a_ia_jx_ix_j} = 0,$$

which is in general cubic, is thus uniquely related to the quadric Q, of which $P(Q)$ is in fact the *polocubic* q. the cubic surface U.† The cubic surface $P(Q)$ will be referred to as the *P-cubic* of the quadric Q. Clearly every P-cubic contains all the vertices of the base pentahedron.

When the quadric Q consists of the plane x_0 and any other plane, *e.g.* when the quadric is expressed by

$$Q \equiv x_0u \equiv x_0(a_1x_1+a_2x_2+a_3x_3+a_4x_4) = 0,$$

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* *Camb. Math. J.*, 6 (1851), 198.

† *I.e.*, $P(Q)$ is the locus of a point X, such that the quadric Q is outpolar to the polar quadric (q. U) of X.

then for the P-cubic of Q we have

$$P(Q) \equiv -\frac{1}{a_0 x_0}\left(\frac{a_1}{a_1 x_1}+\frac{a_2}{a_2 x_2}+\frac{a_3}{a_3 x_3}+\frac{a_4}{a_4 x_4}\right) = 0.$$

The P-cubic of Q is in the present case the T-cubic of the plane u. This was to have been expected, since, if the numbers a_1, a_2, a_3, a_4 have the same meaning in (1 . 21), (1 . 31), then two conjugate points for the transformation (1 . 21) are in general such that their mixed polar plane, $q.U$, is the plane x_0. (1 . 32)

Again, when the quadric Q consists of a pair of coincident planes, so that the quadric is expressed by an equation of the form

$$Q \equiv v^2 \equiv (b_0 x_0+b_1 x_1+b_2 x_2+b_3 x_3+b_4 x_4)^2 \equiv -\Sigma(b_i-b_j)^2 x_i x_j = 0,$$

then, for the P-cubic of Q,

$$P(Q) \equiv \Sigma \frac{(b_i-b_j)^2}{a_i a_j x_i x_j} = 0, \qquad (1 . 33)$$

both the sums being taken for all unequal values of i, j from 0 to 4 inclusive. In this case the surface $P(Q)$, the so-called "pure" polocubic ($q.U$) of the plane v, will be termed the *P-cubic* of that plane, and denoted by $P(v^2)$.

2. *The three-dimensional analogue of Malgouzou's Theorem.*

2 . 1. Given the planes

$$u \equiv a_1 x_1+a_2 x_2+a_3 x_3+a_4 x_4 = 0,$$

$$w \equiv a_1^2 x_1+a_2^2 x_2+a_3^2 x_3+a_4^2 x_4 = 0,$$

then, for the P-cubic of u,

$$P(u^2) \equiv \frac{1}{a_0 x_0}\left(\frac{a_1^2}{a_1 x_1}+\frac{a_2^2}{a_2 x_2}+\frac{a_3^2}{a_3 x_3}+\frac{a_4^2}{a_4 x_4}\right)+\Sigma \frac{(a_i-a_j)^2}{a_i a_j x_i x_j},$$

where the sum is taken for all unequal values of i, j from 1 to 4 inclusive, and therefore

$$P(u^2) \equiv \left(\frac{1}{a_0 x_0}+\frac{1}{a_1 x_1}+\frac{1}{a_2 x_2}+\frac{1}{a_3 x_3}+\frac{1}{a_4 x_4}\right).T(w)-\{T(u)\}^2 = 0. \quad (2 . 11)$$

Thus the P-cubic of the plane u touches the T-cubic of the plane w at all the common points, other than points on the edges of the tetrahedron [A], of the T-cubics of the planes u, w; *i.e.* (1.23) at every point of the twisted cubic which is the transform of the join of u, w. The two cubic surfaces $P(u^2)$, $T(w)$ have further in common every point of either which lies on the plane x_0.

Now the plane w is similarly related to all the eight planes

$$[u] \equiv a_1x_1 \pm a_2x_2 \pm a_3x_3 \pm a_4x_4 = 0;$$

thus, using the word *transform* in the sense of (1.23), we have the following theorem, analogous to that of Malgouzou, for the general cubic surface U : *Given the T-cubic, $T(w)$, of any plane w of general position; then there are eight planes, $[u]$, such that the P-cubic, $P(u^2)$, contains all the points common to $T(w)$ and the plane x_0, and also touches $T(w)$ at every point of a twisted cubic, namely the transform of the join of u, w.* (2.12)

It is well known that the equation

$$\frac{1}{a_0x_0} + \frac{1}{a_1x_1} + \frac{1}{a_2x_2} + \frac{1}{a_3x_3} + \frac{1}{a_4x_4} = 0$$

expresses the Hessian of the cubic surface U. The theorem that the Hessian and the polocubic $P(u^2)$ touch at all their common points [which are also points of $T(u)$] thus follows immediately from the equation (2.11). To this fundamental theorem the theorem (2.12) is therefore complementary. (2.13)

2.2. Again, the condition of contact of the plane

$$p \equiv l_1x_1 + l_2x_2 + l_3x_3 + l_4x_4 = 0$$

with the cubic surface

$$T(w) \equiv \frac{a_1^2}{a_1x_1} + \frac{a_2^2}{a_2x_2} + \frac{a_3^2}{a_3x_3} + \frac{a_4^2}{a_4x_4} = 0$$

is $$a_1\sqrt{(l_1/a_1)} + a_2\sqrt{(l_2/a_2)} + a_3\sqrt{(l_3/a_3)} + a_4\sqrt{(l_4/a_4)} = 0;$$

if this condition is satisfied, then the plane p touches the surface $T(w)$ at the point (x_1, x_2, x_3, x_4), where

$$x_i = \frac{a_i}{\sqrt{(a_il_i)}} \quad (i = 1, 2, 3, 4).$$

If this point also lies on the surface

$$T(u) \equiv \frac{a_1}{a_1 x_1} + \frac{a_2}{a_2 x_2} + \frac{a_3}{a_3 x_3} + \frac{a_4}{a_4 x_4} = 0,$$

then $$\sqrt{(l_1/a_1)} + \sqrt{(l_2/a_2)} + \sqrt{(l_3/a_3)} + \sqrt{(l_4/a_4)} = 0,$$

i.e. the plane p touches the cubic surface

$$T(x_0) \equiv \frac{1}{a_1 x_1} + \frac{1}{a_2 x_2} + \frac{1}{a_3 x_3} + \frac{1}{a_4 x_4} = 0,$$

which is at once the T- and the P-cubic of the plane x_0.

But (2 . 11) the planes which touch the surface $T(w)$ at its common points (other than those on the edges of [A]) with the surface $T(u)$ are the common tangent planes of the surfaces $P(u^2)$, $T(w)$ at their points of contact. The argument continues to hold when any one of the eight planes

$$a_1 x_1 \pm a_2 x_2 \pm a_3 x_3 \pm a_4 x_4 = 0$$

is substituted for u. Thus : *The planes which touch the four-nodal cubic surface $T(w)$ at its points of contact with any of the eight cubic surfaces $[P(u^2)]$ also touch a second four-nodal cubic surface, viz. $T(x_0)$.* (2 . 21)

3. *A three-dimensional analogue of Feuerbach's Theorem.*

3 . 1. In two dimensions, Feuerbach's theorem may be obtained by applying that of Malgouzou to the degenerate three-nodal cubic, *i.e.* the plane triangle. Similarly in three dimensions an analogue of Feuerbach's theorem is obtained by applying the theorem (2 . 12) to the four-nodal cubic surface expressed by the equation

$$x_2 x_3 x_4 + x_3 x_1 x_4 + x_1 x_2 x_4 + x_1 x_2 x_3 = 0. \qquad (3 . 11)$$

The exactness of the analogy depends on the circumstance that every "pure" poloconic q. the surface (3 . 11) touches every edge of the base-tetrahedron.* This may be proved by reducing the equation (3 . 11) to Sylvester's form (1 . 31) ; towards which we now proceed.

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* A direct consequence of the fundamental theorem mentioned in (2 . 13), and of the theorem that the Hessian of the surface (3 . 11) contains all the edges of the base tetrahedron. See (3 . 15), footnote.

Let $$y_0 = -2x_0,$$

and let $$y_i = x_0 + 2x_i \quad (i = 1, 2, 3, 4),$$

so that $$y_0 + y_1 + y_2 + y_3 + y_4 = 0. \tag{3.12}$$

The tetrahedron, [B], of which the faces are respectively expressed by the four equations $y_i = 0$ $(i = 1, 2, 3, 4)$, may be geometrically specified as follows: Let A_{ij} denote the point which is the harmonic conjugate, *q.* A_i, A_j, of the point common to the plane x_0 and the edge A_iA_j of the tetrahedron [A]; then there is a quadric surface which touches A_iA_j at A_{ij}, and [B] is the polar tetrahedron of [A] *q.* that quadric $(i, j = 1, 2, 3, 4)$. Thus [B] is in perspective with [A], the plane x_0 being the axial plane of perspective; and, if B_i denote that vertex of [B] which does not lie in the plane y_i, then the edge B_kB_l of [B] meets the edge A_iA_j of [A] at A_{ij} $(i, j, k, l = 1, 2, 3, 4)$.

Now $$24x_1x_2x_3x_4 \cdot \sum_{i=1}^{4} x_i^{-1} \equiv \tfrac{1}{4}y_0^3 + \sum_{i=1}^{4} y_i^3, \tag{3.13}$$

and thus the equation

$$U' \equiv \tfrac{1}{4}y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 = 0$$

expresses, in Sylvester's form, the cubic surface (3.11).

The transformation specified by the equations

$$y_1y_1' = y_2y_2' = y_3y_3' = y_4y_4'$$

(which may be termed the *y-transformation*) bears to the surface U' a relation of the same type as that which the transformation (1.21) bears to the surface U (1.31); the plane y_0 (which is the plane x_0) being the mixed polar plane, *q.* U', of any pair of conjugate points (of general position) for the y-transformation.

The "pure" polocubic, *q.* U', of any plane being termed the *P'-cubic* of that plane, we shall examine the following case of the theorem (2.12): The P'-cubic of each of the planes expressed by the eight equations

$$[u'] \equiv a_1y_1 \pm a_2y_2 \pm a_3y_3 \pm a_2y_4 = 0$$

touches the y-transform of the plane expressed by

$$w' \equiv a_1^2y_1 + a_2^2y_2 + a_3^2y_3 + a_4^2y = 0$$

at every point of a twisted cubic.

Now (cf. 2 . 11) for the P'-cubic of the plane expressed by

$$u' \equiv a_1y_1+a_2y_2+a_3y_3+a_4y_4 = 0,$$

we have
$$P'(u'^2) \equiv \frac{4}{y_0} \cdot \sum_{i=1}^{4} \frac{a_i^2}{y_i} + \Sigma\Sigma \frac{(a_i-a_j)^2}{y_iy_j}$$

$$\equiv \left(\frac{4}{y_0} + \sum_{i=1}^{4} \frac{1}{y_i}\right) \cdot \sum_{i=1}^{4} \frac{a_i^2}{y_i} - \left(\sum_{i=1}^{4} \frac{a_i}{y_i}\right)^2 = 0, \qquad (3.14)$$

the double sum being taken for all unequal values of i, j from 1 to 4 inclusive. For every point of any edge A_iA_j of [A] we have

$$\frac{4}{y_0} + \sum_{i=1}^{4} \frac{1}{y_i} = 0;^*$$

A_iA_j therefore touches $P'(u'^2)$; and it is easy to show that for the point of contact

$$x_i(s_1-a_i) = x_j(s_1-a_j),$$

where
$$2s_1 = a_1+a_2+a_3+a_4, \qquad (3.15)$$

Again, the y-transform of the plane

$$w' \equiv a_1^2y_1+a_2^2y_2+a_3^2y_3+a_4^2y_4 = 0$$

is the cubic surface

$$T'(w') \equiv \frac{a_1^2}{y_1} + \frac{a_2^2}{y_2} + \frac{a_3^2}{y_3} + \frac{a_4^2}{y_4} = 0; \qquad (3.16)$$

and from (3 . 12) the points common to the surface $T'(w')$ and the plane x_0 also lie on the surface

$$W \equiv \frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \frac{a_3^2}{x_3} + \frac{a_4^2}{x_4} = 0. \qquad (3.17)$$

Thus, if a_1, a_2, a_3, a_4 be properly chosen, the section by the plane x_0 of the surface $T'(w')$ may be any plane cubic containing the meets of the plane x_0 with the edges of the tetrahedron [A].

The application of the theorem (2 . 12) to the four-nodal cubic surface U' thus yields the projective form of the analogue of Feuerbach's theorem as follows: *Given any plane x_0 and any tetrahedron* [A]; *then a second*

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* Hence the Hessian of the surface U' contains all the edges of [A].

tetrahedron [B] *is determined. Further, given any cubic curve C, in the plane* x_0, *containing the meets of that plane with the six edges of* [A]: *then there are eight cubic surfaces,* $[P'(u'^2)]$, *which touch all the edges of* [A], *circumscribe* [B], *and contain the cubic C; and there is a four-nodal cubic surface,* $T'(w')$, *which contains C and touches each of the eight cubic surfaces along one of eight twisted cubics.* (3.18)

The eight twisted cubics of the above theorem are (cf. 2.12) the y-transforms of the joins of the eight planes $[u']$ with the plane w'. The twisted cubics are therefore part of the intersections of the eight surfaces

$$[T'(u')] \equiv \frac{a_1}{y_1} \pm \frac{a_2}{y_2} \pm \frac{a_3}{y_3} \pm \frac{a_4}{y_4} = 0$$

with the surface $T'(w')$. (3.19)

3.2. We now proceed to consider the analogues, in projective space of three dimensions, of the subsidiary theorems mentioned in the introduction.

It follows immediately from (3.15) that: *The point of contact of the cubic surface* $P'(u'^2)$ *with any edge of the tetrahedron* [A] *is the projection, from the opposite edge, of the point expressed by the equations*

$$x_1(s_1-a_1) = x_2(s_1-a_2) = x_3(s_1-a_3) = x_4(s_1-a_4),$$

where

$$2s_1 = a_1+a_2+a_3+a_4;$$

and similarly, changing appropriate signs, for the remaining seven of the cubic surfaces $[P'(u'^2)]$. (3.21)

Again, it may be proved without difficulty that the two common points, other than that on the plane x_0, of the surface $T'(w')$ and the edge A_iA_j of [A], respectively satisfy the equations

$$x_i = x_j,$$

and

$$x_i(s_2-a_i^2) = x_j(s_2-a_j^2),$$

where

$$2s_2 = a_1^2+a_2^2+a_3^2+a_4^2.$$

Thus: *The two common points (not on the plane* x_0*) of the four-nodal cubic surface* $T'(w')$ *and any edge of the tetrahedron* [A] *are the projections, from the opposite edge, of the two points (G, H respectively) expressed by the equations*

$$x_1 = x_2 = x_3 = x_4,$$

and $$x_1(s_2-a_1^2) = x_2(s_2-a_2^2) = x_3(s_2-a_3^2) = x_4(s_2-a_4^2),$$

where $$2s_2 = a_1^2+a_2^2+a_3^2+a_4^2. \qquad (3.22)$$

From (3.22) it follows that: *The twelve points (other than those on the plane x_0) common to the surface $T'(w')$ and the six edges of* [A] *lie on the quadric surface expressed by*

$$\sum_{i=1}^{4}(s_2-a_i^2)\,x_i^2-\Sigma\Sigma(2s_2-a_i^2-a_j^2)\,x_ix_j = 0,$$

where the double sum is taken for all unequal values of i, j from 1 to 4 inclusive. The pole, q. this quadric, of the plane x_0 is the point G of (3.22). (3.23)

Lastly, from (2.2), the planes which touch the surface $T'(w')$ at its points of contact with any of the eight surfaces $[P'(u'^2)]$ also touch the four-nodal cubic surface expressed by

$$\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4} = 0,$$

viz. the y-transform, which is also the P'-cubic, of the plane x_0. Again (cf. 3.15), this surface touches the edge A_iA_j of [A] at the point for which $x_i = x_j$; *i.e.* at the point A_{ij} (3.13). Thus: *The planes which touch the surface $T'(w')$ at its points of contact with any of the eight surfaces $[P'(u'^2)]$ also touch the four-nodal cubic surface of which the nodes are at the vertices of the tetrahedron* [B], *and which touches the edge A_iA_j of* [A] *at the point A_{ij} and meets that edge again on the plane x_0 ($i, j = 1, 2, 3, 4$).* (3.24)

3.3. We arrive at the Euclidean case by (i) identifying the plane x_0 with the plane at infinity, and (ii) regarding the numbers a_1, a_2, a_3, a_4 as the measures of the areas of the corresponding faces of the tetrahedron [A].

From (i) the point A_{ij} (3.13) becomes the middle point of the edge A_iA_j of [A]. The tetrahedron [B] is now that which is polar to [A] q. the quadric which touches all the edges of [A] at their respective middle points; *i.e.* the vertices of [A], [B] are the eight vertices of a parallelepiped, of which, therefore, [A], [B] are the two inscribed tetrahedra. The point G (3.22) becomes the common centroid of [A], [B].

Using (ii), the surface W (3.17) becomes the isogonal transform, q. [A], of the plane at infinity, and the section of W by that plane is now therefore the anallagmatic curve at infinity for the isogonal transformation. Thus the surface W is now* the *pedal cubic* of the tetrahedron [A], *i.e.* the locus of a point of which the orthogonal projections on the four faces of [A] are in a plane. If the plane be termed the *pedal plane* of the point, then it is known that the point at infinity, which is the isogonal conjugate of the given point, is in the direction perpendicular to the pedal plane of that point. The surface $T'(w')$ (3.16) becomes the pedal cubic of [B]. (3.31)

The eight points $(a_1, \pm a_2, \pm a_3, \pm a_4)$ become the in- and ex-centres of the tetrahedron [A]. The eight surfaces $[T'(u')]$ of (3.19) become the isogonal transforms, q. the tetrahedron [B], of the eight planes

$$\frac{y_1}{a_1} \pm \frac{y_2}{a_2} \pm \frac{y_3}{a_3} \pm \frac{y_4}{a_4} = 0,$$

which (3.12) are respectively parallel to the eight planes

$$\frac{x_1}{a_1} \pm \frac{x_2}{a_2} \pm \frac{x_3}{a_3} \pm \frac{x_4}{a_4} = 0,$$

viz. the polar planes, q. the tetrahedron [A], of the in- and ex-centres of that tetrahedron.

We may now write the analogue of Feuerbach's theorem in the following Euclidean form: *Let* [A], [B] *denote the two tetrahedra inscribed in any given parallelepiped. Then there are eight cubic surfaces* (*the* I-cubics *of the tetrahedron* [A]), *each of which touches all the edges of* [A], *is circumscribed to* [B], *and contains the* (*cubic*) *curve at infinity which is anallagmatic for the isogonal transformation q. either* [A] *or* [B]; *each of the I-cubics touches the pedal cubic of* [B] *along one of eight twisted cubics; and the eight twisted cubics are the isogonal transforms, q.* [B], *of the lines at infinity in the polar planes, q.* [A], *of the in- and ex-centres of* [A]. (3.32)

Thus each of the I-cubics is associated with one of the eight in- and ex-centres of [A]; and it may easily be verified that: *The polar plane of any in- or ex-centre of* [A], *q. the associated I-cubic, is the plane at infinity* (cf. 5.14). (3.33)

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* Steiner, 1845. See *Arch. d. Math. u. Phys.* (3), 4 (1908), 275–7.

Of the subsidiary theorems (3 . 21–4) it need only be said that, in the present (Euclidean) case, the centre of the quadric of (3 . 23) is the common centroid of [A], [B]; and that the tangent planes at the points of contact of the eight *I*-cubics of [A] with the pedal cubic of [B] all touch a four-nodal cubic surface, of which the nodes are at the vertices of [B], and which touches all the edges of [A] at their respective middle points.

Further, it follows from (3 . 31) and the specification (3 . 32) of the twisted cubics of contact that: *The pedal planes, q.* [B], *of the points of contact of the pedal cubic of* [B] *with any one of the I-cubics of* [A] *are all perpendicular to the polar plane, q.* [A], *of the corresponding in- or ex-centre of* [A]. (3 . 34)

II. *n dimensions.*

4. *Projective space.*—4 . 1. In space of n dimensions, the theorems with which we are concerned still form part of the theory of the cubic form

$$U \equiv a_0x_0^3+a_1x_1^3+a_2x_2^3+\ldots+a_{n+1}x_{n+1}^3,$$

where

$$x_0+x_1+x_2+\ldots+x_{n+1}=0;$$

but when $n>3$ the equation $U=0$ no longer expresses the *general* cubic $(n-1)$-fold.

The transformation (which is now n-ic) corresponding to that of (1 . 21) is determined by the equations

$$a_1x_1x_1' = a_2x_2x_2' = \ldots = a_{n+1}x_{n+1}x_{n+1}'; \qquad (4 . 11)$$

and is such that, in general, the mixed polar flat, q. the $(n-1)$-fold U, of a pair of points conjugate for the transformation is the flat expressed by $x_0=0$.

Let the equations $x_i=0$ express the walls of the *base-simplex* [A]; and let that vertex of [A] which is not contained by the wall x_i be denoted by A_i $(i=1, 2, \ldots, n+1)$. Then the transform, $T(u)$, (under 4 . 11) of a flat, u, is in general an n-ic $(n-1)$-fold containing all the $(n-2)$-folds of the simplex [A]. The $(n-1)$-fold $P(u^2)$, of which the definition is analogous to that of (1 . 33), is also in general n-ic, and contains all the $(n-3)$-folds of [A]. In particular, if the flats u, w

are respectively expressed by

$$u \equiv a_1x_1+a_2x_2+\ldots+a_{n+1}x_{n+1} = 0,$$

$$w \equiv a_1^2x_1+a_2^2x_2+\ldots+a_{n+1}^2x_{n+1} = 0,$$

then the $(n-1)$-folds $T(u)$, $T(w)$, $P(u^2)$ are respectively expressed by

$$T(u) \equiv \frac{a_1}{a_1x_1}+\frac{a_2}{a_2x_2}+\ldots+\frac{a_{n+1}}{a_{n+1}x_{n+1}} = 0,$$

$$T(w) \equiv \frac{a_1^2}{a_1x_1}+\frac{a_2^2}{a_2x_2}+\ldots+\frac{a_{n+1}^2}{a_{n+1}x_{n+1}} = 0,$$

and (cf. 2 . 11)

$$P(u^2) \equiv T(w) \cdot \sum_{i=0}^{n+1} \frac{1}{a_ix_i} - \{T(u)\}^2 = 0.$$

Proceeding similarly with all the flats expressed by the 2^n equations

$$[u] \equiv a_1x_1 \pm a_2x_2 \pm \ldots \pm a_{n+1}x_{n+1} = 0,$$

we obtain the extension of the theorems (2 . 12, 2 . 21) as follows: *Given the n-ic $(n-1)$-fold, $T(w)$, corresponding to any flat, w, of general position; then there are 2^n flats, $[u]$, such that the n-ic $(n-1)$-fold $P(u^2)$ contains all the points common to $T(w)$ and the flat x_0, and also touches $T(w)$ at every point of a $\frac{1}{2}n(n-1)$-ic $(n-2)$-fold, namely the transform of the join of u, w. The flats which touch the $(n-1)$-fold $T(w)$ at its points of contact with any of the 2^n $(n-1)$-folds $[P(u^2)]$ also touch another n-ic $(n-1)$-fold, namely $T(x_0)$.* (4 . 12)

4 . 2. The extension, which need not be explicitly stated in projective form, of the theorem (3 . 18) is obtained by applying (4 . 12) to the cubic $(n-1)$-fold expressed by

$$\Sigma\Sigma\Sigma x_ix_jx_k = 0,$$

where no two of the numbers i, j, k are equal, and $(i, j, k = 1, 2, \ldots, n+1)$.

If $$y_0 = -(n-1)x_0,$$

and if $$y_i = x_0+2x_i \quad (i = 1, 2, \ldots, n+1), \qquad (4 . 21)$$

so that $$y_0+y_1+\ldots+y_{n+1} = 0,$$

then $$24\Sigma\Sigma\Sigma x_i x_j x_k = \frac{1}{(n-1)^2} \cdot y_0^3 + \sum_{i=1}^{n+1} y_i^3,$$

whence the work proceeds as in (3 . 1). It should be observed that, for every point of an edge A_iA_j of [A], $x_k = 0$ for all values of k (except i, j) from 1 to $n+1$ inclusive; and in this case (4 . 21) $y_k = -(x_i+x_j)$ for the $n-1$ given values of k, while $y_0 = (n-1)(x_i+x_j)$. It follows that the n-ic $(n-1)$-folds corresponding to the P'- and T'-cubics (3 . 1) meet every edge of [A] $(n-2)$-ply on the flat x_0. (4 . 22)

5. *Euclidean space.*—5 . 1. We arrive at the Euclidean case by identifying the flat x_0 with the flat at infinity, and regarding a_i as the measure of the content of that simplex (in space of $n-1$ dimensions) which is determined by all the vertices, except A_i, of [A]. The simplex [B], of which the walls are determined by the equations $y_i = 0$ (4 . 21), then becomes the polar simplex of [A] q. the quadric which touches all the edges of [A] at their respective middle points ($i = 1, 2, \ldots, n+1$). Each wall of [A] is thus parallel to a wall of [B], and [A], [B] are similar, their ratio of similitude being $2/(n-1)$. The two simplices have a common centroid, G, which is also their centre of perspective; and the middle point of any edge of [A] is the centroid of a simplex (in space of $n-2$ dimensions) determined by $n-1$ vertices of [B]. (5 . 11)

The locus of a point (in Euclidean space of n dimensions) of which the orthogonal projections on the walls of a given simplex are in a flat is known* to be an n-ic $(n-1)$-fold, viz. the isogonal transform, q. the simplex, of the flat at infinity. The locus may be termed the *pedal n-ic* of the simplex, and the flat corresponding to any point of the locus the *pedal flat* of that point. The point at infinity which is the isogonal conjugate of any point of the locus corresponds to the direction perpendicular to the pedal flat of the latter point. (5 . 12)

The analogue, in Euclidean space of n dimensions, of Feuerbach's theorem may now be stated as follows: *Let* [A], [B] *denote any two simplices related as in* (5 . 11); *then there are* 2^n *n-ic* $(n-1)$*-folds* (*the* I—n-ics *of* [A]), *each of which touches all the edges of* [A], *contains all the* $(n-3)$*-folds of* [B], *and contains the* (*n-ic*) $(n-2)$*-fold at infinity which is anallagmatic for the isogonal transformation q. either* [A] *or* [B]; *each of the I—n-ics touches the pedal n-ic, F, of* [B] *along one of* 2^n $\frac{1}{2}n(n-1)$*-ic* $(n-2)$*-folds; and these* 2^n $(n-2)$*-folds are the isogonal*

* Beltrami, *Mem. Acc. Bologna* (3), 7 (1876), 241 = *Op. Mat.*, 3, 53; W. J. C. Sharp, *Proc. London Math. Soc.*, 19 (1888), 446. For the former reference the writer is indebted to Prof. H. F. Baker.

transforms, q. [B], *of the linear* $(n-2)$*-folds at infinity in the polar flats, q.* [A], *of the in- and ex-centres of* [A]. (5 . 13)

Thus each of the $I-n$-ics is associated with one of the 2^n in- and ex-centres of [A]; and it may be verified that: *The polar flat of any in- or ex-centre of* [A] *q. the associated* $I-n$*-ic is the flat at infinity.* If I denote any in- or ex-centre of [A]; if any line through I meet the associated $I-n$-ic at the n points X_i; and if r_i denote the signed measure of the length of the segment IX_i: it follows that

$$\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=0;$$

and in particular, since (cf. 4 . 22) every line parallel to an edge of [A] meets any of the 2^n $I-n$-ics $(n-2)$-ply at infinity, therefore: *The* $\frac{1}{2}n(n+1)$ *lines through any in- or ex-centre, I, of the simplex* [A] *parallel to the edges of* [A] *meet the associated* $I-n$*-ic at* $n(n+1)$ *points (other than points at infinity); and these points lie on a quadric of which I is the centre.* (5 . 14)

5 . 2. The analogues of the subsidiary theorems mentioned in the introduction are as follows:—

The point of contact of any one of the $I-n$*-ics of the simplex* [A] *with any edge of that simplex is the projection, from the opposite* $(n-2)$*-fold of the simplex, of a certain point; the point which corresponds to the* $I-n$*-ic associated with the in-centre of* [A] *being expressed by the equations*

$$x_1(s_1-a_1)=x_2(s_1-a_2)=\ldots=x_{n+1}(s_1-a_{n+1}),$$

where
$$2s_1=a_1+a_2+\ldots+a_{n+1}. \qquad (5 . 21)$$

Each edge of the simplex [A] *meets the* $(n-1)$*-fold F* $(n-2)$*-ply at infinity. The remaining two points common to F and any edge of* [A] *are the projections, from the opposite* $(n-2)$*-fold of* [A], *of two points, viz. the centroid G and the point H* expressed by the equations*

$$x_1(s_2-a_1^2)=x_2(s_2-a_2^2)=\ldots=x_{n+1}(s_2-a_{n+1}^2),$$

where
$$2s_2=a_1^2+a_2^2+\ldots+a_{n+1}^2. \qquad (5 . 22)$$

* When the simplex is orthocentric, the point H is not in general the orthocentre.

The $n(n+1)$ common points, other than those at infinity, of F and the $\frac{1}{2}n(n+1)$ edges of the simplex [A] *lie on the quadric expressed by the equation*

$$F' \equiv \sum_{i=1}^{n+1} x_i^2(s_2 - a_i^2) - \Sigma\Sigma x_i x_j (2s_2 - a_i^2 - a_j^2) = 0,$$

the double sum being taken for all unequal values of i, j from 1 to $n+1$ inclusive. (5 . 23)

The centre of the quadric F' is the point N of the line GH such that $NH : GH = \frac{1}{4}(n+1)$. (5 . 24)

Let A_{ij} denote the middle point of the edge, determined by the vertices A_i, A_j, of the simplex [A], *G_{ij} the centroid of the simplex (in space of $n-2$ dimensions) determined by the remaining $n-1$ vertices, and A'_{ij} the point, diametrically opposite A_{ij}, of the quadric F'. Then the points G_{ij}, H, A'_{ij} are in line, and*

$$HA'_{ij} : HG_{ij} = \tfrac{1}{2}(n-1) \quad (i, j = 1, 2, \ldots, n+1) \quad (5 . 25)$$

The flats which touch the $(n-1)$-fold F at its points of contact with any of the $I-n$-ics also touch a further n-ic $(n-1)$-fold, which touches each edge of the simplex [A] *at the middle point of that edge, meets each edge again $(n-2)$-ply at infinity, and contains all the $(n-2)$-folds of the simplex* [B]. (5 . 26)

The pedal flats, q. the simplex [B], *of the points of contact of the $(n-1)$-fold F with any one of the $I-n$-ics of* [A] *are all perpendicular to the polar flat, q.* [A], *of that in- or ex-centre of* [A] *with which the given $I-n$-ic is associated.* (5 . 27)

ON THE RIEMANN ZETA-FUNCTION

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1. *Introduction.*

1.1. In order to summarize the results of the present memoir, it is necessary to develop the preliminary ideas and definitions at some length. The great majority of our results depend on the Riemann hypothesis, and to avoid repetition we lay down that the hypothesis is supposed true except where the contrary is explicitly stated.

For $\sigma > \frac{1}{2}$ let $\nu(\sigma)$ be the least value of ξ such that

$$f(s) = \log \zeta(s) = O\{(\log t)^{\xi+\epsilon}\}$$

as $t \to \infty$. We shall call, for shortness, the least ξ corresponding in the same way to any other function $\phi(s)$ the "ν-function of ϕ." It is known that (for f)

$$\nu(\sigma) = 0 \quad (\sigma \geqslant 1), \tag{1.11}$$

$$\nu(\sigma) \leqslant 2(1-\sigma) \quad (\tfrac{1}{2} < \sigma \leqslant 1), \tag{1.12}$$

$$\nu(\sigma) \geqslant b(1-\sigma) \quad (\tfrac{1}{2} < \sigma \leqslant 1), \tag{1.13}$$

where b is a positive constant; and that $\nu(\sigma)$ is a convex, continuous, and decreasing function of σ in the interval $\frac{1}{2} < \sigma \leqslant 1$.* Since one of our principal theorems is that we may take $b = 1$ in (1.13), it is desirable to begin the problem of a lower bound of $\nu(\sigma)$ *de novo*. We assume as known, therefore, in this connexion, only the much less deep

* (1.11) is nearly trivial. The upper bound in (1.12) is due to J. E. Littlewood, *Comptes Rendus* (1912), 263–266; the remaining results to H. Bohr and E. Landau, *Math. Annalen*, 74 (1913), 3–30. We refer in future to this last paper as B.L.

inequality $\nu(\sigma) \geqslant 0$.* The convexity of $\nu(\sigma)$ still involves continuity in $\sigma > \frac{1}{2}$, but the "decrease" of $\nu(\sigma)$ is now to be interpreted in the wide sense, viz. $\nu(\sigma) \leqslant \nu(\sigma')$ when $\sigma > \sigma'$.

1.2. We observe next that the differential coefficients and the (repeated) integrals (from $+\infty$ as the lower limit) of $f(s)$ have the same ν-function in $\sigma > \frac{1}{2}$ as $f(s)$ itself. In fact, let $\phi(s)$ be a derivative or integral of $f(s)$, $\bar{\nu}(\sigma)$ its ν-function. The original arguments applied to f show equally that $\bar{\nu}(\sigma)$ is continuous and decreasing (in the wide sense) in $\sigma > \frac{1}{2}$, and zero for $\sigma \geqslant 1$. Then

$$\phi_1(s) = \int_\infty^s \phi(s)\,ds = \int_\infty^2 \phi(s)\,ds + \int_2^{2+it} \phi(s)\,ds + \int_2^\sigma \phi(\sigma_1+it)\,d\sigma_1$$

$$= O(1) + \int_2^\sigma O\{(\log t)^{\bar{\nu}(\sigma_1)+\epsilon}\}\,d\sigma_1 = O\{(\log t)^{\bar{\nu}(\sigma)+\epsilon}\}.$$

On the other hand

$$|\phi'(s)| \leqslant \operatorname*{Max}_{|s_1-s|\leqslant\epsilon} \left|\frac{\phi(s_1)}{s_1-s}\right| = O\left(\{\log(t+\epsilon)\}^{\bar{\nu}(\sigma-\epsilon)+\epsilon}\right)$$

whence, by the continuity of $\bar{\nu}$,

$$\phi'(s) = O\{(\log t)^{\bar{\nu}(\sigma)+\epsilon}\}.$$

It follows that neither differentiation nor integration increases the ν-function of ϕ, and hence that $\bar{\nu}(\sigma) = \nu(\sigma)$.

Again, consider the function $\Re\phi(s)$, and its ν-function $\nu_R(\sigma)$. Evidently $\nu_R(\sigma) \leqslant \nu(\sigma)$. On the other hand if $\sigma > \frac{1}{2}$, $\epsilon < \sigma - \frac{1}{2}$, $s_0 = s+1$, we have, by Carathéodory's theorem,

$$|\phi(s)| \leqslant \frac{(1+\epsilon)+1}{(1+\epsilon)-1}\left(|\phi(s_0)| + \operatorname{Max}\Re\phi\{s_0+(1+\epsilon)e^{i\theta}\}\right)$$

$$< \left(\frac{2}{\epsilon}+1\right)\left(O(1)+O\left[\{\log(t+1+\epsilon)\}^{\nu_R(\sigma-\epsilon)+\epsilon}\right]\right),$$

from which it follows that

$$\nu_R(\sigma-\epsilon)+\epsilon \geqslant \nu(\sigma)$$

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* $\nu(\sigma) < 0$ involves $f(s) \to 0$, $\zeta(s) \to 1$ as $t \to \infty$, and this is incompatible with the known "mean-value" result

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2\,dt \to \Sigma n^{-2\sigma} > 1 \quad (\sigma > \tfrac{1}{2}).$$

and so* $$\nu_R(\sigma) \geqslant \nu(\sigma), \quad \nu_R(\sigma) = \nu(\sigma).$$

A similar result holds for $\mathfrak{J}\phi(s)$: *the ν-functions of the real and imaginary parts of any derivative or integral of $f(s)$ are equal to $\nu(\sigma)$ for $\sigma > \frac{1}{2}$.*

1.3. It is convenient for formal purposes to define $\nu(\frac{1}{2}) = \lim_{\epsilon \to 0} \nu(\frac{1}{2}+\epsilon)$; ν is then continuous in $\sigma \geqslant \frac{1}{2}$. We define further a function $\beta(\sigma)$, for $\frac{1}{2} \leqslant \sigma < 1$, by

$$\nu(\sigma) = 2(1-\sigma)\,\beta(\sigma). \tag{1.31}$$

By the convexity of ν we have, for $\frac{1}{2} \leqslant \sigma < \sigma' < 1$,

$$2(1-\sigma')\,\beta(\sigma') = \nu(\sigma') \leqslant \frac{1-\sigma'}{1-\sigma}\,\{\nu(\sigma)-\nu(1)\} = 2\beta(\sigma)(1-\sigma'),$$

$$\beta(\sigma') \leqslant \beta(\sigma).$$

Thus $\beta(\sigma)$ is a decreasing function (in the wide sense) in $\frac{1}{2} \leqslant \sigma < 1$. We define

$$\beta(1) = \lim_{\sigma \to 1-0} \beta(\sigma), \tag{1.32}$$

and write $\beta = \beta(\frac{1}{2})$. Then, by (1.12),

$$\beta(1) \leqslant \beta(\sigma) \leqslant \beta \leqslant 1 \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1). \tag{1.33}$$

1.4. The function $\mathfrak{J}f(\sigma+it)$ exists, subject to a convention when s is a zero ρ of $\xi(s)$, even when $\sigma = \frac{1}{2}$. In fact, if $\frac{1}{2}+it$ is not a ρ, $\mathfrak{J}f(\sigma+it)$ tends to a limit, which we call $\pi S(t)$, as $\sigma \to \frac{1}{2}$; if $\frac{1}{2}+it$ is a ρ, we define

$$\pi S(t) = \mathfrak{J}f(\tfrac{1}{2}+it) = \lim_{\epsilon \to 0} \tfrac{1}{2}\{\mathfrak{J}f(\tfrac{1}{2}+it+i\epsilon)+\mathfrak{J}f(\tfrac{1}{2}+it-i\epsilon)\}, \tag{1.41}$$

and this equation is then valid for all t. It is well known that $S(T)$ is connected with $N(T)$, the number of zeros ρ of $\xi(s)$ whose imaginary parts γ lie between 0 and T,† by the relation

$$N(T) = M(T)+\tfrac{7}{8}+O(T^{-1})+S(T), \tag{1.42}$$

where $$M(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}.$$

* Since $\nu(\sigma)$ is continuous: we do not need the continuity of $\nu_R(\sigma)$.

† If T is the ordinate of a ρ, $N(T)$ is defined as a limit in the manner of (1.41).

I have shown elsewhere* that constants $a_1, a_2, \ldots$ exist such that

$$S_0(T) = S(T), \quad S_{n+1}(T) = \int_0^T S_n(t)\,dt + a_{n+1} \quad (n \geqslant 0), \tag{1.43}$$

and

$$S_n(T) = O\left(\frac{\log T}{(\log\log T)^{n+1}}\right) \quad (n \geqslant 0). \tag{1.44}$$

We define a set of indices† $\alpha = \alpha_0, \alpha_1, \alpha_2, \ldots$ by the relations

$$S_n(T) = O\{(\log T)^{\alpha_n+\epsilon}\},$$

$$S_n(T) \neq O\{(\log T)^k\} \quad (k < \alpha_n).$$

It is known‡ that $\alpha > 0$, but, as in the parallel case of (1.13), we shall not anywhere make use of this. I have shown in the paper just referred to that α_n is a convex function of n, and that

$$1 \geqslant \alpha \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \ldots, \tag{1.45}$$

in which inequalities, however, it is contemplated that all α's beyond a certain point may be $-\infty$.

1.5. In § 2 of the memoir we begin by establishing approximate formulae for $\zeta'(s)/\zeta(s)$ which have a certain analogy with the "approximate functional equation" for $\zeta(s)$. § § 3 is devoted to the relations between the α's and β's. We prove, on the one hand, by an adaptation of an old method of Landau, that $\alpha_n \geqslant \beta$ $(n \geqslant 0)$; and, on the other, that $\beta \geqslant \beta(1) \geqslant \frac{1}{2}$. The last theorem depends on the results of § 2 combined with an appeal, following a method first used by Bohr, to theorems of Diophantine approximation.

In § 4 we discuss the behaviour of $\zeta(s)$ on the line $\sigma = 1$; we find that $\overline{\lim}\,\{|\log\zeta(1+it)|/\log\log t\}$ lies between two positive bounds which

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* *Proc. Camb. Phil. Soc.*, 22 (1924), 295–318.

† Observe that although $\mathfrak{Z}f(s)$ has $\beta(\sigma)$ for β-function when $\sigma > \frac{1}{2}$, it in no way follows that $\alpha = \beta$. It is an attractive and not unreasonable conjecture that $\alpha = 1$, $\beta = \frac{1}{2}$.

‡ B.L., 10. Bohr and Landau, in fact, prove rather more, viz. that $S(T) > (\log T)^k$ and $S(T) < -(\log T)^k$ have arbitrarily large solutions whenever k is less than a positive constant c. We shall not consider one-sided inequalities here. See also p. 185 below.

§ See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 21 (1922), 39–74.

differ by a factor $2\beta(1) \leqslant 2$: we thus obtain the exact value if it is true that $\beta(1) = \frac{1}{2}$. The upper bound depends on § 2, the lower bound on Diophantine approximation and the idea embodied in Lemma 4.

Finally in § 5 we study the mean value of $|S_n(T)|$. The results of § 3 show that the "order" of $S_n(T)$ as a function of $\log T$ is at least $\frac{1}{2}$; we prove now (among other things) that

$$\frac{1}{T}\int_0^T |S_n(t)|\,dt = \begin{cases} O(1) & (n > 0), \\ O(\log\log T) & (n = 0), \end{cases}$$

so that the *average* order is zero.

1.6. The O's and o's depend, throughout the memoir, on all relevant parameters. K's denote positive constants, not the same at different occurrences, depending in general on all parameters. A's denote positive absolute constants, not the same at different occurrences.

It is convenient to lay down that the t of $s = \sigma + it$ always satisfies $t \geqslant 0$, and, where we are concerned with asymptotic behaviour, that t is supposed "large," *i.e.* satisfies any inequality $t >$ constant required by the run of the argument.

2. *The approximate functional equation for* $\zeta'(s)/\zeta(s)$.

2.1. The function $\zeta(s)$ satisfies a functional equation which may be expressed *formally* by the pair of equations

$$\zeta(s) = \Sigma n^{-s} = \chi(s)\,\Sigma n^{s-1}, \tag{2.11}$$

where $\chi(s)$ is a combination of elementary functions and Γ-functions. There is a compromise between the two expressions called (by Hardy and the present writer) the "approximate functional equation," viz.:

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi(s) \sum_{n<y} n^{s-1} + R, \tag{2.12}$$

where $2\pi xy = |t|$, $x > K$, $y > K$, and

$$R = O(x^{-\sigma}) + O(\chi y^{\sigma-1}).$$

If $x > C|t|$, where $2\pi C > 1$, the summation in y disappears. But in this case we can improve the error term, and obtain a second form of the approximate functional equation

$$\zeta(s) = \sum_{n<x} n^{-s} + O(x^{-\sigma}) \quad (c|t| < x < c'|t|). \tag{2.121}$$

For the function $\xi'(s)/\xi(s)$ also we have two formulae (this time valid simultaneously for $\sigma > 1$)

$$(2.13) \quad -\frac{\xi'(s)}{\xi(s)} = \Sigma \Lambda(n) n^{-s} = -b + \frac{1}{s-1} + \tfrac{1}{2} \frac{\Gamma'(1+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}s)} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

The formula*

$$(2.14) \quad -\frac{\xi'(s)}{\xi(s)} = \sum_{n \leqslant x}{}' \Lambda(n) n^{-s} - \frac{x^{1-s}}{1-s} - \sum_{q=1}^{\infty} \frac{x^{-2q-s}}{2q+s} + \sum_\rho \frac{x^{\rho-s}}{\rho-s},$$

in some sort a compromise, is valid for $x > 1$, $s \neq 1$, $s \neq -2q$. Its *setting* is quite novel, but the result is in fact an old one.†

While (2.14) corresponds more or less to (2.12), there is another result corresponding more nearly to (2.121), viz.:

$$(2.15) \quad -\frac{\xi'(s)}{\xi(s)} = \sum_{n \leqslant x}{}' \Lambda(n) n^{-s} + O\{x^{1-\sigma+\epsilon}(\log t)^{-\vartheta}\} + O\{x^{\frac{1}{2}-\sigma+\epsilon}(\log t)^{\beta+\vartheta+\epsilon}\},$$

where ϑ is any constant satisfying $0 \leqslant \vartheta < 1$, and the approximation is valid uniformly in $2 \leqslant x \leqslant t$, $\sigma \geqslant \sigma_0 > \frac{1}{2}$.

We may further generalize this by replacing the product $x^{\frac{1}{2}}(\log t)^\beta$ in the second error term by $x^{\sigma_1}(\log t)^{\nu(\sigma_1)}$, where $\frac{1}{2} \leqslant \sigma_1 \leqslant \sigma$.

In the sequel we use actually neither (2.14) nor (2.15). It is a familiar idea in the theory of series that approximations for $\sum^n a_n$, when n is large, bear fairly close relations to approximations for the function $\Sigma a_n e^{-ny}$ when y is small, *e.g.* there are connexions between the convergence of Σa_n and the existence of the "Abel limit." Here we replace (2.14) and (2.15) by formulae of "Abel" type. The new forms are as effective in applications; and, whereas the proof of (2.15) involves considerations of semi-convergence long, difficult, and unrelated to the essentials of the problem, our use of the Abel form raises only the difficulties inherent in the subject.

2.2. THEOREM 1.‡—*As $t \to \infty$,*

$$(2.21) \quad -\frac{\xi'(s)}{\xi(s)} = \Sigma \Lambda(n) n^{-s} e^{-ny} + \sum_\rho y^{s-\rho} \Gamma(\rho - s) + R,$$

where

$$R = O(y^{\sigma-\frac{1}{2}} \log t)$$

uniformly in $\frac{1}{2} \leqslant \sigma \leqslant \frac{9}{8}$, $e^{-\sqrt{t}} \leqslant y \leqslant 1$.

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* The dash denotes that if x is an integer the last term in $\sum_{n \leqslant x}{}'$ is to be taken with a factor $\frac{1}{2}$.

† See E. Landau, *Handbuch*, 353. We refer in future to this treatise as H.B.

‡ It is possible to develop the approximation for R, but I confine myself to the simplest possible case.

We have, by Mellin's integral for e^{-x}

$$\Sigma\Lambda(n)\,n^{-s}e^{-ny} = \Sigma\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Lambda(n)\,n^{-z-s}y^{-z}\Gamma(z)\,dz$$

$$= -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'(s+z)}{\zeta(s+z)}\,y^{-z}\Gamma(z)\,dz.$$

It is easily verified that the integrand tends exponentially to zero (for fixed s) as $|\mathfrak{I}z|\to\infty$, so that we may, on taking account of the residues at singularities between the contours, replace the path of integration by $\mathfrak{R}z = \frac{1}{4}-\sigma$. The relevant singularities are* $z=0$, with residue $\zeta'(s)/\zeta(s)$; the zeros $\rho-s$ of $\zeta(s+z)$, with residue $y^{s-\rho}\Gamma(\rho-s)$; and $z=1-s$, with residue $-y^{s-1}\Gamma(1-s) = O(y^{\sigma-\frac{1}{4}}\log t)$. It only remains to verify that

$$\int_{\frac{1}{4}-\sigma-i\infty}^{\frac{1}{4}-\sigma+i\infty}\frac{\zeta'(s+z)}{\zeta(s+z)}\,y^{-z}\Gamma(z)\,dz = O(y^{\sigma-\frac{1}{4}}\log t).$$

But if $z = \frac{1}{4}-\sigma+i\eta$,

$$\frac{\zeta'(s+z)}{\zeta(s+z)} = O\{\log(t+|\eta|)\},$$

$$y^{-z}\Gamma(z) = O(y^{\sigma-\frac{1}{4}}e^{-K|\eta|}),$$

and the desired result follows, since

$$\int_0^\infty \log(t+\eta)\,e^{-K\eta}\,d\eta = \int_0^t + \int_t^\infty = O(\log t)\int_0^t e^{-K\eta}\,d\eta + \int_t^\infty O(\log\eta)\,e^{-K\eta}\,d\eta$$

$$= O(\log t)+O(1).$$

2·3. It is convenient to take next some comparatively trivial results that will be constantly required.

LEMMA 1.—*As* $y\to 0$,

$$\Sigma\Lambda(n)\,n^{-\sigma}e^{-ny} = (y^{\sigma-1}-1)\,\Gamma(1-\sigma)+O(1) \tag{2·31}$$

(where the first term on the right is to be interpreted as $-\log y$ *when* $\sigma=1$*), uniformly in* $\frac{1}{2}+\delta\leqslant\sigma\leqslant\frac{9}{8}$ *(on the Riemann hypothesis), and uniformly in* $1\leqslant\sigma\leqslant\frac{9}{8}$ *independently of the Riemann hypothesis.*

* The point $z=-1$ is to the left of the contour.

We have

$$s_n = \sum_1^n \{\Lambda(n)-1\} = \psi(n)-n = \begin{cases} O(n^{\frac{1}{2}+\epsilon}), \\ O\left(\dfrac{n}{\log^2 n}\right) \end{cases} \begin{matrix} \\ \text{without the Riemann} \\ \text{hypothesis.} \end{matrix}$$

Hence

$$\Sigma\{\Lambda(n)-1\}n^{-\sigma}e^{-ny} = O(1)+\Sigma s_n\{n^{-\sigma}e^{-ny}-(n+1)^{-\sigma}e^{-(n+1)y}\}$$

$$= O(1)+O\{\Sigma|s_n|(n^{-1-\sigma}e^{-ny}+yn^{-\sigma}e^{-ny})\}$$

and in either of the two cases considered it is easy to see that this is $O(1)$. It is therefore only necessary to prove (2 . 31) with the factor $\Lambda(n)$ suppressed in the left-hand side. But in $\frac{1}{2}+\delta \leqslant \sigma \leqslant \frac{9}{8}$

$$\Sigma n^{-\sigma}e^{-ny} = \int_1^\infty u^{-\sigma}e^{-uy}\,du+O(1)$$

$$= \left[\frac{u^{1-\sigma}-1}{1-\sigma}\,e^{-uy}\right]_1^\infty + \frac{y}{1-\sigma}\int_1^\infty (u^{1-\sigma}-1)\,e^{-uy}\,du+O(1)$$

$$= 0+\frac{y}{1-\sigma}\int_0^\infty (u^{1-\sigma}-1)\,e^{-uy}\,du+O\left(\frac{1}{1-\sigma}\right)\int_0^1 |u^{1-\sigma}-1|\,du+O(1)$$

$$= \frac{y}{1-\sigma}\{y^{\sigma-2}\Gamma(2-\sigma)-y^{-1}\}+O(1) = y^{\sigma-1}\Gamma(1-\sigma)-\frac{1}{1-\sigma}+O(1)$$

$$= (y^{\sigma-1}-1)\,\Gamma(1-\sigma)+\frac{\Gamma(2-\sigma)-1}{1-\sigma}+O(1) = (y^{\sigma-1}-1)\,\Gamma(1-\sigma)+O(1),$$

and this completes the proof.

2 . 4. The following consequence of Theorem 1 is new, and appears difficult to prove in any other way.

THEOREM 2.—*We have, uniformly in* $\frac{1}{2}+\delta \leqslant \sigma \leqslant 1-\delta$,

$$\frac{\zeta'(s)}{\zeta(s)} = O\{(\log t)^{2(1-\sigma)}\}.*$$

We require the following Lemma, to which we shall have to appeal again.

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* The novelty consists in the absence of a factor $\log\log t$ on the right.

LEMMA 2.—*We have, uniformly in* $\frac{1}{2} < \sigma \leqslant \frac{9}{8}$,

$$\sum_{\rho} |\Gamma(\rho-s)| < \frac{A \log(t+2)}{\sigma-\frac{1}{2}}.$$

In the first place

$$|\Gamma(\rho-s)| = |\Gamma(\rho-s+1)|/|\rho-s| \leqslant |\Gamma(\rho-s+1)|/(\sigma-\tfrac{1}{2}).$$

Next,

$$|\Gamma(\rho-s+1)| < Ae^{-A|t-\gamma|},$$

since $\Re(\rho-s+1)$ lies between $\frac{3}{8}$ and 1. Also

$$\sum_{\gamma} e^{-A|t-\gamma|} = \sum_{n=1}^{\infty} \sum_{n-1 \leqslant |t-\gamma| < n} e^{-A|t-\gamma|},$$

and, since the number of terms in the inner summation is less than $A \log(t+n+2)$, we have finally

$$\sum_{\rho} |\Gamma(\rho-s)| < \frac{A}{\sigma-\frac{1}{2}} \sum_{n=1}^{\infty} \log(t+n+2)\, e^{-An} = \frac{A}{\sigma-\frac{1}{2}} \left\{ \sum_{n<t} + \sum_{n \geqslant t} \right\}$$

$$< \frac{A}{\sigma-\frac{1}{2}} \left\{ \log(2t+2) \sum_{n<t} e^{-An} + \sum_{n \geqslant t} \log(2n+2)\, e^{-An} \right\} < \frac{A}{\sigma-\frac{1}{2}} \log(t+2).$$

We return now to Theorem 2. By Theorem 1

(2.41)

$$\frac{\xi'(s)}{\xi(s)} = O\{\Sigma \Lambda(n) n^{-\sigma} e^{-ny}\} + O(y^{\sigma-\frac{1}{2}}) \Sigma |\Gamma(\rho-s)| + O(y^{\sigma-\frac{1}{2}} \log t) + O(1).$$

Hence, by Lemmas 1 and 2,

$$\frac{\xi'(s)}{\xi(s)} = O\{(y^{\sigma-1}-1)\Gamma(1-\sigma)\} + O(y^{\sigma-\frac{1}{2}} \log t) + O(1)$$

$$= O(y^{\sigma-1}) + O(y^{\sigma-\frac{1}{2}} \log t) \quad (\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 1-\delta).$$

In this we take $y = (\log t)^{-2}$, and obtain the result desired.

2.5. We write

$$\Lambda_r(n) = \frac{\Lambda(n)}{\log^r n},$$

$$f(s) = \log \xi(s) = \Sigma \Lambda_1(n)\, n^{-s}.$$

THEOREM 3.—

(2.51) $$f(s) = \Sigma \Lambda_1(n)\, n^{-s} e^{-ny} + O(y^{\sigma-\epsilon})\, y^{-\sigma_1} (\log t)^{\nu(\sigma_1)+\epsilon} + O(1),$$

uniformly in $\frac{1}{2}+\delta \leqslant \sigma \leqslant \frac{9}{8}$, $\frac{1}{2} \leqslant \sigma_1 \leqslant \mathrm{Min}(\sigma, 1)$, *and* $e^{-\sqrt{t}} < y < 1$. *In particular (taking* $\sigma_1 = \frac{1}{2}$)

$$f(s) = \Sigma \Lambda_1(n)\, n^{-s} e^{-ny} + O\{y^{\sigma-\frac{1}{2}-\epsilon}(\log t)^{\beta+\epsilon}\} + O(1). \tag{2.52}$$

We begin by proving a corresponding formula for $\xi'(s)/\xi(s)$, viz.

$$-\frac{\xi'(s)}{\xi(s)} = \Sigma \Lambda(n)\, n^{-s} e^{-ny} + O(y^{\sigma-\epsilon})\, y^{-\sigma_1}(\log t)^{\nu(\sigma_1)+\epsilon}. \tag{2.53}$$

We may suppose that ϵ is small and that σ_1 satisfies $\frac{1}{2}+\epsilon \leqslant \sigma_1 \leqslant \sigma-\epsilon$. For, since $\nu(\sigma_1)$ is continuous, the result is true (by rechoice of ϵ) for the extended range if it is true for the restricted range. We have

$$\Sigma \Lambda(n)\, n^{-s} e^{-ny} = \Sigma \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Lambda(n)\, n^{-s-z}\, y^{-z}\, \Gamma(z)\, dz$$

$$= -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\xi'(s+z)}{\xi(s+z)}\, y^{-z}\, \Gamma(z)\, dz.$$

As in § 2.2, we may replace the path of integration by another vertical line, this time $\Re z = \sigma_1 - \sigma$, taking account of the relevant residues. The singularities are $z = 0$, with residue $\xi'(s)/\xi(s)$, and $z = 1-s$, with residue $-y^{s-1}\Gamma(1-s) = O(y^{\sigma})$.

To prove (2.53), it is therefore enough to show that

$$\int_{\sigma_1-\sigma-i\infty}^{\sigma_1-\sigma+i\infty} \frac{\xi'(s+z)}{\xi(s+z)}\, y^{-z}\, \Gamma(z)\, dz = O(y^{\sigma-\epsilon})\, y^{-\sigma_1}(\log t)^{\nu(\sigma_1)+\epsilon}. \tag{2.54}$$

On the contour of integration, if $z = \xi + i\eta$ and so $-\frac{5}{8} \leqslant \xi \leqslant -\epsilon$,

$$\left|\frac{\xi'(s+z)}{\xi(s+z)}\right| < K\{\log(t+|\eta|)\}^{\nu(\sigma_1)+\epsilon},$$

$$|\Gamma(z)| < Ke^{-K|\eta|},$$

$$\int_{\sigma_1-\sigma-i\infty}^{\sigma_1-\sigma+i\infty} = O(y^{\sigma-\sigma_1}) \int_0^{\infty} e^{-K\eta}\{\log(t+\eta)\}^{\nu(\sigma_1)+\epsilon}\, d\eta$$

$$= O(y^{\sigma-\sigma_1})\left\{\int_0^t e^{-K\eta}(\log t)^{\nu(\sigma_1)+\epsilon}\, d\eta + \int_t^{\infty} e^{-K\eta}(\log \eta)^{\nu(\sigma_1)+\epsilon}\, d\eta\right\}$$

$$= O(y^{\sigma-\sigma_1})(\log t)^{\nu(\sigma_1)+\epsilon},$$

and (2 . 54) follows *a fortiori*. To deduce (2 . 51), we have finally only to keep σ_1 fixed in (2 . 53), and integrate with respect to σ from $\frac{2}{3}$ to σ.

3. *The relations between the α's and the β's.*

3 . 1. Our first object is to prove

THEOREM 4.*— $$\beta \leqslant \operatorname{Max}(0, \alpha_n) \quad (n \geqslant 0).$$

The first lower bound (on the Riemann hypothesis) for $|S(T)|$ was established by Landau,† and he gave two distinct methods of attacking the problem. The first method does not lead (at any rate naturally) to the "one-sided" results‡ that can be proved by the second. Here we do not attack the one-sided problem, and employ the first method. Practically the only novelty of idea, indeed, is the consideration of α_n as well as α.

Let σ be fixed and $\frac{1}{2} < \sigma < 1$, $t > 4$. Then§

$$\frac{\xi'(s)}{\xi(s)} = -\tfrac{1}{2}\log t + O(1) + \Sigma_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{3 . 11}$$

$$\Re \frac{\xi'(s)}{\xi(s)} = -\tfrac{1}{2}\log t + O(1) + \xi\Sigma\{\phi(\gamma_n) + \phi(-\gamma_n)\}, \tag{3 . 12}$$

where we write $\xi = \sigma - \frac{1}{2}$, $\rho = \frac{1}{2} + i\gamma$, γ_n for the n-th *positive* γ, and

$$\phi(u) = \frac{1}{\xi^2 + (t-u)^2}.$$

Evidently

$$|\xi\Sigma\phi(-\gamma_n)| < \Sigma\frac{1}{\gamma_n^2} = O(1). \tag{3 . 13}$$

Also

$$\Sigma\phi(\gamma_n) = \sum_{n=1}^{\infty}\{n-(n-1)\}\phi(\gamma_n) = \sum_{n=1}^{\infty} n\{\phi(\gamma_n) - \phi(\gamma_{n+1})\},$$

since $n\phi(\gamma_{n+1}) \to 0$ as $n \to \infty$. The term on the right is

$$-\Sigma\int_{\gamma_n}^{\gamma_{n+1}} N(u)\,\phi'(u)\,du = -\int_{\gamma_1}^{\infty} N(u)\,\phi'(u)\,du = -\int_1^{\infty} N(u)\,\phi'(u)\,du,$$

* Since the inequality $\beta > 0$ is known, we might have given the theorem in the form $\alpha_n \geqslant \beta$. But since we are discussing the lower bound for β *de novo*, we must postpone a final statement to Theorem 6.

† E. Landau, *Züricher Vierteljahrschrift*, 56 (1911), 125–148. The results are proved later, in sharper form, in B.L.

‡ Of type $S(T) > \phi(T)$, $S(T) < -\phi(T)$, each for some arbitrarily large T.

§ H.B., 318.

since $\gamma_1 > 1$,

$$= -\int_1^\infty M(u)\,\phi'(u)\,du - \int_1^\infty S(u)\,\phi'(u)\,du + O(1). \tag{3.14}$$

Now

$$-\xi\int_1^x M(u)\,\phi'(u)\,du = -[\xi M(u)\,\phi(u)]_1^\infty + \frac{\xi}{2\pi}\int_1^\infty (\log u + A)\,\phi(u)\,du$$

$$= O(1) + \frac{1}{2\pi}\int_1^\infty \frac{\log u}{\xi^2+(t-u)^2}\,\xi\,du$$

$$= O(1) + \frac{1}{2\pi}\int_{\frac{1}{2}t}^{2t} \frac{\log t + O(1)}{\xi^2+(t-u)^2}\,\xi\,du + \int_1^{\frac{1}{2}t} \frac{O(\log t)}{t^2}\,\xi\,du + \int_{2t}^\infty \frac{O(\log u)}{u^2}\,\xi\,du$$

$$= O(1) + \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\log t + O(1)}{\xi^2+(t-u)^2}\,\xi\,du + \int_1^{\frac{1}{2}t} \frac{O(\log t)}{t^2}\,\xi\,du + \int_{2t}^\infty \frac{O(\log u)}{u^2}\,\xi\,du$$

$$= O(1) + \tfrac{1}{2}\log t + O(1) + O(1) + O(1) = \tfrac{1}{2}\log t + O(1). \tag{3.15}$$

From (3.12) to (3.15)

$$\mathfrak{R}\,\frac{\zeta'(s)}{\zeta(s)} = -\xi\int_1^\infty S(u)\,\phi'(u)\,du + O(1). \tag{3.16}$$

Integration by parts gives

$$\mathfrak{R}\,\frac{\zeta'(s)}{\zeta(s)} = \xi\big[\{-S_1(u)+A\}\,\phi'(u)\big]_1^\infty + \xi\int_1^\infty S_1(u)\,\phi''(u)\,du$$

$$= O(1) + \xi\int_1^\infty S_1(u)\,\phi''(u)\,du,$$

the process may be repeated indefinitely, and we have generally,

$$\mathfrak{R}\,\frac{\zeta'(s)}{\zeta(s)} = O(1) + (-1)^{n+1}\xi\int_1^\infty S_n(u)\,\phi^{(n+1)}(u)\,du. \tag{3.17}$$

Now if $$\psi(v) = \frac{1}{1+v^2}.$$

we have, for $n > 0$,

$$|\psi^{(n)}(v)| = O\Big(\frac{1}{1+v^2}\Big).$$

as $v \to \pm\infty$. Consequently

$$|\psi^{(n)}(v)| < \frac{A(n)}{1+v^2}$$

for all v, where $A(n)$ depends only on n. It follows that

$$|\phi^{(n)}(u)| < \frac{A(n)}{\xi^2+(t-u)^2} \quad (n \geqslant 0).$$

If now $k > \text{Max}(a_n, 0)$ (3.17) gives

$$\Re \frac{\zeta'(s)}{\zeta(s)} = O(1)+\int_1^\infty O\{\log^k(u+t)\} \frac{\xi\, du}{\xi^2+(t-u)^2}$$

$$= O(1)+O(\log^k t)\int_{\frac{1}{2}t}^{2t} \frac{\xi\, du}{\xi^2+(t-u)^2}+O(\log^k t)\int_1^{\frac{1}{2}t} \frac{\xi\, du}{t^2}+\int_{2t}^\infty \frac{O(\log^k u)}{u^2}\,\xi\, du$$

$$= O(1)+O(\log^k t) = O(\log^k t).$$

But the ν-function of the left side is $\nu(\sigma)$; hence

$$\nu(\sigma) \leqslant k.$$

This is true whenever $\frac{1}{2} < \sigma < 1$, $k > \text{Max}(a_n, 0)$. Since $\nu(\sigma) \to \beta$ as $\sigma \to \frac{1}{2}$, we conclude that

$$\beta \leqslant \text{Max}(a_n, 0).$$

3.2. We prove next

THEOREM 5.— $$\beta \geqslant \beta(1) \geqslant \tfrac{1}{2}.$$

Let $x = y^{-1-\delta}$. By a theorem of Dirichlet there exists a $t = t(y) > 2\pi$ such that

$$\left|\left\{\frac{t}{2\pi}\log n\right\}\right| \leqslant \frac{1}{x} \quad (n \leqslant x),* \tag{3.21}$$

and

$$\frac{t}{2\pi} < (x+1)^{x+1}. \tag{3.22}$$

We require now

LEMMA 3.—*The t of* (3.21) *satisfies*

$$y > e^{-\sqrt{t}} \tag{3.23}$$

for sufficiently small y.†

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* $\{u\}$ denotes in general the number $u-p$, where p is the integer nearest u; and denotes $+\frac{1}{2}$ if u differs from an integer by $\frac{1}{2}$.

† This result is, in fact, independent of the Riemann hypothesis.

Suppose that, on the contrary, $y \leqslant e^{-\sqrt{t}}$ (for some arbitrarily small y). Then, with $t = t(y)$ and $u = 2/\log(x-1)$, we have

$$\Re\,\zeta(1+u+it) = \Big(\sum_{n\leqslant x} + \sum_{n>x}\Big) n^{-1-u}\cos(t\log n)$$

$$\geqslant \sum_{n\leqslant x} n^{-1-u}\cos\frac{2\pi}{x} - \sum_{n>x} n^{-1-u} \geqslant \tfrac{1}{2}\sum_{n\leqslant x} n^{-1-u} - \sum_{n>x} n^{-1-u}$$

$$\geqslant \tfrac{1}{2}\sum_{n=1}^{\infty} n^{-1-u} - \tfrac{3}{2}\sum_{n>x} n^{-1-u} \geqslant \frac{1}{2u} - A - \frac{3}{2}\frac{(x-1)^{-u}}{u} = \frac{1}{2u}(1-3e^{-2}) - A$$

$$(3.24)\quad > \frac{A}{u} > K\log x > K\log\frac{1}{y}$$

$$(3.25)\quad > K\sqrt{t}.$$

Now when y is small, x is large; hence $t\to\infty$ by (3.24) (since $t > 2\pi$). But then (3.25) is impossible, and this proves the Lemma.

We may now employ Theorem 3 with the present meanings of y and $t = t(y)$, and for fixed σ, σ_1 satisfying $\frac{1}{2} < \sigma_1 < \sigma < 1$, remembering that $t\to\infty$ as $y\to 0$. Taking the real part, we have

$$\Re\log\zeta(s) = \sum_{n\leqslant x}\Lambda_1(n)n^{-\sigma}\cos(t\log n)e^{-ny} + \sum_{n>x}\Lambda_1(n)n^{-\sigma}\cos(t\log n)e^{-ny}$$

$$+ O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon} + O(1)$$

$$> \sum_{n\leqslant x}\Lambda_1(n)n^{-\sigma}\{1+O(x^{-1})\}e^{-ny} + O\Big(\sum_{n>x} e^{-ny}\Big) + O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon} + O(1)$$

$$> \frac{1}{\log x}\sum_{n\leqslant x}\Lambda(n)n^{-\sigma}e^{-ny} + O(x^{-1})\sum_{n\leqslant x} n^{-\sigma}$$

$$+ O\Big(\sum_{n>x} e^{-ny}\Big) + O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon} + O(1)$$

$$> \frac{1}{\log x}\sum_{n=1}^{\infty}\Lambda(n)n^{-\sigma}e^{-ny} + O\Big(\sum_{n>x} e^{-ny}\Big) + O(1) + O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon}$$

$$> \frac{1}{\log x}\{Ky^{\sigma-1} + O(1)\} + O\Big(\frac{e^{-xy}}{y}\Big) + O(1) + O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon},$$

by Lemma 1,

$$(3.26)\quad > y^{\sigma-1+\epsilon} + O(y^{\sigma-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon},$$

when y is small enough. Now, by (3 . 22),

$$y = x^{-1/(1+\delta)} < (\log t)^{-(1-\epsilon)/(1+\delta)}.$$

Hence by (3 . 26), if ϵ is less than $\sigma-\sigma_1$ and $1-\sigma$.

$$\text{(3 . 27)} \qquad \Re \log \zeta(s) \geqslant (\log t)^{1-\sigma+\eta} + O\{(\log t)^{\sigma_1-\sigma+\nu(\sigma_1)+\eta'}\},$$

where η and η' are functions of ϵ and δ which tend to zero with ϵ and δ. Now (3 . 27) holds for arbitrarily small ϵ, δ, and arbitrarily large t (depending on y and δ), and the left-hand side is $O\{(\log t)^{\nu(\sigma)+\epsilon}\}$. It is evident on a little consideration, therefore, that we must have *either*

$$\text{(3 . 28)} \qquad \nu(\sigma) \geqslant 1-\sigma,$$

or else

$$\sigma_1-\sigma+\nu(\sigma_1) \geqslant 1-\sigma.$$

Since the second alternative reduces to the first when $\sigma_1 \to \sigma$ we must have (3 . 28) in any case, and this is $\beta(\sigma) \geqslant \frac{1}{2}$.

We may now sum up what is known concerning the α's and β's.

THEOREM 6.—

$$\tfrac{1}{2} \leqslant \beta(1) \leqslant \beta \leqslant \ldots \leqslant \alpha_{n+1} \leqslant \alpha_n \leqslant \ldots \leqslant \alpha \leqslant 1.$$

4. *The function* $\zeta(1+it)$.

4 . 1. The theorems of this section are of two types, concerned respectively with upper bounds and lower bounds. We prove first

THEOREM 7.—

$$|\log \zeta(1+it)| \leqslant \log\log\log t + \log c + \epsilon \quad [t > t_0(\epsilon)],$$

where

$$c = 2\beta(1)e^C \leqslant 2\beta e^C \leqslant 2e^C,$$

and C is Euler's constant. In particular

$$|\zeta(1+it)| < (1+\epsilon)c \log\log t \quad (t > t_0),$$

and the same inequality is satisfied by $1/|\zeta(1+it)|$.

We have from Theorem 3, with $\sigma = 1$, $\frac{1}{2} < \sigma_1 < 1$,

$$\text{(4 . 11)} \qquad |\log \zeta(1+it)| \leqslant \Sigma \Lambda_1(n) n^{-1} e^{-ny} + O(y^{1-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon}$$

$$\text{(4 . 12)} \qquad < \sum_{n < y^{-1-\delta}} \Lambda_1(n) n^{-1} + o(1) + O(y^{1-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon},$$

since $$\sum_{n>y^{-1-\delta}} \Lambda_1(n)\, n^{-1} e^{-ny} < \frac{e^{-y^{-\delta}}}{1-e^{-y}} = o(1).$$

Now

$$0 \leqslant -\log \prod_{p<X}(1-p^{-1}) - \sum_{n<X} \Lambda_1(n)\, n^{-1} \leqslant \sum_{\substack{p<X \\ p^m \geqslant X}} p^{-m}$$

$$\leqslant \sum_{\sqrt{X} \leqslant p < X} (p^{-2}+p^{-3}+\ldots) < 2 \sum_{n>\sqrt{X}} n^{-2} = o(1)$$

as $X \to \infty$. Hence from (4 . 12), writing $X = y^{-1-\delta}$

$$|\log \zeta(1+it)| < -\log \prod_{p<X}(1-p^{-1}) + o(1) + O(y^{1-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon}$$

$$= \log\left[\{1+o(1)\}\, e^C \log X\right] + o(1) + O(y^{1-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon},$$

since*

$$\prod_{p<X}(1-p^{-1}) \sim \frac{e^{-C}}{\log X}. \tag{4 . 13}$$

Thus

$$|\log \zeta(1+it)| < \log\log(1/y) + C + \log(1+\delta) + o(1) + O(y^{1-\sigma_1-\epsilon})(\log t)^{\nu(\sigma_1)+\epsilon}$$

In this we choose $\epsilon < 1-\sigma_1$ and y such that

$$y^{-(1-\sigma_1-\epsilon)} = (\log t)^{\nu(\sigma_1)+2\epsilon}.$$

Then the O-term is $o(1)$,

$$\log\log\frac{1}{y} = \log\log\log t + \log\frac{\nu(\sigma_1)+2\epsilon}{1-\sigma_1-\epsilon},$$

and so

$$\overline{\lim_{t\to\infty}}\, \{|\log \zeta(1+it)| - \log\log\log t\} \leqslant C + \delta + \log\frac{\nu(\sigma_1)+2\epsilon}{1-\sigma_1-\epsilon}.$$

If we make ϵ and δ tend to zero in this, the right-hand side becomes $C+\log\{2\beta(\sigma_1)\}$, and in this we may finally give $\beta(\sigma_1)$ its minimum value $\beta(1)$.

4 . 2. We turn now to the result of opposite type.

THEOREM 8.—*We have, independently of the Riemann hypothesis,*

$$\overline{\lim_{t\to\infty}}\, \frac{|\zeta(1+it)|}{\log\log t} \geqslant e^C.$$

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* H.B., 140.

Thus, comparing with Theorem 7, the upper limit lies between e^C and $2e^C$, and e^C is the exact value if it is true that $\beta(1) = \frac{1}{2}$.

For the proof of Theorem 8 we require a new Lemma.

LEMMA 4.—*Given $\delta > 0$, there exists a positive integer N, and positive $a_1, a_2, \ldots, a_N$, such that*

$$\sum_{n=1}^{N} a_n = 1$$

and

$$\sum_{n=1}^{N} a_n \cos n\phi > -\delta \quad (0 \leqslant \phi \leqslant 2\pi).$$

If $0 < r < 1$, we have

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\phi = \frac{1-r^2}{1-2r\cos\phi + r^2} > 0.$$

The series is uniformly convergent in the range of ϕ; hence there exists a $\nu = \nu(r)$, such that

$$1 + 2\sum_{n=1}^{N} r^n \cos n\phi > -1 \quad (N \geqslant \nu).$$

Let now

$$a_n = r^n \Big/ \sum_{n=1}^{N} r^n \quad (1 \leqslant n \leqslant N).$$

Then $\Sigma a_n = 1$ and

$$\sum_{n=1}^{N} a_n \cos n\phi > -1 \Big/ \sum_{n=1}^{N} r^n = -\frac{1-r}{1-r^{N+1}}. \tag{4.21}$$

If we choose $r = 1 - \frac{1}{2}\delta$, and $N > \nu(r)$ and so large that $r^{N+1} < \frac{1}{2}$, (4.21) leads to the inequality $\Sigma a_n \cos n\phi > -\delta$, and the lemma is proved.

4.3. Let now

$$\lambda = \overline{\lim_{t\to\infty}} \, |\zeta(1+it)/\log\log t|.$$

Given $x > 4$ (large), there exists, by Dirichlet's theorem, a $t = t(x) > 2\pi$ such that

$$\left| \left(\frac{t}{2\pi} \log n \right) \right| \leqslant \frac{1}{x} \quad (n \leqslant x)$$

and

$$\frac{t}{2\pi} < (x+1)^{x+1},$$

so that

$$0 < \log \frac{t}{2\pi} < (x+1)\log(x+1). \tag{4.31}$$

Let now u be fixed, and $0 < u \leqslant 1$,

$$\sigma = 1 + \frac{u}{\log x} < 2,$$

and let δ be a (small) positive number, N and $a_1, a_2, \ldots, a_N$ the associated numbers of Lemma 4. Finally let

$$s_m = \sigma + mit, \quad \xi_m = \xi(s_m) \quad (m \leqslant N).$$

In what follows, O's refer to $x \to \infty$; the parameters δ, N, $a_1, \ldots, a_N$, u are all fixed.

We have, for $m \leqslant N$,

$$\log \xi_m = -\sum_{p \leqslant x} \log(1-p^{-s_m}) - \sum_{p > x} \log(1-p^{-s_m}),$$

$$\Re \log \xi_m = -\sum_{p \leqslant x} \log(1-p^{-\sigma}) + \Re \sum_{p \leqslant x} \{\log(1-p^{-\sigma}) - \log(1-p^{-s_m})\} + \Re \sum_{\substack{p>x \\ r>0}} \frac{1}{r} p^{-rs_m}. \tag{4.32}$$

In the second term on the right

$$\Im s_m \log p \equiv O\left(\frac{m}{x}\right) = O\left(\frac{1}{x}\right) \pmod{2\pi},$$

and the second term itself is of the form

$$\sum_{p \leqslant x} O(x^{-1}) p^{-\sigma} = O(x^{-1}) \sum_{p \leqslant x} p^{-1} = o(1).$$

It follows from (4.32) (since $\Sigma a_m = 1$) that

$$\sum_{m=1}^{N} a_m \Re \log \xi_m$$

$$= -\log \prod_{p \leqslant x} (1-p^{-\sigma}) + o(1) + \sum_{r=1}^{\infty} \frac{1}{r} \sum_{p>x} p^{-r\sigma} \sum_{m=1}^{N} a_m \cos(rm\, t \log p)$$

$$\geqslant -\log \prod_{p \leqslant x} (1-p^{-\sigma}) + o(1) - \delta \sum_{r=1}^{\infty} \frac{1}{r} \sum_{p>x} p^{-r\sigma}. \tag{4.33}$$

The coefficient of $-\delta$ on the right is

$$-\sum_{p>x}\log(1-p^{-\sigma}) < 2\sum_{p>x}p^{-\sigma} = 2\sum_{n>x}\{\pi(n)-\pi(n-1)\}n^{-\sigma}$$

$$\leqslant 2\sum_{n>x}\pi(n)\{n^{-\sigma}-(n+1)^{-\sigma}\}$$

$$\leqslant 2\int_x^\infty \pi(v)\frac{d}{dv}(-v^{-\sigma})\,dv < \frac{A}{\log x}\int_x^\infty v\frac{d}{dv}(-v^{-\sigma})\,dv$$

$$= \frac{A\sigma}{\log x}\frac{x^{1-\sigma}}{\sigma-1} < \frac{A}{\log x}e^{-u}\frac{\log x}{u} = \frac{Ae^{-u}}{u}.$$

From (4 . 33) we have, therefore,

$$\Sigma a_m \mathfrak{R}\log\xi_m \geqslant -\log\prod_{p\leqslant x}(1-p^{-\sigma})+o(1)-\frac{A\delta e^{-u}}{u}. \tag{4 . 34}$$

The right-hand side tends to infinity with x; hence, since $t>2\pi$,

$$t = t(x)\to\infty \quad (x\to\infty). \tag{4 . 35}$$

Now for fixed u we have

$$\overline{\lim_{x\to\infty}}\,\mathfrak{R}\log\frac{\xi_m}{\log\log t} = \log\overline{\lim_{x\to\infty}}\frac{|\xi(\sigma+mit)|}{\log\log(mt)}$$

$$\leqslant \log\overline{\lim_{T\to\infty}}\frac{|\xi(1+iT)|^*}{\log\log T} = \log\lambda.$$

Hence, from (4 . 34) and (4 . 35),

$$\log\lambda = \Sigma a_m\log\lambda \geqslant \Sigma a_m\overline{\lim_{x\to\infty}}\,\mathfrak{R}\log\left(\frac{\xi_m}{\log\log t}\right) \geqslant \overline{\lim_{x\to\infty}}\,\Sigma a_m\mathfrak{R}\log\left(\frac{\xi_m}{\log\log t}\right)$$

$$\geqslant \overline{\lim_{x\to\infty}}\log\left(\frac{\prod_{p\leqslant x}(1-p^{-\sigma})^{-1}}{\log\log t}\right)-\frac{A\delta e^{-u}}{u} \geqslant \overline{\lim_{x\to\infty}}\log\left(\frac{\prod_{p\leqslant x}(1-p^{-\sigma})^{-1}}{\log x}\right)-\frac{A\delta e^{-u}}{u},$$

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* This is a result of well known type. Let $f(s) = \zeta(s)/\log\log(s+hi)$ where $h>4$. On $\sigma=1$, $t\geqslant 0$, we have

$$|f(s)| \leqslant \frac{|\zeta(s)|}{\log\log t} < \lambda+\epsilon \quad (t>t_0).$$

Also on $\sigma=2$,

$$|f(s)| = o(1) < \lambda+\epsilon \quad (t>t_1).$$

We can choose h so that $|f(s)|<\lambda+\epsilon$ also on the remainder of the boundary of the strip bounded by $\sigma=1$, $\sigma=2$, and $t=1$. Since $f(s)=O(e^t)$ in the interior of the strip, we have, by a theorem of Phragmén and Lindelöf, $|f|<\lambda+\epsilon$ in the interior, and so

$$\overline{\lim_{t\to\infty}}\frac{|\zeta(s)|}{\log\log t} = \overline{\lim_{t\to\infty}}\frac{|\zeta(s)|}{\log\log(t+h)} \leqslant \lambda,$$

as $s\to\infty$ in the strip in any manner.

since, by (4 . 31), $\log\log t < \{1+o(1)\}\log x$. Hence, by (4 . 13)

$$\log\lambda - C \geqslant \overline{\lim_{x\to\infty}} \log \prod_{p\leqslant x}\left(\frac{1-p^{-1}}{1-p^{-\sigma}}\right) - \frac{A\delta e^{-u}}{u}. \tag{4 . 36}$$

The first term on the right is

$$\overline{\lim} \sum_{p\leqslant x} \log\left(1-\frac{p^{-1}-p^{-\sigma}}{1-p^{-\sigma}}\right) > -A\ \overline{\lim} \sum_{p\leqslant x}(p^{-1}-p^{-\sigma}). \tag{4 . 37}$$

Now for $p \leqslant x$

$$\frac{p^{-1}-p^{-\sigma}}{p^{-1}} = 1-\exp\left(-\frac{u\log p}{\log x}\right) < Au\frac{\log p}{\log x},$$

$$\sum_{p\leqslant x}(p^{-1}-p^{-\sigma}) \leqslant \frac{Au}{\log x}\sum_{p\leqslant x}\frac{\log p}{p} \leqslant \frac{Au}{\log x}\,A\log x = Au.$$

It follows, therefore, finally from (4 . 36) and (4 . 37) that

$$\log\lambda - C \geqslant -Au - A\delta\frac{e^{-u}}{u}.$$

Since the right-hand side can be made arbitrarily near zero by choice first of u and then of δ, we must have

$$\log\lambda - C \geqslant 0,$$

and our proof is completed.

5. *The mean values of the* $|S_n(T)|$.

5 . 1. We begin with a number of Lemmas.

LEMMA 5.—*We have, uniformly in* $e^{-\nu t} < y < 1$ *and* $\frac{1}{2} < \sigma \leqslant \frac{9}{8}$,

$$\log\xi(s) = \Sigma\Lambda_1(n)\,n^{-s}e^{-ny} + O(\log t)\,I_1(\xi,\eta) + O(1),$$

where

$$\xi = \sigma-\tfrac{1}{2}, \quad \eta = \log\frac{1}{y}, \quad I_1(\xi,\eta) = \int_\xi^\infty e^{-u\eta}\frac{du}{u}.$$

This specializes Theorem 3 by taking $\sigma_1 = \frac{1}{2}$ [and replacing $\nu(\sigma_1)$ by its upper bound 1], and refines upon it in the error terms and in the range of σ. It is, however, in reality more closely related to Theorem 1, of which (through Lemma 2) it is a simple consequence. Theorem 1

gives, in fact,

$$-\frac{\xi'(s)}{\xi(s)} = \Sigma\Lambda(n)\,n^{-s}e^{-ny}+\Sigma y^{s-\rho}\,\Gamma(\rho-s)+O(e^{-\xi\eta}\log t)+O(1)$$

$$= \Sigma\Lambda(n)\,n^{-s}e^{-ny}+O\left(\frac{e^{-\xi\eta}}{\xi}\log t\right)+O(1)$$

by Lemma 2. Integrating this from $\frac{9}{8}$ to σ with respect to σ, and observing that the integral of $e^{-\xi\eta}/\xi$ is less than $I_1(\xi, \eta)$ we obtain the desired result.

5 . 2. LEMMA 6.—*Let*

$$I_1(\xi, \eta) = \int_\xi^\infty e^{-u\eta}\frac{du}{u}, \qquad I_n(\xi, \eta) = \int_\xi^\infty I_{n-1}(u, \eta)\,du.$$

Then, uniformly for $\xi \geqslant 0$, *we have as* $\eta \to \infty$,

(5 . 21) $$I_1(\xi, \eta) = O\{|\log(\xi\eta)|+1\}\,e^{-\xi\eta},$$

(5 . 22) $$I_n(\xi, \eta) \leqslant I_n(0, \eta) = O(\eta^{-(n-1)}) \quad (n > 1).$$

We have $I_1(\xi, \eta) = f_1(\xi\eta)$, where

$$f_1(u) = \int_u^\infty e^{-v}\frac{dv}{v}.$$

If $u \leqslant 1$, $$f_1(u) < \int_u^1 \frac{dv}{v}+\int_1^\infty e^{-v}\,dv = O\left(\log\frac{1}{u}+1\right).$$

If $u > 1$, $$f_1(u) < \int_u^\infty e^{-v}\,dv = O(e^{-u}).$$

Hence, for $u > 0$,

$$f_1(u) = O(|\log u|+1)\,e^{-u}.$$

Further, when $n > 1$,

$$f_n(u) = \int_u^\infty f_{n-1}(v) = \ldots = \int_u^\infty du_1\int_{u_1}^\infty du_2 \ldots \int_{u_{n-2}}^\infty f_1(u_{n-1})\,du_{n-1}$$

is evidently continuous in $u \geqslant 0$. Also

$$I_n(\xi, \eta) = \eta^{-(n-1)}f_n(\xi\eta) \leqslant \eta^{-(n-1)}f_n(0).$$

5 . 3. LEMMA 7.—*If* $a_n = O(1)$, $y < \frac{1}{2}$, *then*

$$\frac{1}{H}\int_T^{T+H}\left|\sum_1^\infty a_n n^{-s}e^{-ny}\right|^2 dt = \Sigma|a_n|^2 n^{-2\sigma}e^{-2ny} + O\left(\frac{1}{Hy}\log\frac{1}{y}\right)$$

uniformly in $0 < H \leqslant T$, $\sigma \geqslant \frac{1}{2}$. *Similarly, if* $a_n = O(\log n)$, *the formula holds with a remainder term*

$$O\left(\frac{1}{Hy}\log^3\frac{1}{y}\right).$$

The left side is

$$\sum_{m,\,n=1}^\infty \frac{1}{H}\int_T^{T+H} a_m\bar{a}_n(mn)^{-\sigma}e^{-(m+n)y}\left(\frac{n}{m}\right)^{it} dt = \sum_{m=n} + \sum_{m\neq n}. \tag{5 . 31}$$

Clearly

$$\sum_{m=n} = \Sigma|a_n|^2 n^{-2\sigma}e^{-2ny}. \tag{5 . 32}$$

Also

$$\sum_{m\neq n} = O\left(\frac{1}{H}\right)\sum'_{m,\,n}(mn)^{-\frac{1}{2}}e^{-(m+n)y}\Big/\left|\log\frac{n}{m}\right|, \tag{5 . 33}$$

$$\Sigma'_{m,\,n} \leqslant \sum_{m=1}^\infty m^{-\frac{1}{2}}e^{-my}\left\{\sum'_{\frac{1}{2}m\leqslant n\leqslant 2m} m^{-\frac{1}{2}}\Big/\left|\log\left(1-\frac{m-n}{m}\right)\right|\right.$$

$$\left. + \sum_{n<\frac{1}{2}m} n^{-\frac{1}{2}}e^{-ny}/A + \sum_{n>2m} n^{-\frac{1}{2}}e^{-ny}/A\right\}$$

$$\leqslant \sum_{m=1}^\infty m^{-1}e^{-my}\sum'_{\frac{1}{2}m\leqslant n\leqslant 2m}\frac{Am}{|m-n|} + A\left(\sum_{m=1}^\infty m^{-\frac{1}{2}}e^{-my}\right)^2$$

$$\leqslant \sum_{m=1}^\infty A\log(m+1)e^{-my} + A(\Sigma m^{-\frac{1}{2}}e^{-my})^2 \tag{5 . 34}$$

$$= O\left(\frac{1}{y}\log\frac{1}{y}\right) + O\left(\frac{1}{y}\right) = O\left(\frac{1}{y}\log\frac{1}{y}\right).$$

The first part of the Lemma follows from (5 . 31) to (5 . 34).

In the second part we have a pair of logarithms running throughout the remainder terms, but the proof is substantially unaltered, since

$$\Sigma\log^3(m+1)e^{-my} = O\left(\frac{1}{y}\log^3\frac{1}{y}\right),$$

$$\Sigma m^{-\frac{1}{2}}\log(m+1)e^{-my} = O\left(y^{-\frac{1}{2}}\log\frac{1}{y}\right).$$

5 . 4. THEOREM 9.—*If* $0 \leqslant a \leqslant 1$, *then, as* $T \to \infty$,

$$\frac{1}{H}\int_T^{T+H} |S(t)|\,dt = O(\log\log T) \quad (T^a \leqslant H \leqslant T), \tag{5 . 41}$$

$$\frac{1}{T}\int_0^T |S(t)|\,dt = O(\log\log T), \tag{5 . 42}$$

$$\frac{1}{H}\int_T^{T+H} |S_n(t)|^2\,dt = O(1) \quad (n > 0,\ T^a \leqslant H \leqslant T), \tag{5 . 43}$$

$$\frac{1}{T}\int_0^T |S_n(t)|^2\,dt = O(1) \quad (n > 0). \tag{5 . 44}$$

We observe first that (5 . 42) follows from (5 . 41). Thus, taking $H = T$ and replacing T by $\frac{1}{2}T$,

$$\int_{\frac{1}{2}T}^{T} |S(t)|\,dt < AT\log\log T \quad (T > 4).$$

Writing $\frac{1}{2}T$, $\frac{1}{2^2}T$, ... in succession for T and adding, we have, if ν is defined by $2^{-\nu-1}T \leqslant 4 < 2^{-\nu}T$,

$$\int_4^T |S(t)|\,dt \leqslant \sum_{n=1}^{\nu} \int_{2^{-n}T}^{2^{-n+1}T} |S(t)|\,dt < A\log\log T \sum_{n=1}^{\nu} 2^{-n+1}T < AT\log\log T,$$

and so obtain (5 . 42). Similarly (5 . 44) follows from (5 . 43).

Again, in proving (5 . 41) and (5 . 43), we may suppose $H = T^a$, and the constant a as small as we please. For if $H_1 = T^a$ and $T^a \leqslant H \leqslant T$, let

$$H = mH_1 + \vartheta H_1, \quad 0 \leqslant \vartheta < 1.$$

If (5 . 41) is proved when $H = H_1$, we have (O's being uniform in r and m)

$$\int_{T+(r-1)H_1}^{T+rH_1} |S(t)|\,dt = O(H_1)\log\log(T+H) = O(H_1\log\log T) \quad (1 \leqslant r \leqslant m),$$

$$\int_{T+mH_1}^{T+H} |S(t)|\,dt = O(H_1\log\log T),$$

and, by addition,

$$\int_T^{T+H} |S(t)|\,dt = O\{(m+1)H_1\log\log T\} = O(H\log\log T)$$

since $m \geqslant 1$; and this is the general (5 . 41). Similarly for (5 . 43).

After these preliminaries we take up first the problem of (5 . 43) with $H = T^a$: that of (5 . 41) is much harder. Lemma 5 gives on integration

$$\int_\sigma^\infty \log \xi(\sigma_1+it)\,d\sigma_1 = \int_\sigma^2 + O(1) = \Sigma \Lambda_2(n)\, n^{-s} e^{-ny} + O(\log T)\, I_2(\xi, \eta) + O(1),$$

and generally, by repeating the process,

$$(5 . 45) \qquad U_r(\sigma) = \int_\sigma^\infty d\sigma_1 \int_{\sigma_1}^\infty \dots \int_{\sigma_{r-1}}^\infty \log \xi(\sigma_r+it)\,d\sigma_r$$

$$= \Sigma \Lambda_{r+1}(n)\, n^{-s} e^{-ny} + O(\log T)\, I_{r+1}(\xi, \eta) + O(1).$$

I have shown, in the paper referred to above, that (when $r > 0$) $S_r(t)$ is equal to one of the four values $\pm \Im U_r(\frac{1}{2})$, $\pm \Re U_r(\frac{1}{2})$. Hence, by (5 . 45), when $r > 0$, $T \leqslant t \leqslant T+H$,

$$S_r(t) = O\Big(\Sigma \Lambda_{r+1}(n)\, n^{-(\frac{1}{2}+it)} e^{-ny}\Big) + O(\log T)\, I_{r+1}(0, \eta) + O(1)$$

$$= O\Big(\Sigma \Lambda_{r+1}(n)\, n^{-(\frac{1}{2}+it)} e^{-ny}\Big) + O(\eta^{-r} \log T) + O(1)$$

by Lemma 6. We choose $y = T^{-\frac{1}{2}a}$, so that $\eta = \frac{1}{2}a \log T$. Then in $T \leqslant t \leqslant T+H$, since $\eta^{-r} \log T \leqslant \eta^{-2} \log T = O(1)$, we have

$$S_r(t) = O(\Sigma \Lambda_{r+1}(n)\, n^{-(\frac{1}{2}+it)} e^{-ny}) + O(1).$$

By Lemma 7, since $\Lambda_{r+1}(n) = O(1)$,

$$\frac{1}{H} \int_T^{T+H} |S_r(t)|^2\,dt = O\Big(\frac{1}{H}\Big) \int_T^{T+H} |\Sigma \Lambda_{r+1}(n)\, n^{-(\frac{1}{2}+it)} e^{-ny}|^2\,dt + O(1)$$

$$= O\Big(\Sigma \Lambda_{r+1}^2(n)\, n^{-1} e^{-2ny}\Big) + O\Big(\frac{1}{Hy} \log \frac{1}{y}\Big) + O(1)$$

$$= O\Big(\Sigma \frac{1}{n \log^2 n}\, e^{-ny}\Big) + O(T^{-\frac{1}{2}a} \log T) + O(1) = O(1),$$

the result desired.

5 . 5. It remains to prove (5 . 41), in which we may suppose $H = T^a$, $a < 1$. Let us take $y = T^{-\frac{1}{2}a}$ (as before), and let

$$\sigma_1 = \tfrac{1}{2} + \xi_1 = \tfrac{1}{2} + \frac{4 \log \log T}{a \log T}.$$

Then $K < \xi_1 \eta < K \log T$, and by Lemma 6

$$I_1(\xi_1, \eta) \log T = O(e^{-\xi_1 \eta} \log \log T \log T) = O(e^{-2 \log \log T} \log^2 T) = O(1).$$

It follows from Lemma 5 that, for $T \leqslant t \leqslant T+H$,

$$\log \xi(\sigma_1+it) = \Sigma \Lambda_1(n)\, n^{-\sigma_1 - t} e^{-ny} + O(1).$$

Hence also

$$(5.51) \quad \bar{S}(t) = \bar{S}(t, T) = \mathfrak{J} \log \xi(\sigma_1+it) = O\Big(\Sigma \Lambda_1(n)\, n^{-\sigma_1 - it} e^{-ny}\Big) + O(1).$$

By Lemma 7

$$\frac{1}{H}\int_T^{T+H} |\bar{S}(t)|^2 dt = O\Big(\frac{1}{H}\Big)\int_T^{T+H} |\Sigma \Lambda_1(n)\, n^{-\sigma_1 - it} e^{-ny}|^2 dt + O(1)$$

$$= O\Big(\Sigma \Lambda_1^2(n)\, n^{-2\sigma_1} e^{-2ny}\Big) + O\Big(\frac{1}{Hy}\log\frac{1}{y}\Big) + O(1).$$

The second term on the right is $O(1)$. The series on the right is less than

$$\Sigma \Lambda_1(n)\, n^{-2\sigma_1} = \log \xi(2\sigma_1) = \log \frac{1}{2\sigma_1 - 1} + O(1) = O(\log\log T).$$

Hence
$$\frac{1}{H}\int_T^{T+H} |\bar{S}(t)|^2 dt = O(\log\log T),$$

$$(5.52) \qquad \frac{1}{H}\int_T^{T+H} |\bar{S}(t)|\, dt = O(\sqrt{\log\log T}) = O(\log\log T).$$

5.6. To prove (5.41) it is now sufficient to prove

$$(5.61) \qquad \frac{1}{H}\int_T^{T+H} |S(t) - \bar{S}(t)|\, dt = O(\log\log T).$$

Now

$$(5.62) \quad S(t) - \bar{S}(t) = \int_{\sigma_1}^{\frac{1}{2}} \mathfrak{J} \frac{\xi'(\sigma+it)}{\xi(\sigma+it)}\, d\sigma$$

$$= -\xi_1 \mathfrak{J} \frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)} + \int_{\sigma_1}^{\frac{1}{2}} \mathfrak{J}\Big(\frac{\xi'(\sigma+it)}{\xi(\sigma+it)} - \frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)}\Big)\, d\sigma.$$

In the first term on the right we have, by Lemma 2,

$$-\frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)} = \Sigma \Lambda(n)\, n^{-\sigma_1 - it} e^{-ny} + O\Big(\frac{e^{-\xi_1 \eta}}{\xi_1}\log T\Big) + O(1),$$

$$= \Sigma \Lambda(n)\, n^{-\sigma_1 - it} e^{-ny} + O(1).$$

Hence

$$\frac{1}{H}\int_T^{T+H}\left|\xi_1\mathfrak{J}\frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)}\right|^2dt=O\left(\frac{\xi_1^2}{H}\right)\int_T^{T+H}|\Lambda(n)n^{-\sigma_1-it}e^{-ny}|^2dt+O(1)$$

$$=O(\xi_1^2)\Sigma\Lambda^2(n)n^{-2\sigma_1}e^{-2ny}+O\left(\frac{\xi_1^2}{Hy}\log^3\frac{1}{y}\right)+O(1),$$

by Lemma 7,

$$=O(\xi_1^2)\left|\frac{d}{d\sigma_1}\frac{\xi'(2\sigma_1)}{\xi(2\sigma_1)}\right|+O(1)=O(\xi_1^2)\left(\frac{1}{2\sigma_1-1}\right)^2+O(1)=O(1),$$

$$\frac{1}{H}\int_T^{T+H}\left|\xi_1\mathfrak{J}\frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)}\right|dt=O(1)=O(\log\log T),$$

and our problem reduces finally to proving that

$$(5.63)\qquad P=\frac{1}{H}\int_T^{T+H}dt\left|\int_{\sigma_1}^{\frac{1}{2}}\mathfrak{J}\left\{\frac{\xi'(\sigma+it)}{\xi(\sigma+it)}-\frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)}\right\}d\sigma\right|=O(\log\log T).$$

Now, by (3.11),

$$\int_{\sigma_1}^{\frac{1}{2}}\left\{\mathfrak{J}\frac{\xi'(\sigma+it)}{\xi(\sigma+it)}-\mathfrak{J}\frac{\xi'(\sigma_1+it)}{\xi(\sigma_1+it)}\right\}d\sigma=\int_{\sigma_1}^{\frac{1}{2}}\left\{\Sigma_\rho\mathfrak{J}\left(\frac{1}{s-\rho}-\frac{1}{s_1-\rho}\right)+O(1)\right\}d\sigma$$

$$=\int_{\sigma_1}^{\frac{1}{2}}(\Sigma'+\Sigma'')\mathfrak{J}\left(\frac{1}{s-\rho}-\frac{1}{s_1-\rho}\right)d\sigma+O(1),$$

where Σ', Σ'' are taken over the γ's for which respectively $|t-\gamma|\leqslant 1$ and $|t-\gamma|>1$. It is well known* that

$$\Sigma''\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)=O(\log t)\quad(\tfrac{1}{2}\leqslant\sigma\leqslant 2).$$

Hence

$$\int_{\sigma_1}^{\frac{1}{2}}\Sigma''\mathfrak{J}\left(\frac{1}{s-\rho}-\frac{1}{s_1-\rho}\right)d\sigma=\int_{\sigma_1}^{\frac{1}{2}}O(\log t)\,d\sigma=O(\xi_1\log T)=O(\log\log T).$$

Next we have

$$\int_{\sigma_1}^{\frac{1}{2}}\Sigma'\mathfrak{J}\left(\frac{1}{s-\rho}-\frac{1}{s_1-\rho}\right)d\sigma=\int_0^{\xi_1}\Sigma'\left(\frac{t-\gamma}{\xi^2+(t-\gamma)^2}-\frac{t-\gamma}{\xi_1^2+(t-\gamma)^2}\right)d\xi$$

$$=\Sigma'\left(\arctan\frac{\xi_1}{t-\gamma}-\frac{(t-\gamma)\xi_1}{\xi_1^2+(t-\gamma)^2}\right)=\Sigma'\psi(\gamma,t),$$

* H.B., 339.

say. Hence

$$P \leqslant \frac{1}{H}\int_T^{T+H}\left(O(\log\log T)+\Sigma'|\psi(\gamma,t)|\right)dt$$

$$\leqslant O(\log\log T)+\frac{1}{H}\Sigma_1\int_{\gamma-1}^{\gamma+1}|\psi(\gamma,t)|\,dt,$$

where Σ_1 is taken over the γ's for which $T-1 \leqslant \gamma \leqslant T+H+1$. But

$$\int_{\gamma-1}^{\gamma+1}|\psi(\gamma,t)|\,dt = 2\int_{\gamma}^{\gamma+1} = 2\int_0^1\left|\arctan\frac{\xi_1}{u}-\frac{u\xi_1}{\xi_1^2+u^2}\right|du$$

$$= 2\xi_1\int_0^{1/\xi_1}\left|\arctan\frac{1}{v}-\frac{v}{1+v^2}\right|dv < 2\xi_1\int_0^{\infty} = O(\xi_1),$$

the integral to ∞ being evidently convergent. Thus finally

$$P = O\left(\frac{1}{H}\,\xi_1\,\Sigma_1 1\right)+O(\log\log T).$$

$\Sigma_1 1$ is the number of γ's between $T-1$ and $T+H+1$, and since there are at most $O(\log t)$ γ's between t and $t+1$, and $\log(T+H+1) = O(\log T)$, it follows that

$$\frac{1}{H}\,\xi_1\,\Sigma_1 1 = O\left(\frac{\xi_1}{H}\,H\log T\right) = O(\log\log T),$$

$$P = O(\log\log T).$$

This completes the proof.

THE LINES OF ELECTRIC FORCE DUE TO A MOVING POINT CHARGE

By MAXIME KOURENSKY.

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1. The determination of the lines of force leads to the integration of the Riccati equation

$$2(c^2-\xi'^2-\eta'^2-\zeta'^2)\frac{d\phi}{d\tau} = [\xi''(\xi'-i\eta')-(\xi'+c)(\xi''-i\eta'')]\phi^2$$
$$+2[i(\xi''\eta'-\xi'\eta'')-c\zeta'']\phi+\xi''(\xi''+i\eta'')-(\xi'-c)(\xi''+i\eta''), \quad (1)$$

where ξ, η, ζ denote the coordinates of a pole moving in space, and are functions of the time τ; one and two dashes denote the first and second derivatives with respect to τ; c is the velocity of light and $i = \sqrt{(-1)}$.

According to Bateman,* the integration of equation (1) leads to the integration of the system of three equations

$$(c^2-\xi'^2-\eta'^2-\zeta'^2)\frac{dl}{d\tau} = (c-l\xi'-m\eta'-n\zeta')\xi''+(\xi'-cl)(l\xi''+m\eta''+n\zeta''),$$

$$(c^2-\xi'^2-\eta'^2-\zeta'^2)\frac{dm}{d\tau} = (c-l\xi'-m\eta'-n\zeta')\eta''+(\eta'-cm)(l\xi''+m\eta''+n\zeta''), \quad (2)$$

$$(c^2-\xi'^2-\eta'^2-\zeta'^2)\frac{dn}{d\tau} = (c-l\xi'-m\eta'-n\zeta')\zeta''+(\zeta'-cn)(l\xi''+m\eta''+n\zeta''),$$

which determine the direction-cosines l, m, n of the line joining the point charge at time τ to the point (Ξ, H, Z) on the line of electric force at time $T > \tau$, and are consistent with the equation

$$l^2+m^2+n^2 = 1.$$

Equations (2) represent a generalization of the equations given by Darboux in *Leçons sur la théorie générale des surfaces*, 1, chh. 2, 3. An

* H. Bateman, "On a Porism connected with the theory of Maxwell's equations, etc.", *Amer. J. of Math.*, 37 (1915), 192–194.

investigation of the lines of force from the integral of equation (1) is useful in the development of the theory of Faraday, Maxwell, J. J. Thomson, Lienard, Wiechert, Heaviside, Abraham, etc.*

It is readily seen that the integration of equation (1) leads also to the integration of the three equations which determine the three parameters of the lines of electric force, given by Bateman in 1922.

The investigations of conformal transformations of space of four dimensions (Lie, Darboux, Minkowski, Bateman, Cunningham), of which the transformations in the theory of relativity are particular cases, can be used to find the lines of force for several types of motion of the pole, when they are known for a particular kind of motion.†

Equation (1) was integrated in the case of rectilinear motion of the pole (Bateman), and of uniform motion in a circle (Murnaghan). Applying conformal transformations, the lines of force for motion in a circular helix can be obtained (Murnaghan), and from the lines of force for a stationary pole can be deduced those corresponding to uniformly accelerated motion of the pole in a circle in space of four dimensions (Leigh Page).

2. It is easily seen that if the factor 2 on the left-hand side of equation (1) is omitted, the general integral will be

$$\phi = \frac{C(c-\xi')+(\xi'+i\eta')}{C(\xi'-i\eta')+(c+\xi')},$$

C being an arbitrary constant.

Thus the solution of equation (1) is a particular case of the following problem: given an integral of Riccati's equation

$$\frac{dy}{dx} = Py^2+Qy+R, \tag{3}$$

to find the integral of the equation

$$S\frac{dz}{dx} = Pz^2+Qz+R. \tag{4}$$

* The literature is quoted in a paper by F. D. Murnaghan, "The Lines of Electric Force", *Amer. J. of Math.*, 39 (1917), 147–162.

† H. Bateman, "On Lines of Electric Induction", *Proc. London Math. Soc.* (2), 21 (1921), 256–270.

We can pass from equation (3) to equation (4), and conversely, by means of a general linear substitution

$$y = \frac{az+\beta}{z+\gamma}^{*} \quad (a\gamma-\beta \neq 0),$$

the formulae of transformation being

$$\begin{aligned} Pa^2+Qa+R-a' &= P(a\gamma-\beta)/S, \\ S\gamma'+P\gamma^2-Q\gamma+R &= SP(a\gamma-\beta), \\ S\beta' &= [(S+1)Q+(Sa-\gamma)P]\beta+(S\gamma-a)R. \end{aligned} \tag{5}$$

When α and γ are found, β can be obtained from the last equation by quadratures.

The first two equations of system (5) give

$$S\gamma'+P\gamma^2-Q\gamma+R = S^2(Pa^2+Qa+R-a'). \tag{6}$$

Putting $$S\gamma'+P\gamma^2-Q\gamma = 0,$$

we shall have the required functions a, β, γ for a particular Riccati equation for which the function α satisfies the equation

$$\frac{dt}{dx} = Pt^2+Qt+RT \tag{7}$$

of type (3). Every equation (3) can be put in the form (7) by a substitution

$$y = \frac{at+b}{t+c} \quad (ac-b \neq 0),$$

where

$$\begin{aligned} Pa^2+Qa+R-a' &= P(ac-b), \\ c'+Pc^2-Qc+RT &= P(ac-b), \\ b' &= [2Q+P(a-c)]b-R(Ta-c). \end{aligned}$$

* The usual substitution $y = (az+\beta)/(\gamma z+\delta)$ $(a\delta-\beta\gamma \neq 0)$ gives formulae which are more symmetrical but more complicated, and are not treated here.

When S and T are constants, this problem is connected with the interesting theorem, due to J. de Tilly,* that all linear equations of the second order can be integrated if the following problem can be solved: knowing an integral of either of the equations

$$t''-t'^2+kf(x)=0, \quad t''-t'^2+f(x)+k=0,$$

to find an integral of the equation

$$t''-t'^2+f(x)=0,$$

in which $f(x)$ is, throughout, a function of k as well as of x. If we put $t'=z$, de Tilly's equations reduce respectively to the Riccati equations in the canonical form

$$\frac{dz}{dx}=z^2-kf(x), \quad \frac{dz}{dx}=z^2-f(x)-k, \quad \frac{dz}{dx}=z^2-f(x).$$

3.1. I write the general integral of Riccati's equation (3) in the form

$$y=\frac{C\omega_1+\omega_2\omega_3}{C+\omega_3}, \tag{8}$$

where ω_1, ω_2 are particular solutions, and ω_3 is the function known in the theory of linear equations of the second order as the Schwarzian function, satisfying the equation

$$\{\omega, x\}=2J,$$

where J is the invariant of the normal canonical form $u'=u^2+J$ of the Riccati equation, or of the associated linear equation $v''+Jv=0$. The function ω_3 is determined from (3) by the relations

$$\omega_3=\exp\Big(\int P(\omega_1-\omega_2)dx\Big)=(y_3-\omega_1)/(\omega_2-y_3),$$

where y_3 is a third particular solution of equation (3).

When the integral (8) of equation (3) is found, *i.e.* when the functions ω_1, ω_2, ω_3 are known, the general integrals of equations (4) and (7) are

$$z=\frac{C\Omega_1+\Omega_2\Omega_3}{C+\Omega_3}, \quad t=\frac{CO_1+O_2O_3}{C+O_3}, \tag{9}$$

* J. de Tilly, "Sur l'équation de Riccati et sa double généralisation", *Bull. de l'Académie de Belgique* (3), 9 (1885), 216–235.

i.e. the functions Ω_1, Ω_2, Ω_3 and O_1, O_2, O_3 are determined by the equations

$$\left\{\begin{aligned} &(\Omega_1'-\Omega_2')\frac{\Omega_3}{\Omega_3'}-(\Omega_1+\Omega_2)=(\omega_1'-\omega_2')\frac{\omega_3}{\omega_3'}-(\omega_1+\omega_2),\\ &(\Omega_1\Omega_2'-\Omega_2\Omega_1')\frac{\Omega_3}{\Omega_3'}+\Omega_1\Omega_2=(\omega_1\omega_2'-\omega_2\omega_1')\frac{\omega_3}{\omega_3'}+\omega_1\omega_2,\\ &\frac{(\Omega_1-\Omega_2)\Omega_3}{\Omega_3'}=\frac{S(\omega_1-\omega_2)\omega_3}{\omega_3'}, \end{aligned}\right. \tag{10}$$

$$\left\{\begin{aligned} &(O_1-O_2)\frac{O_3}{O_3'}=(\omega_1-\omega_2)\frac{\omega_3}{\omega_3'},\\ &\frac{O_1'-O_2'}{O_1-O_2}-\frac{O_1+O_2}{O_1-O_2}\frac{O_3'}{O_3}=\frac{\omega_1'-\omega_2'}{\omega_1-\omega_2}-\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\frac{\omega_3'}{\omega_3},\\ &\frac{O_1O_2'-O_2O_1'}{O_1-O_2}+\frac{O_1O_2}{O_1-O_2}\frac{O_3'}{O_3}=T\left[\frac{\omega_1\omega_2'-\omega_2\omega_1'}{\omega_1\omega_2}+\frac{\omega_1\omega_2}{\omega_1-\omega_2}\frac{\omega_3'}{\omega_3}\right]; \end{aligned}\right. \tag{11}$$

and the two particular solutions of equations (4) and (7) must satisfy

$$\left\{\begin{aligned} &\frac{(\Omega_1-\Omega_2)'}{\Omega_1-\Omega_2}\cdot\frac{S(\omega_1-\omega_2)\omega_3}{\omega_3'}-(\Omega_1+\Omega_2)=(\omega_1-\omega_2)'\frac{\omega_3}{\omega_3'}-(\omega_1+\omega_2),\\ &\left(\frac{\Omega_2}{\Omega_1}\right)'\frac{\Omega_1^2}{\Omega_1-\Omega_2}\cdot\frac{S(\omega_1-\omega_2)\omega_3}{\omega_3'}+\Omega_1\Omega_2=\left(\frac{\omega_2}{\omega_1}\right)'\omega_1^2\frac{\omega_3}{\omega_3'}+\omega_1\omega_2, \end{aligned}\right. \tag{10'}$$

$$\left\{\begin{aligned} &\frac{(O_1-O_2)'}{O_1-O_2}-(O_1+O_2)\frac{\omega_3'}{(\omega_1-\omega_2)\omega_3}=\frac{(\omega_1-\omega_2)'}{\omega_1-\omega_2}-\frac{(\omega_1+\omega_2)\omega_3'}{(\omega_1-\omega_2)\omega_3},\\ &\left(\frac{O_2}{O_1}\right)'\frac{O_1^2}{O_1-O_2}+O_1O_2\frac{\omega_3'}{(\omega_1-\omega_2)\omega_3}=T\left[\left(\frac{\omega_2}{\omega_1}\right)'\frac{\omega_1^2}{\omega_1-\omega_2}+\omega_1\omega_2\frac{\omega_3'}{(\omega_1-\omega_2)\omega_3}\right]. \end{aligned}\right. \tag{11'}$$

The solution of Bateman's equation can be reduced to the determination of the functions Ω_1, Ω_2, Ω_3 from systems (10) and (10′), by putting

$$\omega_1=\frac{c-\bar{\zeta}'}{\bar{\xi}'-i\eta'},\qquad \omega_2=\frac{\bar{\xi}'+i\eta'}{c+\bar{\zeta}'},\qquad \omega_3=\frac{c+\bar{\zeta}'}{\bar{\xi}'-i\eta'};\qquad S=2.$$

The third particular solution is in this case

$$y_3=\frac{\omega_2\omega_3+\omega_1}{\omega_3+1}=\frac{c-\bar{\zeta}'+\bar{\xi}'+i\eta'}{c+\bar{\zeta}'+\bar{\xi}'-i\eta'}.$$

4. I have given an integrable case of Riccati's equation

$$\frac{dY}{dx} = P_1 Y^2 + Q_1 Y + R_1,$$

when the coefficients satisfy the relation

$$P_1 + Q_1 + R_1 = 0. \tag{12}$$

Then

$$Y = \frac{C + \int (P_1 + R_1) \exp\left(\int (P_1 - R_1)dx\right) dx - \exp\left(\int (P_1 - R_1)dx\right)}{C + \int (P_1 + R_1) \exp\left(\int (P_1 - R_1)dx\right) dx + \exp\left(\int (P_1 - R_1)dx\right)}. \tag{13}$$

This can be applied to every equation

$$\frac{dy}{dx} = Py^2 + Qy + R \tag{14}$$

by means of the formulae

$$Y = y/\omega, \quad P_1 = P\omega, \quad R_1 = R/\omega, \quad P\omega + R/\omega + Q = \omega'/\omega, \tag{15}$$

where ω is a particular solution of equation (14).

This case of integrability can be applied to equation (1) in order to determine the lines of electric force when the motion of an electron in space is limited by constraints, and also for unconstrained motion.

The relation (12) between the coefficients of equation (1) does not depend on the term

$$2(c^2 - \xi'^2 - \eta'^2 - \zeta'^2),$$

and leads to

$$2(\xi' - c)^2 \frac{d}{d\tau} \frac{\zeta' - i\eta'}{\xi' - c} = 0,$$

i.e. to obtain the integral (13), we must have one of the relations

$$\xi - c\tau = \text{const.}, \quad (\zeta' - i\eta')/(\xi' - c) = \text{const.} \tag{16}$$

The second of these conditions gives a relation

$$\zeta - i\eta = C_1(\xi - c\tau) + C_2 \tag{17}$$

between the coordinates ξ, η, ζ, C_1 and C_2 being arbitrary constants.

With condition (17), the integral ϕ_1 will be obtained from (13) by substituting

$$\begin{aligned} P_1 - R_1 &= -2[c\xi'' + i(\eta'\zeta'' - \eta''\zeta')], \\ P_1 + R_1 &= 2i[c\eta'' + i(\zeta'\xi'' - \zeta''\xi')]; \end{aligned} \tag{18}$$

and the third symmetric function is

$$Q_1 = -2[c\xi'' + i(\xi'\eta'' - \xi''\eta')].$$

Putting $\omega = -1, i, -i$ in (15), we see that condition (12) is also satisfied when the motion of the point is such that

$$\begin{aligned} \xi + i\eta &= C_3(\xi + c\tau) + C_4, \\ \xi + i\xi &= C_5(\eta - c\tau) + C_6, \\ \xi - i\xi &= C_7(\eta + c\tau) + C_8, \end{aligned} \tag{19}$$

corresponding to the integrals ϕ_2, ϕ_3, ϕ_4 which are obtained from the general integral ϕ_1, with condition (17), as follows :

$$\phi_2 = -\phi_1, \quad \phi_3 = i\phi_1, \quad \phi_4 = -i\phi_1.$$

In particular, by a suitable choice of arbitrary constants, we get the relations

$$\begin{aligned} \xi + i\eta &= \kappa(i\xi + ic\tau), \\ \xi - i\eta &= \kappa^{-1}(i\xi - ic\tau), \end{aligned} \tag{20}$$

$$\begin{aligned} \xi + i\xi &= \frac{1-\kappa}{1+\kappa}(i\eta - ic\tau), \\ \xi - i\xi &= \frac{1+\kappa}{1-\kappa}(i\eta + ic\tau), \end{aligned} \tag{21}$$

where κ is a constant. The change of coordinate axes in space of four dimensions, given by the formulae

$$X = i\xi, \quad Y = i\eta, \quad Z = \zeta, \quad T = ic\tau,$$

leads to the four real mutually perpendicular hyperplanes through the origin.

Multiplying the pairs of equations (20) and (21), we get the fundamental equation of optics, the equation of the light wave

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2.$$

In the particular case of Darboux's equations*

$$\frac{d\alpha}{dt} = \beta r - \gamma q, \quad \frac{d\beta}{dt} = \gamma p - \alpha r, \quad \frac{d\gamma}{dt} = \alpha q - \beta p,$$

where $$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

and $$\frac{d\sigma}{dt} = \frac{q+ip}{2}\sigma^2 - ir\sigma + \frac{q-ip}{2},$$

the integrable cases (12) lead to the relations

$$q = \pm ir, \quad p = \pm ir.$$

I propose to publish a physical, mechanical, and geometrical interpretation in another paper.

5. Let us examine in detail one of the relations (17) and (19), the first, for instance.

Putting $C_1 = 0$, we get the constraining relation

$$\xi - i\eta = C_2.$$

Choosing a new system of coordinates given by the formulae

$$x = \xi, \quad y = i\eta, \quad z = \zeta, \quad t = c\tau,$$

we shall have the integral of the Riccati equation for the motion of the pole in the plane

$$z - y = C_2,$$

the integral being obtained from (13) by substituting for P_1, R_1 from (18).

Since C_2 is an arbitrary constant, the integral obtained will satisfy equation (1) for the motion of the pole in any plane parallel to (22), such planes forming thin sheets which fill up space. Putting $C_1 = \infty$, and choosing suitable values for C_2, we shall have the unconstraining relation already given in (16),

$$\xi - c\tau = \text{const.},$$

which allows the pole to move freely in space, and for which the integral found still holds consistently with condition (17); it contains the arbitrary time τ, and two other arbitrary quantities C_1, C_2, and limits to some

* Cf. G. Darboux, *Théorie générale des surfaces*, 1 (Paris, 1887), 19, 22.

extent the motion of the pole. Thus formulae (13) and (18) give

$$\phi = \frac{C+2i\int[c\eta''+i(\xi'\xi''-\xi''\xi')]\,e^{-2\int[c\xi''+i(\eta'\zeta''-\eta''\zeta')]\,d\tau}\,d\tau - e^{-2\int[c\xi''+i(\eta'\zeta''-\eta''\zeta')]d\tau}}{C+2i\int[c\eta''+i(\xi'\xi''-\xi''\xi')]\,e^{-2\int[c\xi''+i(\eta'\zeta''-\eta''\zeta')]\,d\tau}\,d\tau + e^{-2\int[c\xi''+i(\eta'\zeta''-\eta''\zeta')]d\tau}};$$

and following Bateman and calculating ψ by changing i to $-i$ in the expression for ϕ, we have the values of l, m, n from the formulae

$$l = \frac{\phi+\psi}{1+\phi\psi}, \qquad m = -i\,\frac{\phi-\psi}{1+\phi\psi}, \qquad n = \frac{1-\phi\psi}{1+\phi\psi},$$

and obtain the equations of the lines of electric force for the motion of an electron in space :

$$\Xi = \xi+c(T-\tau)l,$$
$$\mathrm{H} = \eta+c(T-\tau)m,$$
$$\mathrm{Z} = \zeta+c(T-\tau)n.$$

THE ALLIED SERIES OF A FOURIER SERIES

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. *Introduction.*

1.1. Suppose that $f(t)$ is integrable in the sense of Lebesgue or, as we shall say, integrable (L), and that its Fourier series is

$$\tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos nt+b_n\sin nt)=A_0+\sum_1^\infty A_n=\Sigma A_n. \tag{1.11}$$

Then the series

$$\sum_1^\infty(b_n\cos nt-a_n\sin nt)=\sum_1^\infty B_n=\Sigma B_n \tag{1.12}$$

is called the *allied* or *conjugate* series of the Fourier series.

The allied series of a Fourier series is not necessarily itself a Fourier series. Thus

$$\sum_2^\infty\frac{\cos nx}{\log n}$$

is a Fourier series, while

$$\sum_2^\infty\frac{\sin nx}{\log n},$$

although convergent for all values of x, is not a Fourier series, since its sum function is not integrable (L) in any interval which includes the origin.*

* It has been proved by Young [W. H. Young, "The Fourier series of bounded functions", *Proc. London Math. Soc.* (2), 12 (1913), 41-70 (46)], that $\Sigma a_n\cos nx$ is a Fourier series if a_n is convex (*i.e.* $a_n-2a_{n+1}+a_{n+2}\geqq 0$) and tends to zero. S. Szidon ["Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen", *Math. Zeitschrift*, 10 (1921), 121–127] has shown that Young's condition cannot be replaced by $a_n-a_{n+1}\geqq 0$. Both Young and Szidon give a number of other interesting theorems of a similar character.

The fact that this particular sine-series is not a Fourier series was first noticed by Fatou, and is referred to in Lebesgue's *Leçons sur les séries trigonométriques*, 124. See also O. Perron, "Einige elementare Funktionen, welche sich in eine trigonometrische, aber nicht Fouriersche Reihe entwickeln lassen", *Math. Annalen*, 87 (1922), 84-89; and E. C. Titchmarsh, "Principal value Fourier series", *Proc. London Math. Soc.* (2), 23 (1925), xli-xliii (*Records* for March 13, 1924).

The formula for the sum of the allied series is

$$g(x) = \frac{1}{2\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\cot\tfrac{1}{2}t\,dt \tag{1.13}$$

or

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt, \tag{1.14}$$

it being, of course, understood that, in (1.14), $f(x)$ is defined by periodicity outside its original interval of definition. In these formulae the integrals which occur are generally not Lebesgue integrals, but "generalized" or, as we shall say, "Cauchy" integrals, defined as limits of the type

$$\lim_{\epsilon\to 0}\int_\epsilon^\pi, \qquad \lim_{\epsilon\to 0,\ T\to\infty}\int_\epsilon^T.$$

1.2. The first accurate discussion of a convergence problem for the allied series is due to Pringsheim.* Pringsheim bases his work upon the *absolute* convergence of the integral (1.13), and his criteria are therefore much less general than those with which we shall be concerned. The more delicate theory of the convergence or summability of the series was initiated in two papers of Young,† though there is an implicit discussion of one problem in Fatou's well known memoir,‡ to which we shall refer further in a moment. Young proves (i) that the series is summable $(C, 1)$ if the integral (1.13) exists, in Cauchy's sense, and

$$\int_0^u |f(x+t)-f(x-t)|\,dt = o(u); \tag{1.21}$$

and (ii) that the series is convergent if (1.13) exists and

$$\frac{1}{u}\int_0^u \{f(x+t)-f(x-t)\}\,dt$$

* A. Pringsheim, "Ueber das Verhalten von Potenzreihen auf dem Convergenzkreise", *Münchener Sitzungsberichte*, 30 (1900), 37–100 (79–100).

† W. H. Young, "Konvergenzbedingungen für die verwandte Reihe einer Fourierschen Reihe", *Münchener Sitzungsberichte*, 41 (1911), 361–371; and "On the convergence of a Fourier series and of its allied series", *Proc. London Math. Soc.* (2), 10 (1912), 254–272. See also a later paper, "On the convergence of the derived series of Fourier series", *ibid.*, 17 (1918), 195–236.

‡ P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Math.*, 30 (1906), 335–400.

is of bounded variation, a theorem corresponding to de la Vallée-Poussin's theorem* for Fourier series and including the tests which correspond to the classical tests, for Fourier series, of Jordan and Dini.

1.3. The condition (1.21) is certainly satisfied for all values of x in which $|f(x)-q|$ is the derivative of its integral whatever be q. We shall call this set the *Lebesgue set*. The set complementary to the Lebesgue set has measure zero, so that (1.21) is true almost always.

In the Lebesgue set the Fourier series is summable $(C,1)$,† and indeed (C,δ) for every positive δ,‡ so that the series is summable (C,δ) almost always. It is natural to ask whether the allied series possesses the same property. The answer is less immediate, but is to be found, so far as summability $(C,1)$ is concerned, in a recent dissertation of Plessner.§ The first step in this direction was taken by Fatou, who proved that, if x is a point of continuity of $f(x)$, the integral (1.13) exists, as a Cauchy integral, if and only if the harmonic function

$$\Sigma(b_n \cos nx - a_n \sin nx)\, r^n$$

tends to a limit when $r \to 1$; or, in other words, if and only if the allied series is summable by Abel's limit, or summable (A). Plessner shows that Fatou's conclusion remains valid in all points of the Lebesgue set, and so almost always; and that it holds for summability $(C,1)$ as well as for summability (A). The problem is thus reduced to that of proving that the integral (1.13) exists for almost all values of x.

This is in fact true; but it is not necessarily true in the Lebesgue set, or even at a point of continuity of $f(x)$;¶ and it has never been proved generally and directly. It has been proved directly by Besikovitch** for functions of integrable square, and by M. Riesz and

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* Ch. J. de la Vallée-Poussin, "Un nouveau cas de convergence des séries de Fourier", *Rend. di Palermo*, 31 (1911), 296–299; see also his *Cours d'analyse*, 2 (ed. 2), 149.

† H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, 61 (1905), 251–280.

‡ G. H. Hardy, "On the summability of Fourier's series", *Proc. London Math. Soc.* (2), 12 (1913), 365–372.

§ A. Plessner, "Zur Theorie der konjugierten trigonometrischen Reihen", *Mitteilungen des Math. Seminars der Univ. Giessen*, 10 (1923), 1–36.

‖ *Loc. cit.*, 360.

¶ Example to the contrary: $x=0$, $f(x)=(\log x)^{-1}$ for $x>0$, $f(x)=0$ for $x \leqslant 0$.

** A. Besikovitch, "Sur la nature des fonctions à carré sommable et des ensembles mesurables", *Fundamenta Math.*, 4 (1923), 172–195. The actual theorem, for functions of integrable square, is due to Lusin: see N. Lusin, "Sur la convergence des séries trigonométriques de Fourier", *Comptes Rendus*, June 2, 1913.

Titchmarsh* for functions of which any power higher than the first is integrable; but Plessner's proof is indirect and based upon the theory of analytic functions, and no alternative has yet been found.

1 . 4. In § 2 we prove first that, *in the Lebesgue set, the allied series is either summable by every Cesàro mean of positive order or summable by no Cesàro mean* (nor indeed by Abel's limit). There is nothing very novel in such a theorem, nor in the method of proof, but it is one which, as the natural analogue for the allied series of the generalizations of Fejér's theorem, should be stated explicitly and proved. We have only to combine this theorem with Plessner's to reach the main conclusion of this section, viz. that *the allied series is summable* (C, δ) *for every positive* δ *and almost all values of* x.

In §§ 3–4, which form the most substantial part of the paper, we solve for the allied series the problem which we solved for Fourier series in a recent memoir,† viz. that of finding the necessary and sufficient conditions for summability of the series by the aggregate of Cesàro's means. Suppose that $\phi(t)$ is integrable (L) in (ϵ, a) for every positive ϵ, and that

$$\chi(\epsilon) = \int_\epsilon^a \phi(t)\, dt. \tag{1 . 41}$$

Then we say that

$$\int_0^a \phi(t)\, dt$$

exists in the Cesàro sense, or *is summable* (C), if $\chi(\epsilon)$ tends to a limit in the Cesàro sense when $\epsilon \to 0$, *i.e.* if

$$\frac{1}{\epsilon} \int_0^\epsilon \frac{d\epsilon_1}{\epsilon_1} \int_0^{\epsilon_1} \frac{d\epsilon_2}{\epsilon_2} \dots \int_0^{\epsilon_{r-2}} \frac{d\epsilon_{r-1}}{\epsilon_{r-1}} \int_0^{\epsilon_{r-1}} \chi(\epsilon_r)\, d\epsilon_r \tag{1 . 42}$$

tends to a limit for some value of r.‡ This being so, we prove that

* In investigations as yet unpublished.

† G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1923), 67–96. We refer to this memoir as F.S.

‡ We alter the form of the definition which we gave in our preliminary notice of this paper [*Proc. London Math. Soc.* (2), 22 (1924), xliii–xlv, *Records* for June 14, 1923], in order to avoid the questions discussed briefly in our note "The equivalence of certain integral means" (*ibid.* xl–xliii). The theorem is true with either definition.

the necessary and sufficient condition that the allied series should be summable (C) is that the integral (1.13) *should be summable (C).*

Finally, in § 5, we discuss, less systematically, certain questions concerning the integral

$$\int_0^a \frac{f(x+t)-f(x)}{t}\,dt. \tag{1.43}$$

There is a striking contrast between this integral and the integral

$$\int_0^a \frac{f(x+t)-f(x-t)}{t}\,dt, \tag{1.44}$$

which dominates the discussion of the allied series. The latter is convergent, for almost all x, for every integrable f, whereas (1.43) may diverge for almost all x even when f is continuous.

The integral (1.43) is connected with the series

$$\Sigma \frac{s_n-s}{n}, \tag{1.45}$$

where s_n is the sum of the first $n+1$ terms of the Fourier series of $f(x)$, and s is an appropriate number independent of n, in much the same way that (1.44) is connected with the allied series. We give one or two theorems to illustrate this connexion, but have not thought it worth while to prove a theorem analogous to that of § 3, though such a theorem is doubtless true. We conclude by giving a sufficient condition that (1.43) should converge almost everywhere, and represent a function of integrable square, viz. that

$$\Sigma(a_n^2+b_n^2)(\log n)^2 \tag{1.46}$$

should be convergent.

2. *The summability (Cδ) of the allied series.*

2.1. THEOREM 1.—*In a point of the Lebesgue set, the allied series is either summable by every Cesàro mean of positive order, or summable by no Cesàro mean.*

It is convenient to work not with the actual Cesàro means, but with

the equivalent Rieszian means.* We write

$$B_\omega^\delta = \sum_{1 \leqslant n < \omega} \left(1 - \frac{n}{\omega}\right)^\delta B_n, \tag{2.11}$$

$$\mathfrak{B}_\omega^\delta = \sum_{1 \leqslant n < \omega} \left(1 - \frac{n}{\omega}\right)^\delta n B_n; \tag{2.12}$$

and we have to show that, if x is in the Lebesgue set, and ΣB_n is summable by some Cesàro or Rieszian mean, then

$$B_\omega^\delta \to f(x), \tag{2.13}$$

when $\omega \to \infty$, for every positive δ.

2.2.—LEMMA α.—*If $\delta > 0$ and x is in the Lebesgue set, then*

$$\mathfrak{B}_\omega^\delta = o(\omega). \tag{2.21}$$

We may plainly suppose that $\delta \leqslant 2$. We have

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \sim \Sigma B_n \sin nt, \tag{2.22}$$

the series being the Fourier series of the function in t. We multiply (2.22) by

$$\gamma(t) = \frac{d}{dt}\{t^{-1-\delta} C_{1+\delta}(\omega t)\}, \tag{2.23}$$

where

$$C_r(t) = \frac{t^r}{\Gamma(r+1)}\left\{1 - \frac{t^2}{(r+1)(r+2)} + \frac{t^4}{(r+1)\ldots(r+4)} - \ldots\right\}, \tag{2.24}$$

and integrate term by term over the interval $(0, \infty)$. When $\omega t = u$ is large, we have

$$u^{-1-\delta} C_{1+\delta}(u) = \frac{A}{u^2} + O\left(\frac{1}{u^3}\right) + \frac{B \sin(u - \frac{1}{2}\delta\pi)}{u^{1+\delta}} + O\left(\frac{1}{u^{2+\delta}}\right),$$

where A and B are constants, while asymptotic formulae for the

* See M. Riesz, "Une méthode de sommation équivalent à la méthode des moyennes arithmétiques", *Comptes Rendus*, 22 Nov., 1909. Riesz does not give all the details of the proof: a complete account will appear in the second volume of the second edition of Hobson's *Theory of functions of a real variable*.

derivatives of the function can be found by formal differentiation.* It follows that

$$(2.25)\qquad \gamma(t) = \omega^{2+\delta}\frac{d}{du}\{u^{-1-\delta}C_{1+\delta}(u)\}$$

$$= \omega^{2+\delta}\left\{-\frac{2A}{u^3}+O\left(\frac{1}{u^4}\right)+\frac{B\cos(u-\frac{1}{2}\delta\pi)}{u^{1+\delta}}+O\left(\frac{1}{u^{2+\delta}}\right)\right\},$$

and that its derivative is of the form obtained by formal differentiation. Hence $\gamma(t)$ and $\gamma'(t)$ both possess absolutely convergent integrals up to infinity, and the legitimacy of the integration results from a theorem of Young.†

We obtain on integration

$$(2.26)\qquad \int_0^\infty \gamma(t)\psi(t)\,dt = \Sigma B_n\int_0^\infty \gamma(t)\sin nt\,dt$$

$$= -\Sigma nB_n\int_0^\infty t^{-1-\delta}C_{1+\delta}(\omega t)\cos nt\,dt$$

$$= -\frac{\pi}{2\Gamma(1+\delta)}\sum_{n\leqslant\omega}(\omega-n)^\delta nB_n,$$

since‡

$$\int_0^\infty t^{-q}C_q(t)\cos xt\,dt = \frac{\pi}{2\Gamma(q)}(1-x)^{q-1}\quad(x\leqslant 1),\qquad =0\quad(x>1).$$

Hence

$$(2.27)\qquad \mathfrak{B}_\omega^\delta = -\frac{2\Gamma(1+\delta)}{\pi\omega^\delta}\int_0^\infty \gamma(t)\psi(t)\,dt.$$

---

* The asymptotic expansions for $C_r(u)$ and associated functions, for real values of u, are easily derived, by standard methods, from the formula

$$C_r(u) = \frac{u^r}{\Gamma(r)}\int_0^1 \cos ux\,(1-x)^{r-1}\,dx.$$

† W. H. Young, "On the integration of Fourier series", *Proc. London Math. Soc.* (2), 9 (1911), 449–462 (454). See also G. H. Hardy, "Notes on some points in the integral calculus (55)", *Messenger of Math.*, 51 (1922), 186–192; and other papers of Young there referred to.

‡ W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 53 (1912), 161–177 (166).

2.3. We write

$$(2.31)\quad \frac{1}{\omega^\delta}\int_0^\infty \gamma(t)\,\psi(t)\,dt = \frac{1}{\omega^\delta}\int_0^{1/\omega} + \frac{1}{\omega^\delta}\int_{1/\omega}^1 + \frac{1}{\omega^\delta}\int_1^\infty = J_1+J_2+J_3,$$

say; and

$$(2.32)\quad \chi(t) = \int_0^t |\psi(u)|\,du = o(t)$$

(since x is in the Lebesgue set). In J, $\omega t < 1$ and $\gamma(t) = O(\omega^{3+\delta}t)$,* so that

$$(2.33)\quad J_1 = O\left(\frac{1}{\omega^\delta}\int_0^{1/\omega}\omega^{2+\delta}.\,\omega t\,.\,|\psi(t)|\,dt\right)$$

$$= O\left\{\omega^2\chi\left(\frac{1}{\omega}\right)\right\} = o(\omega).$$

In J_2, $u = \omega t$ is greater than 1, and so†

$$\gamma(t) = O\{\omega^{2+\delta}(\omega t)^{-1-\delta}\} = O(\omega t^{-1-\delta}),$$

by (2.25). Hence

$$(2.34)\quad J_2 = O\left(\omega^{1-\delta}\int_{1/\omega}^1 |\psi|\,t^{-1-\delta}\,dt\right)$$

$$= O\left\{\omega^{1-\delta}\left[\chi(1)-\omega^{1+\delta}\chi\left(\frac{1}{\omega}\right)+(1+\delta)\int_{1/\omega}^1 \chi(t)\,t^{-2-\delta}\,dt\right]\right\}$$

$$= O(\omega^{1-\delta})+o(\omega)+o\left(\omega^{1-\delta}\int_{1/\omega}^1 t^{-1-\delta}\,dt\right)$$

$$= O(\omega^{1-\delta})+o(\omega) = o(\omega).$$

Finally

$$(2.35)\quad J_3 = O(\omega^{1-\delta})\int_1^\infty |\psi|\,t^{-1-\delta}\,dt = O(\omega^{1-\delta}) = o(\omega).$$

From (2.27), (2.31), (2.32), (2.33), and (2.34) we deduce

$$\mathfrak{B}_\omega^\delta = o(\omega),$$

which proves the lemma.

---

* Since $$t^{-1-\delta}C_{1+\delta}(\omega t) = A\omega^{1+\delta}+B\omega^{3+\delta}t^2+\ldots.$$

† It is here that we suppose $\delta \leqslant 2$.

2.4. LEMMA β.—*If* ΣB_n *is summable* (C, r), *where* $r \geqslant 1$, *and*

$$\mathfrak{B}_\omega^{r-1} = o(\omega), \tag{2.41}$$

then ΣB_n *is summable* $(C, r-1)$.

For $$B_\omega^r - B_\omega^{r-1} = -\frac{1}{\omega}\mathfrak{B}_\omega^{r-1} = o(1).$$

2.5. Theorem 1 follows at once from Lemmas α and β. And if we combine Theorem 1 with the theorem of Plessner, that the allied series is summable $(C, 1)$ for almost all values of x, we obtain

THEOREM 2.—*The allied series is almost always summable* (C, δ) *for every positive* δ.

We add one remark before leaving this part of the subject. Theorem 1 does not contain quite all that we stated at the beginning of § 1.4, since we have said nothing about summability by Abel's limit. But here there is in reality nothing new to prove, since, after Plessner, summability by Abel's limit, in a point of the Lebesgue set, implies and is implied by the existence of the integral (1.13), and this, after Young, implies summability $(C, 1)$.

3. *Necessary and sufficient conditions for the summability* (C) *of the allied series of a function of integrable square.*

3.1. We pass to the problem of finding necessary and sufficient conditions for summability by *some* Cesàro mean at a particular point; and, as in our treatment of the same problem for the Fourier series, we consider first the case in which f^2 is integrable.*

THEOREM 3.—*The necessary and sufficient condition that the allied series should be summable* (C), *for a particular value of* x, *is that the integral* (1.13) *should be summable* (C). *The sum of the series is in this case equal to the value of the integral.*

As we explained in § 1.4, we say that

$$\chi(\epsilon) = \int_\epsilon^\alpha \phi(t)\,dt \to A \quad (C, r) \tag{3.11}$$

* Only trivial changes are needed when we are given that $f^{1+\delta}$ $(\delta > 0)$ is integrable.

or that

$$\int_0^a \phi(t)\,dt = A \quad (C, r), \tag{3.12}$$

or that the integral is summable (C, r), to sum A, if

$$\chi_r(\epsilon) = \frac{1}{\epsilon}\int_0^{\epsilon}\frac{d\epsilon_1}{\epsilon_1}\int_0^{\epsilon_1}\frac{d\epsilon_2}{\epsilon_2}\ldots\int_0^{\epsilon_{r-1}}\chi(\epsilon_r)\,d\epsilon_r \to A. \tag{3.13}$$

We shall say that the integral (3.12) is *convergent*, or that $\phi(t)$ possesses a "Cauchy integral" down to 0, or is integrable (C), if $\chi(\epsilon)$ tends to a limit in the ordinary sense. It is to be understood, in all these definitions, that $\phi(t)$ is integrable (L) in any interval which does not include the origin. Finally, we shall say that $\chi \to A$ $(C, -1)$, or that the integral (3.12) is equal to A $(C, -1)$, if the integral is convergent and

$$\phi(t) = o\left(\frac{1}{t}\right).$$

Since $\Sigma B_n \sin nt$ is the Fourier series of the function $\psi(t)$ of § 2.2, our problem may be restated as follows: we have to prove that, *if* $\Sigma B_n \sin nt$ *is the Fourier series of an odd function* $\psi(t)$, *then the necessary and sufficient condition for the summability of* ΣB_n *is that the integral*

$$\frac{1}{\pi}\int_0^{\pi}\psi(t)\cot\tfrac{1}{2}t\,dt \tag{3.14}$$

should be summable (C), *and that the sum of the series is then equal to the value of the integral.*

We reduce the problem thus, and then write $f(t)$ for $\psi(t)$ and a_n for B_n, so that now

$$f(t) \sim \Sigma a_n \sin nt. \tag{3.15}$$

3.2. LEMMA γ.—*If*

$$\frac{1}{\pi}\int_0^{\pi} f(t)\cot\tfrac{1}{2}t\,dt \tag{3.21}$$

is summable $(C, -1)$, *i.e. if* (i) *it is convergent and* (ii) $f(t) = o(1)$, *then* Σa_n *is summable* $(C, 1)$; *and its sum is equal to the value of the integral.*

This lemma, which plays the part played by Fejér's theorem in the corresponding problem for the Fourier series, is included in the theorem of Young already referred to.*

* P. 212, f.n. †.

LEMMA 8.*—*If*

$$\frac{2}{\pi}\int_0^\infty \frac{f(t)}{t}\,dt$$

is convergent, as

$$\lim_{\epsilon\to 0,\ T\to\infty}\int_\epsilon^T,$$

then the integral (3.21) *is also convergent (as a Cauchy integral), and the two integrals are equal.*

We have

$$\int_0^\infty \frac{f(t)}{t}\,dt = \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{f(t)}{t}\,dt = \tfrac{1}{2}\int_{-\pi}^{\pi}\frac{f(t)}{t}\,dt + \tfrac{1}{2}\sum_{-\infty}^{\infty}{}'\int_{(2k-1)\pi}^{(2k+1)\pi}\frac{f(t)}{t}\,dt$$

$$= \tfrac{1}{2}\int_{-\pi}^{\pi}\frac{f(t)}{t}\,dt + \tfrac{1}{2}\sum_{-\infty}^{\infty}{}'\int_{-\pi}^{\pi}\frac{f(t)}{t+2k\pi}\,dt,$$

$\sum_{-\infty}^{\infty}{}'$ denoting $\lim_{K\to\infty}\left(\sum_{-K}^{-1}+\sum_{1}^{K}\right)$.

This series converges uniformly in $(-\pi, \pi)$ to the sum

$$\tfrac{1}{2}\cot\tfrac{1}{2}t - \frac{1}{t}.$$

We may therefore integrate term by term, when we obtain the result of the lemma.

LEMMA ϵ.—*If* Σa_n *is summable* $(C, -1)$, *then the integral* (3.21) *is convergent, and equal to the sum of the series.*

We have

$$\int_\epsilon^\infty \frac{f(t)}{t}\,dt = \Sigma a_n\int_\epsilon^\infty \frac{\sin nt}{t}\,dt = \Sigma a_n\int_{n\epsilon}^\infty \frac{\sin w}{w}\,dw, \tag{3.22}$$

the term-by-term integration being justified by the theorems proved in the papers of Young and Hardy referred to on p. 217.† Now

$$S(\epsilon) = \Sigma a_n\int_0^{n\epsilon}\frac{\sin w}{w}\,dw = \Sigma a_n\rho(n\epsilon)$$

exists for every positive ϵ, and tends to zero with ϵ.

---

* The result of this lemma is, of course, familiar.

† See footnote †.

For, in the first place,

$$\rho(n\epsilon) = \tfrac{1}{2}\pi - \int_{n\epsilon}^{\infty} \frac{\sin w}{w}\, dw = \tfrac{1}{2}\pi + O\left(\frac{1}{n}\right)$$

if ϵ is fixed, and

$$\tfrac{1}{2}\pi \Sigma a_n, \quad \Sigma a_n O\left(\frac{1}{n}\right) = \Sigma o\left(\frac{1}{n^2}\right)$$

are convergent. Hence $S(\epsilon)$ exists for every $\epsilon > 0$. Finally

$$S(\epsilon) = \sum_{n \leqslant 1/\epsilon} o\left(\frac{1}{n}\right) O(n\epsilon) + \tfrac{1}{2}\pi \sum_{n > 1/\epsilon} a_n + \sum_{n > 1/\epsilon} o\left(\frac{1}{n}\right) O\left(\frac{1}{n\epsilon}\right)$$

$$= \epsilon \sum_{n \leqslant 1/\epsilon} o(1) + o(1) + \frac{1}{\epsilon} \sum_{n > 1/\epsilon} o\left(\frac{1}{n^2}\right) = o(1).$$

Thus $S(\epsilon)$ exists and tends to zero. It follows that the left-hand side of (3 . 22) tends to a limit, and that

$$\int_0^{\infty} \frac{f(t)}{t}\, dt = \Sigma a_n \int_0^{\infty} \frac{\sin w}{w}\, dw = \tfrac{1}{2}\pi \Sigma a_n. \tag{3 . 23}$$

The lemma follows from (3 . 23) and Lemma δ.

This lemma plays the part played in our discussion of the Fourier series by a theorem of Fatou.*

3 . 3. LEMMA ξ.—*Suppose that* $\phi(t)$ *and*

$$\phi^{(1)} = \frac{\phi_1}{t} = \frac{1}{t}\int_0^t \phi\, du, \quad \phi^{(2)} = \frac{(\phi^{(1)})_1}{t}, \quad \phi^{(3)} = \frac{(\phi^{(2)})_1}{t}, \quad \ldots, \tag{3 . 31}$$

are integrable (C). *Then the existence of any one of the integrals*

$$\int_0 \frac{\phi}{t}\, dt \ (C, r), \quad \int_0 \frac{\phi^{(1)}}{t}\, dt \ (C, r-1),$$

$$\ldots, \quad \int_0 \frac{\phi^{(r)}}{t}\, dt \ (C, 0), \quad \int_0 \frac{\phi^{(r+1)}}{t}\, dt \ (C, -1)$$

implies that of the remainder.

In the case at present under consideration ϕ, $\phi^{(1)}$, ... are all integrable (L), and indeed functions of integrable square.† We state the

* See F.S., Lemma 12.

† See F.S., Lemma 10.

lemma more generally in view of § 4. It is plainly sufficient to prove that, if $r \geqslant 0$, the existence of either of the first two integrals implies that of the other.

Let

$$j(\epsilon) = \int_\epsilon^a \frac{\phi}{t}\,dt, \qquad k(\epsilon) = \int_\epsilon^a \frac{\phi^{(1)}}{t}\,dt. \tag{3.32}$$

Then
$$\epsilon j(\epsilon) = \epsilon \int_\epsilon^\delta \frac{\phi}{t}\,dt + \epsilon \int_\delta^a \frac{\phi}{t}\,dt = \int_\epsilon^\theta \phi\,dt + \epsilon \int_\delta^a \frac{\phi}{t}\,dt,$$

where $0 < \epsilon < \theta < \delta < a$. Choosing first δ and then ϵ appropriately, we see that

$$\epsilon j(\epsilon) = o(1). \tag{3.33}$$

Next, if we suppose $0 < \eta < \epsilon$, and integrate by parts, we obtain

$$j_1(\epsilon, \eta) = \int_\eta^\epsilon j(u)\,du = \int_\eta^\epsilon du \int_u^a \frac{\phi}{w}\,dw = \epsilon \int_\epsilon^a \frac{\phi}{w}\,dw - \eta \int_\eta^a \frac{\phi}{w}\,dw + \int_\eta^\epsilon \phi\,dw.$$

Making $\eta \to 0$, and using (3.33), we obtain

$$j_1(\epsilon) = \int_0^\epsilon j(u)\,du = \epsilon j(\epsilon) + \phi_1(\epsilon). \tag{3.34}$$

Finally

$$k(\epsilon) = \int_\epsilon^a \frac{\phi_1}{t^2}\,dt = -\frac{\phi_1(a)}{a} + \frac{\phi_1(\epsilon)}{\epsilon} + \int_\epsilon^a \frac{\phi}{t}\,dt; \tag{3.35}$$

and the combination of (3.34) and (3.35) gives

$$k(\epsilon) = \frac{j_1(\epsilon)}{\epsilon} - \frac{\phi_1(a)}{a}. \tag{3.36}$$

It follows from (3.36) that the assertions

$$k(\epsilon) \to A \ \ (C, r-1), \qquad j(\epsilon) \to A + \frac{\phi_1(a)}{a} \ \ (C, r)$$

are equivalent, so long as $r \geqslant 1$; and this proves the lemma except when $r = 0$.

If $j(\epsilon)$ tends to a limit, so does $j_1(\epsilon)/\epsilon$ and therefore, by (3 . 36), $k(\epsilon)$. Also $\phi_1(\epsilon) = o(\epsilon)$, by (3 . 34), and so

$$\frac{\phi_1(t)}{t^2} = o\left(\frac{1}{t}\right). \tag{3 . 37}$$

That is to say, $k(\epsilon)$ tends to a limit $(C, -1)$. Finally, if $k(\epsilon)$ tends to a limit $(C, -1)$, then $j_1(\epsilon)/\epsilon$ tends to a limit by (3 . 36), and $j(\epsilon)$ tends to a limit by (3 . 34) and (3 . 37). This completes the proof.

It will be observed that, in Lemma ξ, there is no reference to the actual values of the various integrals considered. For this reason we shall find it necessary to modify the form of the lemma in a moment.

3 . 4. We proceed to the proof of Theorem 3, for a function of integrable square.

We introduce the functions

$$\psi_0 = f, \quad \psi_1 = \tfrac{1}{2}\cot\tfrac{1}{2}t\int_0^t \psi_0(u)\,du + \gamma_1 \sin^3 t, \tag{3 . 41}$$

$$\psi_2 = \tfrac{1}{2}\cot\tfrac{1}{2}t\int_0^t \psi_1(u)\,du + \gamma_2 \sin^3 t, \quad \ldots,$$

the γ's being constants whose values will be fixed in a moment. These functions correspond to the functions

$$f,\ f^{(1)},\ f^{(2)},\ \ldots,$$

defined as in Lemma ξ. If any ψ_s exists, so does the corresponding $f^{(s)}$, and conversely, and the difference between the two functions is $o(t)$.* We may therefore apply the results of Lemma ξ to either set of functions indifferently.

In the special case now considered, all the functions ψ_s are of integrable square. We have

$$\sum_n^N \frac{a_\nu}{\nu} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sum_n^N \frac{\sin \nu t}{\nu}\,dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(t) \sum_n^N \cos \nu t\,dt, \tag{3 . 42}$$

where

$$f_1(t) = \int_0^t f(u)\,du,$$

* See F.S., 85. It is important to observe that this argument holds even when f is merely integrable (C).

so that $f_1(-\pi) = f_1(\pi)$. Thus

(3.43)

$$\sum_{n}^{N} \frac{a_\nu}{\nu} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})t - \sin(n-\frac{1}{2})t}{\sin\frac{1}{2}t} f_1(t)\,dt$$

$$\to \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin nt \cot\tfrac{1}{2}t f_1(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \psi_1 \sin nt\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t)\cos nt\,dt - \frac{\gamma_1}{\pi}\int_{-\pi}^{\pi} \sin^3 t \sin nt\,dt$$

$$= a_n^1 + b'_{n,1} + b''_{n,1} = a_n^1 + b_{n,1},$$

say; so that $$a_{n,1} = \sum_{1}^{\infty} \frac{a_\nu}{\nu} = a_n^1 + b_{n,1}.$$

Here a_n^1 is the Fourier sine coefficient of the odd function $\psi_1(t)$. As regards $b_{n,1}$, we observe first that $b'_{n,1}$ (defined for $n \geqslant 1$) is the Fourier cosine coefficient of the even function $-\frac{1}{2}f_1(t)$. This function is an integral, and vanishes for $t = 0$; and its Fourier series is

$$-\tfrac{1}{2}\sum_{1}^{\infty} \frac{a_n}{n}(1-\cos nt) = -\tfrac{1}{2}\sum_{1}^{\infty} \frac{a_n}{n} + \tfrac{1}{2}\sum_{1}^{\infty} \frac{a_n}{n}\cos nt,$$

so that its first coefficient $b'_{0,1}$ is $-\Sigma a_n/n$. It follows that

$$\tfrac{1}{2}b'_{0,1} + \sum_{1}^{\infty} b'_{n,1} = 0 \quad (C, -1),$$

(3.44) $$\sum_{1}^{\infty} b'_{n,1} = \tfrac{1}{2}\sum_{1}^{\infty} \frac{a_n}{n} \quad (C, -1).$$

Next, $b''_{n,1}$ vanishes except when $n = 1$ and $n = 3$, while $b''_{1,1} = -\frac{3}{4}\gamma_1$ and $b''_{3,1} = \frac{1}{4}\gamma_1$. Hence $\Sigma b''_{n,1}$ is summable $(C, -1)$, to sum $-\frac{1}{2}\gamma_1$. If then we choose γ_1 so that

(3.45) $$\gamma_1 = \sum_{1}^{\infty} \frac{a_n}{n},$$

the series $\Sigma b_{n,1}$ will be summable $(C, -1)$ to sum zero. In other words, when γ_1 is defined by (3.45), $a_{n,1}$ differs from a_n^1, the Fourier coefficient of ψ_1, by the general term of a series summable $(C, -1)$ to sum 0.

We may now (as in F.S., 87–88) repeat the argument, choosing $\gamma_2, \gamma_3, \ldots$ appropriately at successive stages,* and we conclude that, if $a_{n,s}$ is defined as there, then $a_{n,s}$ *differs from* a_s^n, *the Fourier coefficient of* ψ_s, *by the general term of a series summable* $(C, -1)$ *to sum* 0.

3.5. In order to complete the proof, we require one further lemma (the modified form of Lemma ξ referred to at the end of § 3.3). We state it, like Lemma ξ, in a more general form than that in which it is required immediately.

LEMMA η.—*Suppose that* $\psi_0, \psi_1, \psi_2, \ldots$ *are defined by* (3.41) (*the* γ*'s being chosen as in* § 3.4), *and that* $\psi_0 = f$ *is integrable* (C). *Then the existence of any one of the integrals*

$$\frac{1}{\pi}\int_0^\pi \psi_s(t)\cot\tfrac{1}{2}t\,dt \quad (C, r-s;\ \ 0\leqslant s\leqslant r+1)$$

implies that of the remainder. Further, the values of all the integrals are equal.

As regards the *existence* of the integrals, there is nothing new to prove, as Lemma ξ, and the remark at the beginning of § 3.4, contain all that we require. It is therefore only necessary to verify that

$$\frac{1}{\pi}\int_0^\pi \psi_0(t)\cot\tfrac{1}{2}t\,dt = \frac{1}{\pi}\int_0^\pi \psi_1(t)\cot\tfrac{1}{2}t\,dt.\dagger \tag{3.51}$$

If

$$j(\epsilon) = \int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t f(t)\,dt, \quad k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t\,\psi_1(t)\,dt, \tag{3.52}$$

we have

$$j_1(\epsilon) = \int_0^\epsilon j(t)\,dt = \int_0^\epsilon \sec^2\tfrac{1}{2}t\,j(t)\,dt - \int_0^\epsilon \tan^2\tfrac{1}{2}t\,j(t)\,dt. \tag{3.53}$$

Now $tj(t) = o(1)$, as in § 3.3. The second term on the right-hand side of (3.53) is therefore $o(\epsilon^2)$. The first term is

$$2\int_0^\epsilon \frac{d}{dt}(\tan\tfrac{1}{2}t)\,dt\int_t^\pi f(u)\,\tfrac{1}{2}\cot\tfrac{1}{2}u\,du = 2\tan\tfrac{1}{2}\epsilon\int_\epsilon^\pi f(t)\,\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt + \int_0^\epsilon f(t)\,dt;$$

* Thus $\gamma_2 = \Sigma\dfrac{a_n^1}{n}$.

† The first integral being summable (C, r), the second $(C, r-1)$.

and so

$$j_1(\epsilon) = 2\tan\tfrac{1}{2}\epsilon\int_\epsilon^\pi f(t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt+\int_0^\epsilon f(t)\,dt+o(\epsilon^2). \tag{3.54}$$

On the other hand

$$k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{4}\cot^2\tfrac{1}{2}t f_1(t)\,dt+\tfrac{1}{2}\gamma_1\int_\epsilon^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt$$

$$= \int_\epsilon^\pi \tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}t f_1(t)\,dt-\tfrac{1}{4}\int_\epsilon^\pi f_1(t)\,dt+\tfrac{1}{2}\gamma_1\int_\epsilon^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt.$$

The second term here is

$$-\tfrac{1}{4}\int_0^\pi f_1(t)\,dt+o(\epsilon) = -\tfrac{1}{4}\pi\Sigma\frac{a_n}{n}+o(\epsilon),$$

and the third is

$$\tfrac{1}{2}\gamma_1\int_0^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt+o(\epsilon) = \tfrac{1}{4}\pi\gamma_1+o(\epsilon).$$

These two terms, in virtue of (3.45), yield $o(\epsilon)$. Hence

$$k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}t f_1(t)\,dt+o(\epsilon) \tag{3.55}$$

$$= \tfrac{1}{2}\cot\tfrac{1}{2}\epsilon\int_0^\epsilon f(t)\,dt+\int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t f(t)\,dt+o(\epsilon).$$

Combining (3.54) and (3.55), we obtain

$$k(\epsilon)-\tfrac{1}{2}\cot\tfrac{1}{2}\epsilon\; j_1(\epsilon) = o(1) \tag{3.56}$$

and therefore

$$k(\epsilon)-\frac{j_1(\epsilon)}{\epsilon} = o(1). \tag{3.57}$$

This equation takes the place of (3.36); and the remainder of the proof of Lemma ζ now applies to the present lemma. In the present case, however, we can assert what we could not assert before, viz. the equality of the values of all the integrals considered.

3.6. We can now complete the proof of Theorem 3, arguing substantially as in F.S.,* and using the notation and the substance of the fundamental arithmetic theorem (Theorem A 1) proved there, in combination now, however, with Lemmas γ, ϵ, and η. If A is summable (C, r), then A_{r+1} is summable $(C, -1)$, and so A^{r+1} is summable $(C, -1)$.† Hence, by Lemma ϵ,

$$\frac{1}{\pi}\int_0^\pi \psi_{r+1}(t)\cot\tfrac{1}{2}t\,dt \tag{3.61}$$

is summable $(C, 0)$; and therefore, by Lemma η,

$$\frac{1}{\pi}\int_0^\pi \psi_0(t)\cot\tfrac{1}{2}t\,dt \tag{3.62}$$

is summable $(C, r+1)$. Further, the values of all the series and integrals are the same. On the other hand, if (3.62) is summable (C, r), then (3.61) is summable $(C, -1)$, by Lemma η. Hence A^{r+1} is summable $(C, 1)$, by Lemma γ, and therefore A_{r+1} is summable $(C, 1)$; and so A is summable $(C, r+2)$. This completes the proof of Theorem 3, for functions of integrable square.

4. *Proof of Theorem* 3 *in the general case.*

4.1. In the general case, as in the corresponding investigation of the Fourier series, more delicate arguments are required. We begin, as there, by establishing the *sufficiency* of our criterion.

LEMMA θ.—*Suppose that* ϕ *is integrable* (L), *that*

$$j(t) = \int_t^a \frac{\phi}{u}\,du \quad (t > 0),$$

that $j^{(1)}(t) = \frac{1}{t}\int_0^t j(u)\,du$, $j^{(2)}(t) = \frac{1}{t}\int_0^t j^{(1)}(u)\,du$,

$$\ldots,\quad j^{(r)}(t) = \frac{1}{t}\int_0^t j^{(r-1)}(u)\,du$$

---

* F.S., 88.

† See the end of § 3.4.

exist as Cauchy integrals, and that

$$j^{(r)}(t) \to A, \tag{4.11}$$

when $t \to 0$. *Then the functions* $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(r)}$ *of Lemma* ξ *exist, and are integrable* (L), *and*

$$\phi^{(r+1)}(t) \to 0, \tag{4.12}$$

when $t \to 0$.

We have

$$\frac{d}{dt}(tj) = j + tj' = j - \phi \quad (u > 0). \tag{4.13}$$

Now ϕ is by hypothesis integrable (L) down to 0; and so also is j, since (i) j is by hypothesis integrable (C), and (ii)

$$\int_0^t |j|\,dt \leqslant \int_0^t dt \int_t^a \frac{|\phi|}{u}\,du = \int_0^t \frac{|\phi|}{u}\,du \int_0^t dt = \int_0^a |\phi|\,du.$$

Finally $tj \to 0$, by (3.23).* Hence, integrating (4.13), we find

$$tj(t) = \int_0^t j\,du - \int_0^t \phi\,du = j_1(t) - \phi_1(t),$$

or

$$\phi^{(1)} = j^{(1)} - j. \tag{4.14}$$

Again, j and $j^{(1)}$ are, by hypothesis, integrable (C). Hence $\phi^{(1)}$ is integrable (C). Integrating, and dividing by t, we obtain

$$\phi^{(2)} = j^{(2)} - j^{(1)}.$$

Plainly we may repeat the argument. All of the functions

$$\phi^{(1)}, \quad \phi^{(2)}, \quad \ldots, \quad \phi^{(r)}$$

are integrable (C), and

$$\phi^{(s)} = j^{(s)} - j^{(s-1)} \quad (1 \leqslant s \leqslant r). \tag{4.15}$$

* This is so, in fact, whenever ϕ is integrable (C).

Finally

$$\phi^{(r+1)} = \frac{1}{t}\int_0^t j^{(r)}(u)\,du - j^{(r)}(t) \to 0 \tag{4.16}$$

when $t \to 0$, which is (4.12).

We can now appeal to Lemma 13 of F.S., which shows that $\phi^{(1)}$, $\phi^{(2)}$, ..., $\phi^{(r)}$ are integrable (L); and the lemma is completely established.

4.2. There is now no difficulty in proving our criterion sufficient. If

$$\int_0 \frac{f(t)}{t}\,dt$$

is summable (C), the conditions of Lemma θ are satisfied, when $\phi = f$, for some value of r. The functions $f^{(1)}$, $f^{(2)}$, ... are therefore integrable (L), and so also are the functions ψ_1, ψ_2, ... of § 3.4. The argument of §§ 3.4 and 3.5, in so far as the sufficiency of the criterion is concerned, is therefore valid as it stands.

4.3. We proceed to the proof of the *necessity* of the condition, for which additional lemmas are required.

We shall, as in the corresponding argument of F.S., use the terms "Fourier series", "Fourier constant", in a slightly extended sense. If (i) $f(t)$ is odd, and possesses a Cauchy integral over $(0, \pi)$, so that

$$a_n = \int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

exists also as a Cauchy integral*

$$a_n = o(1), \tag{ii}$$

we shall say that a_n is the *Fourier coefficient*, and $\Sigma a_n \sin nt$ the *Fourier series* of $f(t)$, and write

$$f(t) \sim \Sigma a_n \sin nt.$$

---

* As $$\lim\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right) = 2\lim\int_{\epsilon}^{\pi}.$$

These conditions are, of course, satisfied in particular if $f(t)$ is integrable (L).

LEMMA 1.—*If* (i) $\Sigma a_n \sin nt$ *is the Fourier series of* $f(t)$ *(in the sense just explained),*

(ii) $$f_1(t) = \int_0^t f(u)\,du, \quad f_2(t) = \int_0^t f_1(u)\,du, \ \ldots,$$

(iii) Σa_n *is summable* (C, r),

then

$$f_{r+2}(t) = o(t^{r+2}). \tag{4.31}$$

As a change in a_1 alters $f(t)$ by a multiple of $\sin t$, we may suppose without loss of generality that $\Sigma a_n = 0$. We have

$$f_1(t) = \Sigma \frac{a_n}{n}(1-\cos nt),$$

this series being, as the Fourier series of a continuous function, uniformly summable $(C, 1)$, and so, since $a_n = o(1)$, uniformly convergent. By further integrations we obtain

$$f_2(t) = \Sigma \frac{a_n}{n^2}(nt-\sin nt), \quad f_3(t) = \Sigma \frac{a_n}{n^3}(\cos nt - 1 + \tfrac{1}{2}n^2t^2),$$

and

$$f_{r+2}(t) = \Sigma \frac{a_n}{n^{r+2}} s_{r+2}(nt), \tag{4.32}$$

where

$$s_\nu(t) = \left(\int_0^t du\right)^\nu \sin u.$$

It may be verified at once that

$$s_{r+2}(u) \sim Au^{r+3}$$

for small u, and

$$s_{r+2}(u) \sim Au^{r+1}$$

for large u; and that

$$\Delta^k s_{r+2}(nt) = O(t^{r+3}n^{r+3-k}) \quad (nt \leqslant 1,\ 0 \leqslant k \leqslant r+3), \tag{4.33}$$

$$\Delta^k s_{r+2}(nt) = O(t^{r+1}n^{r+1-k}) \quad (nt \geqslant 1,\ 0 \leqslant k \leqslant r+1). \tag{4.34}$$

Now

$$(4.35)\qquad f_{r+2}(t)=\Sigma\frac{a_n}{n^{r+2}}s_{r+2}(nt)=\Sigma A_n\Delta\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right)$$

$$=\Sigma A_n^1\Delta^2\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right)=\ldots=\Sigma A_n^r\Delta^{r+1}\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right),$$

provided only

$$(4.36)\qquad A_n^k\Delta^k\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right)\to 0\quad(0\leqslant k\leqslant r)$$

when t is fixed and $n\to\infty$. Now $A_n^k=o(n^{k+1})$, since $a_n=o(1)$. Hence $A_n^k\Delta^k$ is, by (4.34), a sum of terms of the type

$$o(n^{k+1})\,O(n^{-r-2-\lambda})\,O(n^{r+1-k+\lambda})=o(1),$$

and the condition (4.36) is satisfied.

We have therefore

$$f_{r+2}(t)=\Sigma A_n^r\Delta^{r+1}\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right)=\sum_{k=0}^{r+1}c_{r,k}\sum_n A_n^r\Delta^k\left(s_{r+2}(nt)\right)\Delta^{r+1-k}\left((n+k)^{-r-2}\right),$$

the c's being functions of r and k only. The typical series here is, by (4.33) and (4.34),

$$\sum_{nt\leqslant 1}o(n^r)\,.\,O(t^{r+3}n^{r+3-k})\,O(n^{-2r-3+k})+\sum_{nt>1}o(n^r)\,.\,O(t^{r+1}n^{r+1-k})\,.\,O(n^{-2r-3+k})$$

$$=o\left(t^{r+3}\sum_{nt\leqslant 1}1\right)+o\left(t^{r+1}\sum_{nt>1}\frac{1}{n^2}\right)=o(t^{r+2}),$$

which proves the lemma.

4.4.—LEMMA K.—*If*

$$f(t)\sim\Sigma a_n\sin nt,$$

in the sense of § 4.3, *and* Σa_n *is summable* (C), *then*

$$\int_0 f^{(1)}(t)\,dt=\int_0\frac{f_1(t)}{t}\,dt$$

is convergent.

For

$$(4.41)\qquad \int_\epsilon^a \frac{f_1(t)}{t}\,dt = \frac{f_2(a)}{a} - \frac{f_2(\epsilon)}{\epsilon} + \int_\epsilon^a \frac{f_2(t)}{t^2}\,dt$$

$$= \frac{f_2(a)}{a} + \frac{f_3(a)}{a^2} - \frac{f_2(\epsilon)}{\epsilon} - \frac{f_3(\epsilon)}{\epsilon^2} + 2\int_\epsilon^a \frac{f_3(t)}{t^3}\,dt$$

$$= \ldots = F(a) - F(\epsilon) + (r+1)!\int_\epsilon^a \frac{f_{r+2}(t)}{t^{r+2}}\,dt,$$

where

$$F(\epsilon) = \sum_{k=1}^{r+1} (k-1)!\,\frac{f_{k+1}(\epsilon)}{\epsilon^k}.$$

Since $f_1(t)$ is continuous, every term of $F(\epsilon)$ tends to zero; and the last integral in (4.41) tends to a limit in virtue of Lemma ι.

4.5. We return to the argument of §§ 3.4 and 3.5. We assume (i) that $\Sigma a_n \sin nt$ is the Fourier series of $f(t)$ in the sense of § 4.3, and (ii) that Σa_n is summable (C), and we consider where the argument requires reconsideration.

We observe first that, by Lemma θ, $f^{(1)}(t)$, and therefore $\psi_1(t)$, is integrable (C). Hence (3.43) will be established if we can prove that

$$(4.51)\qquad J(N) = \int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\,f_1(t)\,dt \to 0$$

when $N \to \infty$. As $\psi_1(t)$ is no longer given as integrable (L), this is no longer a consequence of the theorem of Riemann-Lebesgue.

The integral $J(N)$ is substantially the partial sum of the Fourier series of $f_1(t)$ for $t = 0$. As $f_1(t)$ is continuous, and equal to 0, for $t = 0$, it follows from Fejér's theorem that

$$J(N) \to 0 \quad (C, 1).$$

But the series in question has coefficients $o(1/n)$, since $a_n = o(1)$, and is therefore, if summable, convergent. Hence (4.51) holds in the ordinary sense.

Next, our conclusions concerning $\Sigma b_{n,1}$ retain their validity, since $f_1(t)$, though no longer a Lebesgue integral, is continuous, and has

Fourier coefficients of the order required.* We conclude then, as in §3.4, that $a_{n,1}$ *differs from the Fourier coefficient of* $\psi_1(t)$ *by the general term of a series summable* $(C, -1)$ *to sum* 0.

It remains to show that the argument may be repeated. For this we must show that $\psi_1(t)$ possesses the relevant properties of $\psi_0(t)$ or $f(t)$. And this is in fact clear from what precedes. For $\psi_1(t)$ is, as we have seen, integrable (C), so that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\psi_1(t)\sin nt\,dt$$

is convergent; and being, as we have seen, substantially equal to $a_{n,1}$, it tends to zero. Thus $\psi_1(t)$ has a Fourier series $\Sigma a_n^1 \sin nt$ in the sense of § 4.3. Since Σa_n is summable (C), $\Sigma a_{n,1}$ is summable, and therefore Σa_n^1 is summable. These are the properties assumed for $f(t)$, so that our argument is one capable of repetition. This once established, the proof of the necessity of our condition in § 3.5 stands, and Theorem 3 is proved in full generality.

5. *The series* $\Sigma\frac{s_n-s}{n}$ *and the integral* $\int\frac{\phi(t)}{t}\,dt$.

5.1. In this section we consider the series

$$S=\Sigma\frac{s_n-s}{n}, \tag{5.11}$$

where

$$s_n=\sum_0^n A_\nu. \tag{5.12}$$

The interest of this series lies in its relations to the integral

$$\int_0\frac{\phi(t)}{t}\,dt, \tag{5.13}$$

* We take this opportunity of making a small correction of F.S., 94–95. We say there (94, 10 lines from below) that "the proof that $b_{n,1}=o(1/n)$ is unchanged." This proof should in fact be replaced by that stated in the text, as the original proof (though actually valid for $b_{n,1}$) would not, since it depends on the integrability (L) of $\phi(t)$, be suitable for repetition at subsequent stages of the argument.

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}, \tag{5.14}$$

these relations being very similar to those between the allied series and the integral

$$\int_0 \frac{\psi(t)}{t}\,dt = \tfrac{1}{2}\int_0 \frac{f(x+t)-f(x-t)}{t}\,dt. \tag{5.15}$$

We do not propose to work out these relations in detail, but confine ourselves to a few selected theorems sufficient to illustrate the likenesses and differences between the two problems.

5.2. The most striking difference between the two problems is embodied in

THEOREM 4.—*There are continuous functions $f(t)$ for which the integral* (5.13) *diverges, whatever be s, for almost all values of x.*

It is plain that, if $f(t)$ is continuous, (5.13) can converge only if $s = f(x)$; and that, since (5.15) is almost always convergent, the integrals (5.13) and (1.43) converge or diverge together for almost all x.

We take

$$f(t) = \Sigma b^{-\nu}\cos a^{\alpha\nu}\,t, \tag{5.21}$$

where a and b are integers and $a > b > 1$; so that

$$a_n = b^{-\nu} \quad (n = a^{\alpha\nu} = \lambda_\nu), \qquad a_n = 0 \quad (n \neq \lambda_\nu). \tag{5.22}$$

Then

$$\phi(t) = -2\Sigma b^{-\nu}\cos\lambda_\nu x \sin^2\tfrac{1}{2}\lambda_\nu t, \tag{5.23}$$

and, if $c > 0$,

$$\chi(\epsilon) = \int_\epsilon^c \frac{\phi(t)}{t}\,dt = -2\Sigma b^{-\nu}\cos\lambda_\nu x \int_\epsilon^c \frac{\sin^2\frac{1}{2}\lambda_\nu t}{t}\,dt \tag{5.24}$$

$$= -2\Sigma b^{-\nu}\cos\lambda_\nu x\,\omega_\nu(\epsilon),$$

where

$$\omega_\nu(\epsilon) = \int_\epsilon^c \frac{\sin^2\frac{1}{2}\lambda_\nu t}{t}\,dt = \int_{\frac{1}{2}\lambda_\nu\epsilon}^{\frac{1}{2}\lambda_\nu c} \frac{\sin^2 u}{u}\,du. \tag{5.25}$$

If $\frac{1}{2}\lambda_\nu\epsilon \leqslant 1 < \frac{1}{2}\lambda_\nu c$, we have

$$\omega_\nu(\epsilon) = \int_{\frac{1}{2}\lambda_\nu\epsilon}^{1} \frac{\sin^2 u}{u}\,du + \tfrac{1}{2}\int_{1}^{\frac{1}{2}\lambda_\nu c} \frac{1-\cos 2u}{u}\,du \tag{5.26}$$

$$= \tfrac{1}{2}\log\lambda_\nu + O(1) = \tfrac{1}{2}a^\nu \log a + O(1),$$

while if $\frac{1}{2}\lambda_\nu\epsilon > 1$ we have

$$\omega_\nu(\epsilon) = \tfrac{1}{2}\int_{\frac{1}{2}\lambda_\nu\epsilon}^{\frac{1}{2}\lambda_\nu c} \frac{du}{u} + O(1) = \tfrac{1}{2}\log\frac{c}{\epsilon} + O(1) = \tfrac{1}{2}\log\frac{1}{\epsilon} + O(1). \tag{5.27}$$

It follows that

$$\chi(\epsilon) = -\log a \sum_{\frac{1}{2}\lambda_\nu \leqslant 1/\epsilon} \left(\frac{a}{b}\right)^\nu \cos\lambda_\nu x - \log\frac{1}{\epsilon} \sum_{\frac{1}{2}\lambda_\nu > 1/\epsilon} b^{-\nu}\cos\lambda_\nu x + O(1) \tag{5.28}$$

$$= S_1 + S_2 + O(1),$$

say.

5.3. We take

$$\frac{1}{\epsilon} = \tfrac{1}{2}a^{a^{\mu+\frac{1}{2}}} = \tfrac{1}{2}\lambda_{\mu+\frac{1}{2}}. \tag{5.31}$$

Then S_1 is numerically less than

$$\left(\frac{a}{b}\right)^\mu \log a\left(1+\frac{b}{a}+\frac{b^2}{a^2}+\ldots\right) = \left(\frac{a}{b}\right)^\mu \log a\,\frac{a}{a-b};$$

and the sum of the terms of S_2, other than the first (for which $\nu = \mu+1$), is numerically less than

$$(a^{\mu+\frac{1}{2}}\log a - \log 2)\,b^{-\mu-1}\left(\frac{1}{b}+\frac{1}{b^2}+\ldots\right) = (a^{\mu+\frac{1}{2}}\log a - \log 2)\,\frac{b^{-\mu-1}}{b-1}.$$

We have therefore

$$\chi(\epsilon) = -\left(\frac{a}{b}\right)^\mu \log a\left\{\frac{\sqrt{a}}{b}\cos\lambda_{\mu+1}x + \frac{\sqrt{a}}{b}\left[\frac{1}{b-1}\right] + \left[\frac{a}{a-b}\right]\right\} + O(1), \tag{5.32}$$

where $[\alpha]$ denotes generally a number whose modulus is less than α.

Suppose now that b and $\sqrt{a/b}$ are large. Then it is plain from (5 . 32) that, if μ and x have values such that

$$|\cos \lambda_{\mu+1} x| > \tfrac{1}{2},$$

$\chi(\epsilon)$ will be large numerically, and will have the sign opposite to that of $\cos \lambda_{\mu+1} x$. Now the set of limiting values of this function, when $\mu \to \infty$, is, for all values of x which do not lie in a certain set of measure zero, everywhere dense in the interval $(-1, 1)$.* It follows that we can, for almost all values of x, so choose ϵ that $\chi(\epsilon)$ is large, and of either sign: and plainly this establishes Theorem 4.

It may be proved directly that in this case the series (5 . 11) is divergent for almost all x.

We add that Mr. E. C. Titchmarsh has gone further in this direction, and has proved, by direct construction and without using the theory of trigonometrical series, that, *if* $\lambda(t)$ *decreases steadily with* t *and the integral*

$$\int_0 \frac{dt}{\lambda(t)}$$

is divergent, then continuous functions $f(t)$ *exist for which*

$$\int_0 \frac{f(x+t)-f(x)}{\lambda(t)}\, dt$$

is almost always divergent. He has also given examples which show that

$$\int_0 \frac{|f(x+t)-f(x-t)|}{\lambda(t)}\, dt$$

may diverge almost everywhere; and that so also may

$$\int_0 \frac{f(x+t)-f(x-t)}{t\mu(t)}\, dt$$

if $\mu(t)$ is any function which tends steadily to zero with t, so that the results of Besikovitch and Plessner are the best possible of their kind.

* See G. H. Hardy and J. E. Littlewood, "Some problems of diophantine approximation", *Acta Math.*, 37 (1914), 155–190 (181).

5 . 4. THEOREM 5.—*If the integral* (5 . 13) *is convergent, then the series* (5 . 11) *is summable* $(C, 1)$, *to sum*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi-t}{2}\cot\tfrac{1}{2}t-\log|2\sin\tfrac{1}{2}t|\right)\phi(t)\,dt. \tag{5 . 41}$$

The Fourier series of $f(t)$, for $t=x$, is identical with that of $\phi(t)+s$ for $t=0$. It is therefore sufficient to suppose that

$$f(t)\sim\tfrac{1}{2}a_0+\Sigma a_n\cos nt,$$

and that f/t is integrable (C), and to prove that

$$S=\sum_1^\infty\frac{s_n}{n}=\sum_1^\infty\frac{\frac{1}{2}a_0+a_1+\ldots+a_n}{n} \tag{5 . 42}$$

$$=\frac{1}{\pi}\int_0^\pi\left(\frac{\pi-t}{2}\cot\tfrac{1}{2}t-\log|2\sin\tfrac{1}{2}t|\right)f(t)\,dt\quad(C,\,1).$$

We have

$$s_n=\frac{1}{\pi}\int_0^\pi\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}f(t)\,dt$$

$$=\frac{1}{\pi}\int_0^\pi f(t)\cot\tfrac{1}{2}t\sin nt\,dt+\frac{1}{\pi}\int_0^\pi f(t)\cos nt\,dt,$$

$$\sum_1^n\frac{s_\nu}{\nu}=\frac{1}{\pi}\int_0^\pi f(t)\cot\tfrac{1}{2}t\sum_1^n\frac{\sin\nu t}{\nu}\,dt+\frac{1}{\pi}\int_0^\pi f(t)\sum_1^n\frac{\cos\nu t}{\nu}\,dt \tag{5 . 43}$$

$$=S_1+S_2,$$

say. We consider S_2 first. Integrating by parts we have

$$S_2=\frac{1}{\pi}\left\{f_1(\pi)\sum_1^n\frac{(-1)^\nu}{\nu}+\int_0^\pi f_1(t)\sum_1^n\sin\nu t\,dt\right\}=S_2'+S_2'', \tag{5 . 44}$$

say.

In the first place

$$S_2'\to-\frac{1}{\pi}f_1(\pi)\log 2. \tag{5 . 45}$$

Next, we observe that

$$f_1(t) = \int_0^t \frac{f(u)}{u} \,.\, u \,.\, du = t \int_{t'}^t \frac{f(u)}{u}\, du \quad (0 < t' < t),$$

so that $f_1(t) = o(t)$. Hence

$$(5.46) \quad S_2'' = \frac{1}{2\pi} \int_0^\pi f_1(t) \frac{\cos \frac{1}{2}t - \cos(n+\frac{1}{2})t}{\sin \frac{1}{2}t}\, dt$$

$$= \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt - \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t \cos nt\, dt$$

$$+ \frac{1}{2\pi} \int_0^\pi f_1(t) \sin nt\, dt$$

$$\to \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt.$$

Finally, writing

$$(5.47) \qquad \chi(t) = \int_0^t f(u) \cot \tfrac{1}{2}u\, du,$$

and again integrating by parts, we obtain

$$S_1 = -\frac{1}{\pi} \int_0^\pi \chi(t) \sum_1^n \cos \nu t\, dt$$

$$= -\frac{1}{2\pi} \int_0^\pi \chi(t) \frac{\sin(n+\frac{1}{2})t - \sin \frac{1}{2}t}{\sin \frac{1}{2}t}\, dt$$

$$= \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt - \frac{1}{2\pi} \int_0^\pi \chi(t) \cot \tfrac{1}{2}t \sin nt\, dt - \frac{1}{2\pi} \int_0^\pi \chi(t) \cos nt\, dt.$$

The last term tends to zero, since $\chi(t)$ is continuous; and the second term tends to zero $(C, 1)$, by Fejér's theorem, since $\chi(t)$ is continuous and vanishes for $t = 0$. Hence

$$(5.48) \qquad S_1 \to \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt \quad (C, 1).$$

From (5.43)–(5.48) we obtain

$$(5.49) \quad S = -\frac{1}{\pi} f_1(\pi) \log 2 + \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt + \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt \quad (C, 1).$$

If now we make the transformations

$$-\frac{1}{\pi}f_1(\pi)\log 2 = -\frac{1}{\pi}\log 2\int_0^\pi f(t)\,dt,$$

$$\frac{1}{2\pi}\int_0^\pi f_1(t)\cot\tfrac{1}{2}t\,dt = \frac{1}{\pi}\int_0^\pi f_1(t)\,\frac{d}{dt}\log|\sin\tfrac{1}{2}t|\,dt$$

$$= -\frac{1}{\pi}\int_0^\pi \log|\sin\tfrac{1}{2}t|\,f(t)\,dt,$$

$$\frac{1}{2\pi}\int_0^\pi \chi(t)\,dt = \frac{1}{2\pi}\int_0^\pi dt\int_0^t f(u)\cot\tfrac{1}{2}u\,du$$

$$= \tfrac{1}{2}\int_0^\pi f(u)\cot\tfrac{1}{2}u\,du - \frac{1}{2\pi}\int_0^\pi u\,f(u)\cot\tfrac{1}{2}u\,du,$$

we are led to (5 . 42). These partial integrations present no difficulty.

5 . 5. THEOREM 6.—*If the series* (5 . 11) *is summable* $(C,-1)$, *then the integral* (5 . 13) *is summable* $(C,1)$.

To say that (5 . 11) is summable $(C,-1)$ is plainly equivalent to asserting its convergence and also that of the Fourier series of $f(t)$.

If $\epsilon>0$, we have

$$J(\epsilon) = \int_\epsilon^\pi \phi(t)\cot\tfrac{1}{2}t\,dt = \sum_0^\infty A_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\cos nt\,dt - s\int_\epsilon^\pi \cot\tfrac{1}{2}t\,dt$$

$$= \sum_0^\infty s_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\,\{\cos nt-\cos(n+1)t\}\,dt+2s\log\sin\tfrac{1}{2}\epsilon,$$

since

$$s_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\cos nt\,dt\to 0$$

when ϵ is fixed and $n\to\infty$. Hence

$$(5\,.\,51)\quad J(\epsilon) = \sum_0^\infty s_n\int_\epsilon^\pi\{\sin(n+1)t+\sin nt\}\,dt+2s\log\sin\tfrac{1}{2}\epsilon$$

$$= s_0\cos\epsilon+\sum_1^\infty s_n\left\{\frac{\cos n\epsilon}{n}+\frac{\cos(n+1)\epsilon}{n+1}\right\}$$

$$+s_0-\sum_1^\infty s_n\frac{(-1)^n}{n(n+1)}+2s\log\sin\tfrac{1}{2}\epsilon.$$

If in (5 . 51) we replace s_n and s_0 by s, it is easily verified that the coefficient of s reduces to zero, so that

$$J(\epsilon) = J_1(\epsilon)+J_2(\epsilon), \tag{5 . 52}$$

where

$$J_1(\epsilon) = (s_0-s)\cos\epsilon+\sum_1^\infty (s_n-s)\left\{\frac{\cos n\epsilon}{n}+\frac{\cos(n+1)\epsilon}{n+1}\right\}, \tag{5 . 53}$$

and

$$J_2(\epsilon) = (s_0-s)-\sum_1^\infty (s_n-s)\frac{(-1)^n}{n(n+1)} \tag{5 . 54}$$

is independent of ϵ.

Now

$$\frac{1}{\epsilon}\int_0^\epsilon J_1(\eta)\,d\eta = (s_0-s)\frac{\sin\epsilon}{\epsilon}+\sum_1^\infty (s_n-s)\left\{\frac{\sin n\epsilon}{n^2\epsilon}+\frac{\sin(n+1)\epsilon}{(n+1)^2\epsilon}\right\}$$

$$= \sum_1^\infty (s_n-s)\frac{\sin n\epsilon}{n^2\epsilon}+\sum_0^\infty (s_n-s)\frac{\sin(n+1)\epsilon}{(n+1)^2\epsilon}.$$

Each of these series is of the form

$$\sum_1^\infty t_n\frac{\sin n\epsilon}{n\epsilon},$$

where Σt_n is summable $(C, -1)$; and so each tends to the limit Σt_n, by a theorem of Fatou.* Hence

$$J_1(\epsilon)\to\sum_1^\infty\frac{s_n-s}{n}+\sum_0^\infty\frac{s_n-s}{n+1} = s_0-s+2\sum_1^\infty\frac{s_n-s}{n}-\sum_1^\infty\frac{s_n-s}{n(n+1)} \quad (C, 1);$$

$$J(\epsilon)\to 2(s_0-s)+2\sum_1^\infty\frac{s_n-s}{n}-\sum_1^\infty(s_n-s)\frac{1+(-1)^n}{n(n+1)} \quad (C, 1); \tag{5 . 55}$$

which proves the theorem.

We note an obvious corollary which will be useful later.

LEMMA λ.—*If $f(t)$ is of integrable square, and $s = f(x)$, and the series* (5 . 11) *is almost always summable* $(C, -1)$, *and to a function of integrable square, then the integral* (5 . 13) *is almost always summable* $(C, 1)$, *and represents a function of integrable square.*

---

* F.S., 88 (Lemma 12).

This is obvious from (5 . 55), since the three terms on the right-hand side are of integrable square.

5 . 6. Theorems 5 and 6 are plainly the first of two scales of theorems, the next theorem of the scale begun by Theorem 6, for example, being that, *if the series* (5 . 11) *is convergent, then* (5 . 13) *is summable* $(C, 2)$. It is doubtless true that *the necessary and sufficient condition for the summability of* (5 . 11) *is the summability of* (5 . 13); but we have not attempted to work out the proof in detail.

5 . 7. We have seen that (5 . 13) does not necessarily exist almost always even when $f(t)$ is continuous, and we know of no very illuminating additional criterion, sufficient to ensure that it shall do so, which is not obviously trivial.* It is, however, comparatively easy to find a significant criterion that (5 . 13) shall converge almost everywhere *to a function of integrable square.*

THEOREM 7.—*A sufficient condition that the integral* (5 . 13) *should exist as a Cauchy integral, with* $s = f(x)$, *for almost all values of* x, *and represent a function of integrable square, is that the series*

$$\Sigma(|a_n|^2+|b_n|^2)(\log n)^2 \tag{5 . 71}$$

should be convergent.

We require three additional lemmas.

LEMMA μ.—*If* Σa_n *is convergent*† *to the sum* s, *and* $\Sigma a_n \log n$ *is summable* $(C, 1)$, *then*

$$\Sigma \frac{s_n - s}{n},$$

where $s_n = a_1+a_2+\ldots+a_n$, *is convergent.*

* Obviously, if $f(t)$ satisfies a Lipschitz condition $f(t+h)-f(t) = O(|h|^\alpha)$, where $\alpha > 0$, uniformly in t, the integral is uniformly convergent (as a Lebesgue integral). We mean by an "illuminating" criterion one which would throw light on the possibilities of non-absolute convergence.

† This condition is really unnecessary; but it simplifies the proof, and the lemma is as it stands sufficient for our purpose. If Σa_n is not convergent, it is any rate summable $(C, 1)$, and s must be taken to be its Cesàro sum.

We may plainly suppose, without loss of generality, that $a_1 = 0$, $s = 0$. We write

$$t_n = a_2 \log 2 + \ldots + a_n \log n, \quad T_n = t_2 + \ldots + t_n,$$

$$\mathbf{T}_n = \sum_2^{n-1} t_\nu \Delta \frac{1}{\log \nu},^* \qquad \tau_n = \frac{s_2}{2} + \ldots + \frac{s_n}{n}.$$

We have then, by partial summation,

$$s_n = \sum_2^n \frac{t_\nu - t_{\nu-1}}{\log \nu} = \mathbf{T}_n + \frac{t_n}{\log n} \tag{5.72}$$

and

$$\tau_n = s_n \log n - t_n + \sum_2^n s_\nu \omega_\nu, \tag{5.73}$$

where

$$\omega_\nu = \frac{1}{\nu} + \Delta \log \nu = O\left(\frac{1}{\nu^2}\right).$$

From (5.72) and 5.73) we deduce

$$\tau_n = \mathbf{T}_n \log n + \sum_2^{n-1} s_\nu \omega_\nu. \tag{5.74}$$

The last series is plainly convergent when continued to infinity, so that what we have to prove is that $\mathbf{T}_n \log n$ tends to a limit.

We observe first that

$$\mathbf{T}_n = \sum_2^{n-1} t_\nu \Delta \frac{1}{\log \nu} = \sum_2^{n-2} T_\nu \Delta^2 \frac{1}{\log \nu} + T_{n-1} \Delta \frac{1}{\log (n-1)}. \tag{5.75}$$

Now $T_n \sim \mathbf{a}n$, where $\mathbf{a}$ is the $(C, 1)$ sum of $\Sigma a_n \log n$. Hence the second term on the right of (5.75) tends to zero, and the first series is convergent when continued to infinity; so that

$$\mathbf{T}_n \to \sum_2^\infty t_\nu \Delta \frac{1}{\log \nu} = \mathbf{T},$$

say. But $\mathbf{T}$ must be 0, by (5.72); for $s_n \to 0$, and $t_n \to \mathbf{a}$ $(C, 1)$, so that

$$\frac{t_n}{\log n} \to 0 \quad (C, 1),$$

* The operator Δ is defined by the equation

$$\Delta f(\nu) = f(\nu) - f(\nu + 1).$$

from which it follows that $\mathbf{T}_n \to 0$ $(C, 1)$, or that $\mathbf{T} = 0$. We may therefore write (5 . 75) in the form

$$\mathbf{T}_n = -\sum_n^\infty t_\nu \Delta \frac{1}{\log \nu} = T_{n-1} \Delta \frac{1}{\log n} - \sum_n^\infty T_\nu \Delta^2 \frac{1}{\log \nu}.$$

Here the first term is $o(1/\log n)$, while the second is asymptotic to

$$-\mathbf{a} \sum_{n-1}^\infty \nu \Delta^2 \frac{1}{\log \nu} \sim -\mathbf{a} \sum_{n-1}^\infty \frac{1}{\nu (\log \nu)^2} \sim -\frac{\mathbf{a}}{\log n}.$$

Then $\mathbf{T}_n \log n$ tends to a limit, and the lemma is proved.

5 . 8. LEMMA ν.—*In order that*

$$\mathbf{S}_n = \sum_1^n \frac{s_\nu - f(x)}{\nu},$$

where s_ν is the sum of the first $\nu+1$ terms of the Fourier series of $f(t)$, for $t = x$, should converge in mean to a function of integrable square, it is necessary and sufficient that the series (5 . 71) *should be convergent.*

A function $\chi_n(x)$, defined for $a \leqslant x \leqslant b$, is said to converge in mean if

$$\lim \int_a^b \{\chi_\mu(x) - \chi_\nu(x)\}^2 dx = 0$$

when μ and ν tend to infinity. There is then a function $\chi(x)$, of integrable square, such that

$$\lim \int_a^b \{\chi_n(x) - \chi(x)\}^2 dx = 0$$

when $n \to \infty$. And it is possible to choose a sequence $n_1, n_2, \ldots$ of values of n so that $\chi_{n_i}(x) \to \chi(x)$, when $i \to \infty$, for almost all values of x.*

The problem is immediately reducible to the corresponding problems for an even and odd function, which may be treated similarly. We suppose then that $f(x)$ is even and, as plainly we may without loss of

* The notion of convergence in mean is due to E. Fischer ("Sur la convergence en moyenne", *Comptes Rendus*, 13 May, 1907). For a simple account of the fundamentals of the theory see M. Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289-335.

generality, that $a_0 = 0$, so that

$$f \sim \sum_1^\infty a_n \cos nx = \sum_1^\infty A_n, \qquad s_n = A_1 + A_2 + \ldots + A_n.$$

We have

$$\sum_\mu^\nu \frac{s_n - f(x)}{n} = \frac{A_1 + \ldots + A_\mu}{\mu} + \ldots + \frac{A_1 + \ldots + A_\nu}{\nu} - f\sigma_{\mu,\nu}$$

$$= A_1\sigma_{\mu,\nu} + \ldots + A_\mu\sigma_{\mu,\nu} + A_{\mu+1}\sigma_{\mu+1,\nu} + \ldots + A_\nu\sigma_{\nu,\nu} - f\sigma_{\mu,\nu},$$

where

$$\sigma_{\mu,\nu} = \sum_\mu^\nu \frac{1}{n}.$$

Squaring, integrating, and performing some simple reductions, we obtain

$$(5.81) \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} (\mathbf{S}_\nu - \mathbf{S}_{\mu-1})^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\sum_\mu^\nu \frac{s_n - f(x)}{n}\right)^2 dx$$

$$= a_{\mu+1}^2\sigma_{\mu,\mu}^2 + a_{\mu+2}^2\sigma_{\mu,\mu+1}^2 + \ldots + a_\nu^2\sigma_{\mu,\nu-1}^2 + (a_{\nu+1}^2 + a_{\nu+2}^2 + \ldots)\sigma_{\mu,\nu}^2.$$

Suppose first that (5.71) is convergent. Then

$$a_{\nu-1}^2 + a_{\nu+2}^2 + \ldots = o\left(\frac{1}{(\log\nu)^2}\right), \qquad \sigma_{\mu,\nu} = O(\log\nu),$$

so that the last term in (5.81) tends to zero. The remainder of the right-hand side of (5.81) is

$$O\left\{\sum_{\mu+1}^\nu a_n^2 \left(\log\frac{n}{\mu}\right)^2\right\} = O\left(\sum_{\mu+1}^\nu a_n^2 (\log n)^2\right) \to 0.$$

Hence the integral (5.81) tends to 0, and the sufficiency of our criterion is proved. Its necessity is obvious from (5.81), since $\sigma_{\mu,\lambda} \sim \log\lambda$ when μ is fixed and $\lambda \to \infty$.

LEMMA *o.—If x is in the Lebesgue set, and the integral* (5.13) *is summable* $(C,1)$*, then it is convergent as a Cauchy integral.*

It is plainly enough to show that, if

$$j(t) = \int_t^a \frac{\phi(t)}{t}\, dt,$$

then

$$j(t) - \frac{1}{t}\int_0^t j(u)\, du = j(t) - \frac{j_1(t)}{t} \to 0,$$

when x is in the Lebesgue set; and this follows at once from (3 . 34), since

$$\phi_1(t) = \int_0^t \phi(u)\,du = O\left(\int_0^t |\,\phi(u)\,|\,du\right) = o(t).*$$

5 . 9. We can now prove Theorem 7. If the series (5 . 71) is convergent, then $\Sigma A_n \log n$ is almost always summable $(C, 1)$, and ΣA_n is almost always convergent to $f(x)$.†

Hence first, by Lemma μ, the series

$$\mathbf{S} = \sum_1^\infty \frac{s_\nu - f(x)}{\nu}$$

is almost always convergent. It converges in mean, by Lemma ν, to a function of integrable square, so that the square of its sum function $\sigma(x)$ is integrable. Since $s_\nu \to f(x)$ almost always, it is almost always summable $(C, -1)$; and hence, by Theorem 6, the integral (5 . 13), with $s = f(x)$, is almost always summable $(C, 1)$; and therefore, by Lemma o, almost always convergent. Finally, by Lemma λ, its value is of integrable square.

* Obviously we do not use the full force of the condition that x is in the Lebesgue set; and the lemma might be generalized in many ways.

† That ΣA_n converges almost always follows from a theorem proved by Hardy (*loc. cit.* p. 213, f.n.‡ Recently Kolmogouroff and Seliverstoff ("Sur la convergence des séries de Fourier", *Comptes Rendus*, January 14th, 1924), have proved more, viz. that $\Sigma(|\,a_n\,|^2 + |\,b_n\,|^2)(\log n)^{1+\delta}$ is convergent for any positive δ.

ON WELL-POISED SERIES, GENERALIZED HYPERGEOMETRIC SERIES HAVING PARAMETERS IN PAIRS, EACH PAIR WITH THE SAME SUM

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1. *Introduction.*—The generalized hypergeometric series

$$\sum_{n=0}^{\infty} \frac{\Pi_{a,\,b,\,c,\,d,\,e,\,f,\,g}\{\Gamma(a+n)/\Gamma(a)\}}{\Pi_{p,\,q,\,r,\,s,\,t,\,u}\{\Gamma(p+n)/\Gamma(p)\}} \cdot \frac{x^n}{n!}$$

is denoted by

$$(1.1) \qquad {}_7F_6\left[\begin{matrix} a, b, c, d, e, f, g; \\ p, q, r, s, t, u \end{matrix} x\right],$$

and the corresponding notation is used for series with other parameters. When the x is not stated explicitly it is to be taken as $+1$.

We propose to investigate the properties of series which are subject to the conditions

$$x = \pm 1,$$

$$(1.2) \qquad a+1 = b+p = c+q = \ldots.$$

It is convenient to have a name for such series, and the name "well-poised" seems to be appropriate.

In §§ 2–8 the cases in which the sum of a well-poised series can either be expressed in finite terms or as the sum of a hypergeometric series of lower order are discussed. In §§ 10–14 some of the special cases in which such hypergeometric series can themselves be summed, so that the well-poised series can also be summed, are dealt with. In §§ 15, 16 it is shown how important general propositions can be established by reversing the formulae and expressing other hypergeometric series in terms of well-poised series.

The fundamental propositions form natural groups, those in which $x = (-1)^{r-1}$, where r is the number of pairs of parameters in the well-poised series, and those in which $x = (-1)^r$. The properties of the second group are readily obtained as corollaries from those of the first. We may refer to the two groups as the strong and the weak. In each group we have propositions in which there is no restriction on the parameters. These are followed by propositions with one or more restrictions. All the results could be derived from one formula (7 . 7), which is a generalization of the formulae, due to Dougall and to Saalschütz, mentioned in §§ 8 and 16 respectively.

The proof of (7 . 7) is somewhat laborious; it occupies practically the whole of §§ 2–7. It is to be hoped that a shorter proof will be discovered.*

Except where it is essential to the argument, reference to the convergence of series has been omitted. It will be recalled that the conditions that the general hypergeometric series of (1 . 1) may be convergent are :

if $x = +1$, $\quad \Sigma p - \Sigma a > 0$,

if $x = -1$, $\quad \Sigma p - \Sigma a > -1$.

It is to be assumed that such restrictive conditions are imposed on the parameters of the series which are mentioned.

2. *Strong Group. Known Propositions.*—We start with a proposition due to Kummer†

$$(2 . 1) \qquad {}_2F_1\left[\begin{matrix} a, & b; \\ 1+a-b & \end{matrix} \; -1\right] = \frac{\Gamma(1+a-b)\,\Gamma(\frac{1}{2})}{2^a\,\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}a-b+1)},$$

which may also be written in the form

$$(2 . 11) \qquad {}_2F_1\left[\begin{matrix} a, & b; \\ 1+a-b & \end{matrix} \; -1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+a)}.$$

Professor Watson‡ has shown recently that the following proposition,

* See below, p. 263.

† *Journal für Math.*, 15 (1836), 78, Formula 53. This reference is taken from Prof. Watson's paper cited in the next foot-note.

‡ *Proc. London Math. Soc.* (2), 22 (1924), xxxii–xxxiv (*Records* for May 17, 1923).

due to A. C. Dixon,* can be deduced from Kummer's :

$$(2.2)\quad {}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\ 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+\frac{1}{2}a-c)\,\Gamma(1+a-b-c)}.$$

3. *Strong Group. Reduction of* ${}_4F_3$.—The definition of ${}_4F_3$ may be written in the form

$$(3.1)\quad {}_4F_3\left[\begin{matrix} a, & b, & c, & d; \\ 1+a-b, & 1+a-c, & 1+a-d & \end{matrix}\ -1\right] = \sum_{n=0}^{\infty} (-)^n \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(1+a-b)}{n!\ \Gamma(1+a-b+n)\,\Gamma(a)\,\Gamma(b)} \cdot \frac{\Gamma(c+n)\,\Gamma(d+n)\,\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a-c+n)\,\Gamma(1+a-d+n)\,\Gamma(c)\,\Gamma(d)}.$$

Now we know, by Gauss's theorem, that

$$(3.2)\quad \frac{\Gamma(c+n)\,\Gamma(d+n)}{\Gamma(1+a-c+n)\,\Gamma(1+a-d+n)} = \frac{\Gamma(c+n)\,\Gamma(d+n)}{\Gamma(1+a+2n)\,\Gamma(1+a-c-d)}\ F\left[\begin{matrix} c+n, & d+n \\ & 1+a+2n \end{matrix}\right] = \frac{1}{\Gamma(1+a-c-d)} \sum_{p=n}^{\infty} \frac{\Gamma(c+p)\,\Gamma(d+p)}{\Gamma(1+p-n)\,\Gamma(1+a+p+n)}.$$

Hence on substitution and change of order of summation, followed by the quotation of (2.2), we find that

$${}_4F_3\left[\begin{matrix} a, & b, & c, & d; \\ 1+a-b, & 1+a-c, & 1+a-d & \end{matrix}\ -1\right]$$

$$(3.3)\qquad = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma c\,\Gamma d\,\Gamma(1+a-c-d)} \times \sum_{p=0}^{\infty} \frac{\Gamma(c+p)\,\Gamma(d+p)}{\Gamma(1+p)\,\Gamma(1+a+p)}\ {}_3F_2\left[\begin{matrix} a, & b, & -p; \\ 1+a-b, & 1+a+p & \end{matrix}\ 1\right]$$

$$(3.31)\qquad = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma c\,\Gamma d\,\Gamma(1+a-c-d)} \times \sum_{p=0}^{\infty} \frac{\Gamma(c+p)\,\Gamma(d+p)\,\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a-b+p)}{p!\ \Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+\frac{1}{2}a+p)\,\Gamma(1+a-b+p)}.$$

* *Proc. London Math. Soc.* (1), 35 (1903), 284.

This result may be written in the form

$$(3.4)\quad {}_4F_3\left[\begin{matrix} a, & b, & c, & d; \\ 1+a-b, & 1+a-c, & 1+a-d & \end{matrix}\ -1\right]$$
$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}\ {}_3F_2\left[\begin{matrix} 1+\tfrac{1}{2}a-b, & c, & \\ 1+\tfrac{1}{2}a, & 1+a-b & \end{matrix}\ d\right].$$

It will be seen that the well-poised series ${}_4F_3$ has been expressed in terms of a series of type ${}_3F_2$. In general no further reduction is possible.

4. *Strong Group. Reduction of* ${}_5F_4$.—The analysis of ${}_5F_4$ follows the same lines as far as the analogue of (3.3). We find that

$$(4.1)\quad {}_5F_4\left[\begin{matrix} a, & b, & c, & d, & e; \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix}\ 1\right]$$
$$= \frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma d\,\Gamma e\,\Gamma(1+a-d-e)} \sum_{p=0}^{\infty} \frac{\Gamma(d+p)\,\Gamma(e+p)}{\Gamma(1+p)\,\Gamma(1+a+p)}$$
$$\times\ {}_4F_3\left[\begin{matrix} a, & b, & c, & -p; \\ 1+a-b, & 1+a-c, & 1+a+p & \end{matrix}\ -1\right].$$

Substitution from (3.4) shows that

$$(4.2)\quad {}_5F_4 = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(d)\,\Gamma(e)\,\Gamma(1+a-d-e)}$$
$$\times \sum_{p=0}^{x} \frac{\Gamma(d+p)\,\Gamma(e+p)}{p!\ \Gamma(1+a-c+p)}\ {}_3F_2\left[\begin{matrix} 1+\tfrac{1}{2}a-b, & c\,; & \\ 1+\tfrac{1}{2}a, & 1+a-b & \end{matrix}\ -p\right]$$
$$= \frac{\Gamma(1+\tfrac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+\tfrac{1}{2}a-b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(1+a-d-e)}$$
$$\times \sum_{p=0}^{\infty} \sum_{r=0}^{p} \frac{(-)^r\,\Gamma(1+\tfrac{1}{2}a-b+r)\,\Gamma(c+r)\,\Gamma(d+p)\,\Gamma(e+p)}{r!\ (p-r)!\ \Gamma(1+\tfrac{1}{2}a+r)\,\Gamma(1+a-b+r)\,\Gamma(1+a-c+p)}.$$

If it is legitimate to write $p = r+q$ and change the order of summation, then the double sum is equivalent to

$$(4.3)\quad \sum_{r=0}^{\infty} \frac{(-)^r}{r!}\ \frac{\Gamma(1+\tfrac{1}{2}a-b+r)\,\Gamma(c+r)\,\Gamma(d+r)\,\Gamma(e+r)}{\Gamma(1+\tfrac{1}{2}a+r)\,\Gamma(1+a-b+r)\,\Gamma(1+a-c+r)}$$
$$\times\ {}_2F_1\left[\begin{matrix} d+r, & e+r \\ & 1+a-c+r \end{matrix}\right].$$

The series ${}_2F_1$ are in general divergent for all values of r above some definite limit, and it is not legitimate to make the suggested change of order. Further progress is only possible if a restriction is imposed on the parameters. We must make either $(1+\frac{1}{2}a-b)$ or c or d or e a negative integer. If, for example, e is a negative integer, we must write $(-)^p\Gamma(1-e)/\Gamma(1-e-p)$ in place of $\Gamma(e+p)/\Gamma(e)$ in (4 . 1), and restrict the summation to the range from $p=0$ to $p=-e$. The corresponding modifications in (4 . 2) and (4 . 3) are easily made.

Gauss's theorem now gives us

$$(4\,.\,4)\qquad {}_2F_1\left[\begin{matrix} d+r,\ e+r \\ 1+a-c+r \end{matrix}\right] = \frac{\Gamma(1+a-c+r)\,\Gamma(1+a-c-d-e-r)}{\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)}.$$

By substituting from (4 . 3) and (4 . 4) in (4 . 2), we get

$$(4\,.\,5)\qquad {}_5F_4\left[\begin{matrix} a,\ b, & c, & d, & e\,; \\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e & & & \end{matrix}\ 1\right]$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)}$$

$$\times {}_4F_3\left[\begin{matrix} 1+\frac{1}{2}a-b,\ c,\ d,\ e. \\ 1+\frac{1}{2}a,\ c+d+e-a,\ 1+a-b \end{matrix}\right].$$

This formula is subject to the restriction that one of the parameters $1+\frac{1}{2}a-b$, c, d, or e is a negative integer. The series ${}_4F_3$ terminates; the series ${}_5F_4$ does not terminate when the restriction is that $1+\frac{1}{2}a-b$ is a negative integer, but in that case certain factors of the higher terms of the series cancel.

5. *Strong Group. Summations subject to the restriction* $b=1+\frac{1}{2}a$. —The results (3 . 4) and (4 . 5) are still valid when $1+\frac{1}{2}a-b=0$. With that restriction they take the simple forms

$$(5\,.\,1)\qquad {}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ c, & d\,; \\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d & \end{matrix}\ -1\right] = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}$$

and

$$(5\,.\,2)\qquad {}_5F_4\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ c, & d, & e\,; \\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e & & \end{matrix}\ 1\right]$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)}.$$

6. *Strong Group. Reduction of* ${}_6F_5$.—Proceeding to the next order, and following the same method as in § 3, we write down the equation

$$(6.1)\quad {}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\ -1\right]$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(e)\,\Gamma(f)\,\Gamma(1+a-e-f)} \sum_{p=0}^{\infty} \frac{\Gamma(e+p)\,\Gamma(f+p)}{\Gamma(1+p)\,\Gamma(1+a+p)}$$

$$\times {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & -p; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a+p \end{matrix}\ 1\right].$$

We quote (5 . 2) (with $-p$ written in place of e) and get

$$(6.2)\quad {}_6F_5 = \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(e)\,\Gamma(f)\,\Gamma(1+a-e-f)}$$

$$\times \sum_{p=0}^{\infty} \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-c-d+p)\,\Gamma(e+p)\,\Gamma(f+p)}{\Gamma(1+a)\,\Gamma(1+a-c-d)\,\Gamma(1+a-d+p)\,\Gamma(1+a-e+p)\,p\,!}$$

so that

$$(6.3)\quad {}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f; \\ \frac{1}{2}a, & & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\ -1\right]$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(1+a-e-f)}\ {}_3F_2\left[\begin{matrix} 1+a-c-d, & e, & f \\ 1+a-c, & & 1+a-d \end{matrix}\right].$$

7. *Strong Group. Reduction of* ${}_7F_6$.—Again, using the form suggested by the experience gained in § 4, we write

$$(7.1)\quad {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ \frac{1}{2}a, & & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{matrix}\ 1\right]$$

$$= \frac{\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1-f)\,\Gamma(1-g)}{\Gamma(1+a-f-g)}$$

$$\times \sum_{p=0}^{\infty} \frac{1}{\Gamma(1+p)\,\Gamma(1+a+p)\,\Gamma(1-f-p)\,\Gamma(1-g-p)}$$

$$\times {}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & -p; \\ \frac{1}{2}a, & & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+p \end{matrix}\ -1\right]$$

But substitution of $-p$ for f in (6 . 3) gives

$$(7.2)\quad {}_6F_5 = \frac{\Gamma(1+a-e)\,\Gamma(1+a+p)}{\Gamma(1+a)\,\Gamma(1+a-e+p)}\ {}_3F_2\left[\begin{matrix} 1+a-c-d, & e, & -p \\ 1+a-c, & & 1+a-d \end{matrix}\right].$$

Introducing this expression into (7 . 1) we find that

$$(7.3)\quad {}_7F_6 = \sum_{p=0}^{\infty} \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1-f)\,\Gamma(1-g)}{\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1+p)} \times \Gamma(1-f-p)\,\Gamma(1-g-p)\,\Gamma(1+a-e+p) \times {}_3F_2\left[\begin{matrix}1+a-c-d,\ c,\ -p\\ 1+a-c,\ 1+a-d\end{matrix}\right]$$

$$(7.4)\quad = \sum_{p=0}^{\infty}\sum_{r=0}^{p} \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f) \times \Gamma(1+a-g)\,\Gamma(1-e)\,\Gamma(1-f)\,\Gamma(1-g)\,\Gamma(c+d-a)}{r!\,(p-r)!\,\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1-f-p)\,\Gamma(1-g-p) \times \Gamma(1+a-e+p)\,\Gamma(c+d-a-r)\,\Gamma(1-e-r) \times \Gamma(1+a-c+r)\,\Gamma(1+a-d+r)}$$

Putting $p = r+q$, and changing the order of summation, leads to

$$(7.5)\quad {}_7F_6 = \sum_{r=0}^{\infty} \frac{N/[\Gamma(1+a)\,\Gamma(1+a-f-g)]}{r!\,\Gamma(1-f-r)\,\Gamma(1-g-r)\,\Gamma(1+a-e+r)\,\Gamma(c+d-a-r) \times \Gamma(1-e-r)\,\Gamma(1+a-c+r)\,\Gamma(1+a-d+r)} \times {}_2F_1\left[\begin{matrix}f+r,\ g+r\\ 1+a-e+r\end{matrix}\right],$$

N being written for the numerator which appears in (7 . 4).

This change in order of summation cannot be made if the parameters are unrestricted, for the series ${}_2F_1$ is divergent if r exceeds some definite limit.

If, however, e, f, g, or $1+a-c-d$ is a negative integer, the series (7 . 5) terminates. In the early terms, which survive, we can write

$$(7.6)\quad {}_2F_1\left[\begin{matrix}f+r,\ g+r\\ 1+a-e+r\end{matrix}\right] = \frac{\Gamma(1+a-e+r)\,\Gamma(1+a-e-f-g-r)}{\Gamma(1+a-e-f)\,\Gamma(1+a-e-g)},$$

and it follows that

$$(7.7)\quad {}_7F_6\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c,\ d,\ e,\ f,\ g;\\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f,\ 1+a-g\end{matrix}\ 1\right] = \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1+a-e-f)\,\Gamma(1+a-e-g)} \times {}_4F_3\left[\begin{matrix}1+a-c-d,\ e,\ f,\ g\\ e+f+g-a,\ 1+a-c,\ 1+a-d\end{matrix}\right].$$

The relation is subject to the condition that ${}_4F_3$ is a terminating series.

8. *Dougall's Theorem.*—If $1+2a = c+d+e+f+g$, the ${}_4F_3$ of (7 . 7) reduces to

$$ {}_3F_2\left[\begin{matrix} e, & f, & g \\ 1+a-c, & 1+a-d \end{matrix}\right]. $$

In this case the parameters of the ${}_3F_2$ are subject to the two conditions, that $(1+a-c)+(1+a-d)-e-f-g = 1$, and that either e, f, or g is a negative integer. Accordingly we can quote Saalschütz's formula*

$$ (8\cdot 1) \qquad F\left[\begin{matrix} x, & y, & z \\ u, & v \end{matrix}\right] = \frac{\Gamma(u)\,\Gamma(1-v+x)\,\Gamma(1-v+y)\,\Gamma(1-v+z)}{\Gamma(1-v)\,\Gamma(u-x)\,\Gamma(u-y)\,\Gamma(u-z)}, $$

which is valid provided that either x, y, or z is a negative integer, and that $u+v-x-y-z = 1$. We are led to the result

$$ (8\cdot 2) \quad {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{matrix}\ \middle|\ 1\right] $$

$$ = \frac{\begin{matrix}\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g) \\ \times\Gamma(c+d-a)\,\Gamma(c+e-a)\,\Gamma(c+f-a)\,\Gamma(c+g-a)\end{matrix}}{\begin{matrix}\Gamma(1+a)\,\Gamma(c-a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-g) \\ \times\Gamma(1+a-e-f)\,\Gamma(1+a-e-g)\,\Gamma(1+a-f-g)\end{matrix}}. $$

This formula is symmetrical in d, e, f, and g, so it will be valid if either of those parameters is a negative integer, whilst $1+2a = c+d+e+f+g$.

The equation (8 . 2) is equivalent to one discovered† by Dougall and proved‡ by a less direct but much more elegant method.

It is found that our method will not carry us any further; no formula for the reduction of a well-poised series of order 8 has been discovered.

9. *Weak Group.*—By equating one of the parameters to $\frac{1}{2}(1+a)$ we can reduce a "strong" formula to a "weak" one. The first two results, derived from Kummer's formula and Dixon's, (2 . 1) and (2 . 2), are familiar, viz.—

$$ (9\cdot 1) \qquad {}_1F_0[a;\ -1] = 2^{-a}, $$

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* Hardy, *Proc. Camb. Phil. Soc.*, 21 (1923). 497, gives three references: *Math. Zeitschrift*, 35 (1890), 186; 36 (1891), 278 and 321.

† "On Vandermonde's Theorem and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114-132, equation (6). The results 8·1 and 5·2 stated above are also obtained in Dougall's paper, equations (2) and (9).

‡ The proof is quoted by Hardy, *loc. cit.*, 493.

the binomial theorem, and

$$(9.2)\qquad {}_2F_1\left[\begin{matrix}a,\ b\,; \\ 1+a-b\end{matrix}\ 1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1-2b)}{\Gamma(1+a-2b)\,\Gamma(1-b)},$$

a particular case of Gauss's theorem.

Equation (3.4) can be reduced in various ways; to preserve symmetry we exchange b and d, and then write $d=\frac{1}{2}(1+a)$, the result being

$$(9.3)\qquad {}_3F_2\left[\begin{matrix}a,\ b, & c\,; \\ 1+a-b,\ 1+a-c\end{matrix}\ -1\right]$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}\ {}_3F_2\left[\begin{matrix}\frac{1}{2},\ b, & c \\ 1+\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}a\end{matrix}\right].$$

Similarly from (4.5) we derive the equation

$$(9.4)\qquad {}_4F_3\left[\begin{matrix}a,\ b, & c, & d\,; \\ 1+a-b,\ 1+a-c, & 1+a-d\end{matrix}\ 1\right]$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)\,\Gamma(1+a-b-c)}$$

$$\times {}_4F_3\left[\begin{matrix}\frac{1}{2},\ b, & c, & d \\ 1+\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}a, & b+c+d-a\end{matrix}\right],$$

which is subject to the restriction that b, c, or d is a negative integer.

The special cases of § 5 yield the formulae

$$(9.5)\qquad {}_3F_2\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c\,; \\ \frac{1}{2}a,\ 1+a-c\end{matrix}\ -1\right] = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c)}$$

and

$$(9.6)\qquad {}_4F_3\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c, & d\,; \\ \frac{1}{2}a,\ 1+a-c, & 1+a-d\end{matrix}\ 1\right]$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c-d)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-d)\,\Gamma(1+a-c-d)}.$$

In reducing (6.3) or (7.7), we have a choice as to the parameter to equate to $\frac{1}{2}(1+a)$. From the alternatives we select the forms

$$(9.7)\qquad {}_5F_4\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c, & d, & e\,; \\ \frac{1}{2}a,\ 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\ -1\right]$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-e)}\ {}_3F_2\left[\begin{matrix}1+a-c-d,\ e,\ \frac{1}{2}+\frac{1}{2}a \\ 1+a-e, \qquad 1+a-d\end{matrix}\right]$$

and

$$(9.8)\quad {}_6F_5\left[\begin{matrix}a, & 1+\frac{1}{2}a, & c, & d, & e, & f;\\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\;1\right]$$

$$=\frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-e-f)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-e)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-f)\,\Gamma(1+a-e-f)}$$

$$\times {}_4F_3\left[\begin{matrix}1+a-c-d, & e, & f, & \frac{1}{2}+\frac{1}{2}a\\ 1+a-c, & 1+a-d, & \frac{1}{2}-\frac{1}{2}a+e+f\end{matrix}\right],$$

(9.8) being subject to the condition that the ${}_4F_3$ is a terminating series.

Finally we notice the "weak" form of Dougall's theorem, viz.

$$(9.9)\quad {}_6F_5\left[\begin{matrix}a, & 1+\frac{1}{2}a, & c, & d, & e, & f,\\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\;1\right]$$

$$=\frac{\begin{matrix}\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\\ \times\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\,\Gamma(\frac{1}{2}-\frac{1}{2}a+d)\,\Gamma(\frac{1}{2}-\frac{1}{2}a+e)\,\Gamma(\frac{1}{2}-\frac{1}{2}a+f)\end{matrix}}{\begin{matrix}\Gamma(1+a)\,\Gamma(\frac{1}{2}-\frac{1}{2}a)\,\Gamma(1+a-c-d)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-f)\\ \times\Gamma(1+a-d-e)\,\Gamma(1+a-d-f)\,\Gamma(1+a-e-f)\end{matrix}}$$

which holds if $\frac{1}{2}+\frac{3}{2}a-c-d-e-f=0$, and one of the parameters c, d, e, f is a negative integer.

10. *Special Cases. Method of Selection.*—Special cases of the formulae can be written down readily. A large number of examples based on our formula (5.2) have been enunciated* by Dougall and by Ramanujan. The cases which will be dealt with here depend on the known formulae by which ${}_3F_2$ can be summed in finite terms. One of these, Saalschütz's, has been quoted already (8.1 above). The other three, which are closely connected in the theory of the ${}_3F_2$ expansions, are as follows :—

Dixon's Formula (2.2 above).

$$(10.1)\quad {}_3F_2\left[\begin{matrix}x, & y, & z\\ 1+x-y, & 1+x-z\end{matrix}\right]$$

$$=\frac{\Gamma(1+\frac{1}{2}x)\,\Gamma(1+x-y)\,\Gamma(1+x-z)\,\Gamma(1+\frac{1}{2}x-y-z)}{\Gamma(1+x)\,\Gamma(1+\frac{1}{2}x-y)\,\Gamma(1+\frac{1}{2}x-z)\,\Gamma(1+x-y-z)}.$$

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* *Vide* Hardy, *loc. cit.*, 495.

*Watson's Formula.**

$$(10.2)\quad {}_3F_2\left[\begin{matrix} x, & y, & z \\ \frac{1}{2}(x+y+1), & 2z \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+z)\,\Gamma(\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}y)\,\Gamma(\frac{1}{2}-\frac{1}{2}x-\frac{1}{2}y+z)}{\Gamma(\frac{1}{2}+\frac{1}{2}x)\,\Gamma(\frac{1}{2}+\frac{1}{2}y)\,\Gamma(\frac{1}{2}-\frac{1}{2}x+z)\,\Gamma(\frac{1}{2}-\frac{1}{2}y+z)}.$$

and† (subject to the conditions $x+y=1$; $u+v=2z+1$).

$$(10.3)\quad {}_3F_2\left[\begin{matrix} x, & y, & z \\ u, & v \end{matrix}\right] = \frac{\pi\,\Gamma(u)\,\Gamma(v)}{2^{2z-1}\,\Gamma(\frac{1}{2}x+\frac{1}{2}u)\,\Gamma(\frac{1}{2}x+\frac{1}{2}v)\,\Gamma(\frac{1}{2}y+\frac{1}{2}u)\,\Gamma(\frac{1}{2}y+\frac{1}{2}v)}.$$

11.—*Special Cases of* ${}_3F_2(-)$.—Starting from the formula (9.3), we choose the parameters so that Watson's formula, (10.2), may be applicable.

It is found that the conditions are $a=4c-2$; $b=4c-\frac{5}{2}$; and that

$$(11.1)\quad {}_3F_2\left[\begin{matrix} 4c-2, & 4c-\frac{5}{2}, & c\,; \\ & \frac{3}{2}, & 3c-1 \end{matrix}\quad -1\right]$$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(3c-1)}{\Gamma(4c-1)\,\Gamma(\frac{3}{2}-c)}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}, & 4c-\frac{5}{2}, & c \\ & 2c, & 2c-\frac{1}{2} \end{matrix}\right]$$

$$= \frac{\pi\,.\,\sin(c-\frac{3}{4})\pi\,\Gamma(3c-1)}{2^{6c-2}(c-\frac{3}{4})\,\Gamma(\frac{3}{4})\,\Gamma(2c-\frac{3}{4})\,\Gamma(c)}.$$

Formula (10.3) is applicable in two ways, it leads to

$$(11.2)\quad {}_3F_2\left[\begin{matrix} 2c-\frac{1}{2}, & c, & \frac{1}{2}\,; \\ & c+\frac{1}{2}, & 2c \end{matrix}\quad -1\right] = \frac{\pi}{2^{2c-\frac{1}{2}}}\left[\frac{\Gamma(c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{3}{8})\,\Gamma(\frac{1}{2}c+\frac{5}{8})}\right]^2$$

and also to

$$(11.3)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}+x, & \frac{1}{2}-x\,; \\ & 1-x, & 1+x \end{matrix}\quad -1\right]$$

$$= \frac{\pi^2 x}{\sqrt{2}\,.\,\sin\pi x\,\Gamma(\frac{1}{2}x+\frac{5}{8})\,\Gamma(\frac{1}{2}x+\frac{7}{8})\,\Gamma(\frac{5}{8}-\frac{1}{2}x)\,\Gamma(\frac{7}{8}-\frac{1}{2}x)}.$$

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* The special case of (10.2) in which x or y is a negative integer was discovered by Watson. His demonstration is published in *Proc. London Math. Soc.* (2), 23 (1925), xiii–xv (*Records* for November 8, 1923). The more general proposition was established in my paper "A Group of Generalized Hypergeometric Series", etc., *Proc. London Math. Soc.* (2), 23 1925), 113, Formula 31.

† *Loc. cit.*, 114, Formula 34.

12. *Special Cases of* ${}_4F_3(-)$.—Starting from the general formula (3 . 4),

$$(12.1)\quad {}_4F_3\left[\begin{matrix}a, & b, & c, & d, & -1\\ 1+a-b, & 1+a-c, & 1+a-d; & & \end{matrix}\right]$$
$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}\ {}_3F_2\left[\begin{matrix}1+\frac{1}{2}a-b, & c, & d\\ 1+\frac{1}{2}a, & 1+a-b & \end{matrix}\right],$$

we may choose the parameters so as to satisfy the conditions of Saalschütz's formula (8 . 1). To retain symmetry we write

$$a=-2\kappa,\quad b=1-\kappa+n,\quad c=-\kappa+x,\quad d=-\kappa-x,$$

n being a positive integer less than $\kappa+1$, and find

$$(12.2)\quad {}_4F_3\left[\begin{matrix}-2\kappa, & -\kappa+1+n, & -\kappa+x, & -\kappa-x; & -1\\ & -\kappa-n, & 1-\kappa-x, & 1-\kappa+x & \end{matrix}\right]$$
$$= \frac{\Gamma(1-\kappa)\,\Gamma(1+\kappa)\,\Gamma(1+n+x)\,\Gamma(1+n-x)\,\Gamma(1-\kappa+x)\,\Gamma(1-\kappa-x)}{\Gamma(1-2\kappa)\,\Gamma(1-x)\,\Gamma(1+x)\,\Gamma(1+n-\kappa)\,\Gamma(1+n+\kappa)}$$

Watson's formula (10 . 2) in conjunction with (12 . 1) gives two results

$$(12.3)\quad {}_4F_3\left[\begin{matrix}4x+4y, & 6x, & 6y, & \frac{1}{2}+x+y; & -1\\ & 1-2x+4y, & 1+4x-2y, & \frac{1}{2}+3(x+y) & \end{matrix}\right]$$
$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(1+4x-2y)\,\Gamma(1-2x+4y)\,\Gamma(\frac{1}{2}+3x+3y)\,\Gamma(1+x+y)}{\Gamma(\frac{1}{2}+3x)\,\Gamma(\frac{1}{2}+3y)\,\Gamma(1+x-2y)\,\Gamma(1-2x+y)\,\Gamma(1+4x+4y)}$$

and

$$(12.4)\quad {}_4F_3\left[\begin{matrix}a, & d, & \frac{1}{2}a+d, & \frac{1}{2}+\frac{1}{2}a-\frac{1}{2}d; & -1\\ 1+a-d, & 1+\frac{1}{2}a-d, & \frac{1}{2}+\frac{1}{2}a+\frac{1}{2}d & & \end{matrix}\right]$$
$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{1}{4}a-\frac{1}{2}d)\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}d)\,\Gamma(1+\frac{1}{2}a-\frac{1}{2}d)}{2^{\frac{1}{2}a+d}\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{4}a)\,\Gamma(\frac{1}{2}+\frac{1}{4}a+\frac{1}{2}d)\,\Gamma(1+\frac{1}{4}a-d)}.$$

Similarly two results are obtained by the use of (10 . 3) with (12 . 1). One is identical with (12 . 4), but the other is new :

$$(12.5)\quad {}_4F_3\left[\begin{matrix}a, & 1-\frac{1}{2}a, & b, & 1-b; & -1\\ & \frac{3}{2}a, & 1+a-b, & a+b & \end{matrix}\right]$$
$$= \frac{\pi^{\frac{1}{2}}\,\Gamma(\frac{3}{2}a)\,\Gamma(a+b)\,\Gamma(1+a-b)}{2^{3a-1}\,\Gamma(a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{4}a+\frac{1}{2}b)\,\Gamma(1+\frac{1}{4}a-\frac{1}{2}b)}\cdot$$
$$\times\,\Gamma(\tfrac{3}{4}a+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}+\tfrac{3}{4}a-\tfrac{1}{2}b)$$

13. *A Special Case of* ${}_5F_4(-)$.—As an example of the reduction of ${}_5F_4(-)$ we use (9 . 7) and then (10 . 3).

$$(13.1)\quad {}_5F_4\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ \frac{1}{2}-\frac{1}{2}a,\ d, & 1-d; \\ \frac{1}{2}a,\ \frac{1}{2}+\frac{3}{2}a,\ 1+a-d,\ a+d & \end{matrix}\ -1\right]$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{3}{2}a)}{\Gamma(1+a)\,\Gamma(a)}\ {}_3F_2\left[\begin{matrix} a,\ \frac{1}{2}-\frac{1}{2}a, & \frac{1}{2}+\frac{1}{2}a, \\ 1+a-d,\ a+d & \end{matrix}\right]$$

$$= \frac{\pi^{\frac{1}{2}}\,\Gamma(\frac{1}{2}+\frac{3}{2}a)\,\Gamma(1+a-d)\,\Gamma(a+d)}{2^{3a-2}\,\Gamma(1+a)\,\Gamma(\frac{1}{2}a)\,\Gamma(\frac{3}{4}+\frac{1}{4}a-\frac{1}{2}d)\,\Gamma(\frac{3}{4}+\frac{3}{4}a-\frac{1}{2}d)} \cdot \\ \times\Gamma(\tfrac{1}{4}+\tfrac{1}{4}a+\tfrac{1}{2}d)\,\Gamma(\tfrac{1}{4}+\tfrac{3}{4}a+\tfrac{1}{2}d)$$

14. *A Special Case of* ${}_6F_5(-)$.—In like fashion the "strong" formula (6 . 3) gives us, with the help of (10 . 3),

$$(14.1)\quad {}_6F_5\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ \frac{1}{2}+x,\ & \frac{1}{2}-x,\ & \frac{1}{2}+y,\ & \frac{1}{2}-y\,; \\ \frac{1}{2}a,\ & \frac{1}{2}+a-x,\ & \frac{1}{2}+a+x,\ & \frac{1}{2}+a-y,\ \frac{1}{2}+a+y & \end{matrix}\ -1\right]$$

$$= \frac{\pi\,\Gamma(\frac{1}{2}+a+x)\,\Gamma(\frac{1}{2}+a-x)\,\Gamma(\frac{1}{2}+a+y)\,\Gamma(\frac{1}{2}+a-y)}{2^{2a-1}\,\Gamma(a)\,\Gamma(1+a)\,\Gamma\left(\frac{1+a+x+y}{2}\right)\,\Gamma\left(\frac{1+a+x-y}{2}\right)} \cdot \\ \times\Gamma\left(\frac{1+a-x+y}{2}\right)\,\Gamma\left(\frac{1+a-x-y}{2}\right)$$

It will be noticed that (13 . 1) can be derived from (14 . 1) by putting $x = \frac{1}{2}a$, $y = d-\frac{1}{2}$.

15. *Some General Transformations.*—To revert to the general theory, it may be noticed that the formula (6 . 3) can be used to establish relations between series of type ${}_3F_2(+1)$.

From (6 . 3) we learn, by exchange of c, d with e, f, that

$$(15.1)\quad \frac{{}_3F_2\left[\begin{matrix} 1+a-c-d,\ e,\ f \\ 1+a-c, \quad 1+a-d \end{matrix}\right]}{\Gamma(1+a-e-f)\,\Gamma(1+a-c)\,\Gamma(1+a-d)}$$

$$= \frac{{}_3F_2\left[\begin{matrix} 1+a-e-f,\ c,\ d \\ 1+a-e, \quad 1+a-f \end{matrix}\right]}{\Gamma(1+a-c-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}$$

and, by change of parameters, that

$$(15.2)\qquad {}_3F_2\left[\begin{matrix}a,\ b,\ c\\ e,\ f\end{matrix}\right]=\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\,F\left[\begin{matrix}e-a,\ f-a,\ s\\ s+b,\ s+c\end{matrix}\right],$$

where s is equal to $e+f-a-b-c$. This formula (due to Thomae) can be proved in various ways.

To adapt the notation of my recent paper,* we write

$$(15.3)\qquad Fp(0\,;1)=\frac{\Gamma(\delta_{01}+1)}{\Gamma\beta_{20}\,\Gamma\beta_{30}\,\Gamma\beta_{40}\,\Gamma\beta_{50}}\,{}_6F_5\left[\begin{matrix}\delta_{01},\ 1+\frac{1}{2}\delta_{01},\ \alpha_{345},\ \alpha_{452},\ \alpha_{523},\ \alpha_{234}\\ \frac{1}{2}\delta_{01},\ \beta_{20},\ \beta_{30},\ \beta_{40},\ \beta_{50}\end{matrix}\,;\,-1\right].$$

where

$$(15.31)\qquad \delta_{01}=\tfrac{1}{2}-2r_0-r_1,$$

$$(15.32)\qquad \alpha_{345}=\tfrac{1}{2}+r_3+r_4+r_5,$$

and

$$(15.33)\qquad \beta_{20}=1+r_2-r_0,$$

$\sum_0^5 r$ being zero, whilst

$$(15.34)\qquad Fp(0\,;\,4,\,5)=\frac{1}{\Gamma\alpha_{123}\,\Gamma\beta_{40}\,\Gamma\beta_{50}}\,{}_3F_2\left[\begin{matrix}\alpha_{145},\ \alpha_{245},\ \alpha_{345}\\ \beta_{40},\ \beta_{50}\end{matrix}\right].$$

In this notation (6.3) takes the form

$$(15.35)\qquad Fp(0\,;\,1)=Fp(0\,;\,4,\,5),$$

and considerations of symmetry show not only that the 10 functions $Fp(0\,;\,u,\,v)$ are equal, but also that the 5 functions $Fp(0\,;\,m)$ are equal. For many purposes the 15 functions are equivalent, and each may be written $Fp(0)$. The relations which have previously been established between $Fp(0)$, $Fp(1)$, $Fp(2)$, etc., may be regarded as relations between well-poised series. Moreover, if $Fn(0\,;\,1)$ be defined by changing the signs of all the r's involved in $Fp(0\,;\,1)$, we shall be able to write $Fn(0\,;1)=Fn(0\,;4,5)$, and utilize the known relations between the

* F. J. W. Whipple, "A Group of Generalized Hypergeometric Series", etc., *Proc. London Math. Soc.* (2), 23 (1925), 104-114.

functions Fp and Fn. There must be 60 different well-poised series in the complete group of related functions.

To facilitate translation from the Fp notation to hypergeometric series, the following table has been made. The parameters which occur in (15 . 3) being regarded as identical with those on the left of (6 . 3), all the α's, β's, and δ's are expressed in this table as linear functions of a, c, d, e, and f. For brevity t is written instead of $1+2a-c-d-e-f$.

Expressions for α's, β's, and δ's in terms of a, c, d, e, f.

$(t = 1+2a-c-d-e-f)$.

$\alpha_{012} = 1-c,$	$\alpha_{345} = c,$	$\alpha_{023} = e+f-a,$	$\alpha_{145} = 1+a-e-f,$
$\alpha_{013} = 1-d,$	$\alpha_{245} = d,$	$\alpha_{024} = d+f-a,$	$\alpha_{135} = 1+a-d-f,$
$\alpha_{014} = 1-e,$	$\alpha_{235} = e,$	$\alpha_{025} = d+e-a,$	$\alpha_{134} = 1+a-d-e,$
$\alpha_{015} = 1-f,$	$\alpha_{234} = f,$	$\alpha_{034} = c+f-a,$	$\alpha_{125} = 1+a-c-f,$
		$\alpha_{035} = c+e-a,$	$\alpha_{124} = 1+a-c-e,$
		$\alpha_{045} = c+d-a,$	$\alpha_{123} = 1+a-c-d.$
$\beta_{01} = 1-t,$	$\beta_{10} = 1+t,$		
$\beta_{02} = 1-a+c,$	$\beta_{20} = 1+a-c,$	$\beta_{12} = 1+t-a+c,$	$\beta_{21} = d+e+f-a,$
$\beta_{03} = 1-a+d,$	$\beta_{30} = 1+a-d,$	$\beta_{13} = 1+t-a+d,$	$\beta_{31} = c+e+f-a,$
$\beta_{04} = 1-a+e,$	$\beta_{40} = 1+a-e,$	$\beta_{14} = 1+t-a+e,$	$\beta_{41} = c+d+j-a,$
$\beta_{05} = 1-a+f,$	$\beta_{50} = 1+a-f,$	$\beta_{15} = 1+t-a+f,$	$\beta_{51} = c+d+e-a,$
$\beta_{23} = 1+d-c,$	$\beta_{32} = 1+c-d,$	$\beta_{34} = 1+e-d,$	$\beta_{43} = 1+d-e,$
$\beta_{24} = 1+e-c,$	$\beta_{42} = 1+c-e,$	$\beta_{35} = 1+f-d,$	$\beta_{53} = 1+d-f,$
$\beta_{25} = 1+f-c,$	$\beta_{52} = 1+c-f,$	$\beta_{45} = 1+f-e,$	$\beta_{54} = 1+e-f.$
$\delta_{01} = a,$	$\delta_{10} = a-t,$		
$\delta_{02} = t+c,$	$\delta_{20} = t-a+2c,$	$\delta_{12} = c-t,$	$\delta_{21} = 2c-a,$
$\delta_{03} = t+d,$	$\delta_{30} = t-a+2d,$	$\delta_{13} = d-t,$	$\delta_{31} = 2d-a,$
$\delta_{04} = t+e,$	$\delta_{40} = t-a+2e,$	$\delta_{14} = e-t,$	$\delta_{41} = 2e-a,$
$\delta_{05} = t+f,$	$\delta_{50} = t-a+2f,$	$\delta_{15} = f-t,$	$\delta_{51} = 2f-a,$
$\delta_{23} = 1+c-e-f,$	$\delta_{32} = 1+d-e-f,$	$\delta_{34} = 1+d-c-f,$	$\delta_{43} = 1+e-c-f,$
$\delta_{24} = 1+c-d-f,$	$\delta_{42} = 1+e-d-f,$	$\delta_{35} = 1+d-c-e,$	$\delta_{53} = 1+f-c-e,$
$\delta_{25} = 1+c-d-e,$	$\delta_{52} = 1+f-d-e,$	$\delta_{45} = 1+e-c-d,$	$\delta_{54} = 1+f-c-d.$

It has been mentioned that the five functions $Fp(0\,;\,n)$ are equal. We may therefore write

$$Fp(0\,;\,1) = Fp(0\,;\,2), \tag{15 . 4}$$

and by the use of the table find that

$$\frac{\Gamma(1+a)}{\Gamma(1+a-c)}\,{}_6F_5\left[\begin{matrix}a, & 1+\frac{1}{2}a, & c, & d, & e, & f; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\;\;-1\right]$$

$$= \frac{\Gamma(1+t+c)}{\Gamma(1+t)}\,{}_6F_5\left[\begin{matrix}t+c, & 1+\frac{1}{2}(t+c), & c, & 1+a-e-f, & 1+a-d-f, & 1+a-d-e; \\ & \frac{1}{2}(t+c), & 1+t, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\;\;-1\right]. \tag{15.41}$$

Again, the known relation*

(15.5)

$$\frac{\sin\pi\beta_{45}}{\Gamma\alpha_{012}\,\Gamma\alpha_{013}\,\Gamma\alpha_{023}}\,Fp(0)+\frac{\sin\pi\beta_{50}}{\Gamma\alpha_{124}\,\Gamma\alpha_{134}\,\Gamma\alpha_{234}}\,Fp(4)+\frac{\sin\pi\beta_{04}}{\Gamma\alpha_{125}\,\Gamma\alpha_{135}\,\Gamma\alpha_{235}}\,Fp(5)=0,$$

becomes in terms of well-poised series

$$(15.51)\quad \frac{\Gamma(1+a)\,\Gamma(e-a)\,\Gamma(f-a)}{\Gamma(1+a-c)\,\Gamma(1+a-d)}\,{}_6F_5\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ c,\ d,\ e,\ f\\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f\end{matrix};\ -1\right]$$

$$+\frac{\Gamma(1+2e-a)\,\Gamma(a-e)\,\Gamma(f-e)}{\Gamma(1+e-c)\,\Gamma(1+e-d)}\,{}_6F_5\left[\begin{matrix}2e-a,\ 1+\frac{1}{2}(2e-a),\ e+f-a,\ d+e-a,\ c+e-a,\ e\\ \frac{1}{2}(2e-a),\ 1+e-f,\ 1+e-d,\ 1+e-d,\ 1+e-c,\ 1+e-a\end{matrix};\ -1\right]$$

$$+\frac{\Gamma(1+2f-a)\,\Gamma(a-f)\,\Gamma(e-f)}{\Gamma(1+f-c)\,\Gamma(1+f-d)}\,{}_6F_5\left[\begin{matrix}2f-a,\ 1+\frac{1}{2}(2f-a),\ e+f-a,\ d+f-a,\ c+f-a,\ f\\ \frac{1}{2}(2f-a),\ 1+f-e,\ 1+f-d,\ 1+f-c,\ 1+f-a\end{matrix};\ -1\right]$$

$$=0.$$

In (15.5) $Fp(0)$ stands for any of the expressions $Fp(0;\,u,\,v)$ or for any of the expressions $Fp(0;\,m)$. The particular form $Fp(0;\,1)$ has been selected. Similarly $Fp(4;\,1)$ has been selected as the form of $Fp(4)$, and $Fp(5;\,1)$ as the form of $Fp(5)$. The three series in (15.51) have the same "convergency indicator" $4+3a-2(c+d+e+f)$.

It is not necessary to develop the analogous formulae involving Fn's, but it is convenient to have the definition written down, viz.:

$$(15.6)\quad Fn(0;\,1)\equiv\frac{\Gamma(2-\delta_{01})}{\Gamma\beta_{02}\,\Gamma\beta_{03}\,\Gamma\beta_{04}\,\Gamma\beta_{05}}\times F\left[\begin{matrix}1-\delta_{01},\ 1+\frac{1}{2}(1-\delta_{01}),\ \alpha_{012},\ \alpha_{013},\ \alpha_{014},\ \alpha_{015}\\ \frac{1}{2}(1-\delta_{01}),\ \beta_{02},\ \beta_{03},\ \beta_{04},\ \beta_{05}\end{matrix};\ -1\right].$$

The particular case in which the r's of (15.31), (15.32), (15.33) are such that

$$(15.71)\qquad r_0+r_3=r_1+r_4=r_2+r_5=0$$

is of interest as showing how a well-poised series can be summed in terms of gamma-functions. In this case we know† that

$$(15.72)\quad Fp(0;\,1)=\frac{2^{2r_0}\pi}{\Gamma(\frac{1}{2}-r_0)\,\Gamma(1-\frac{1}{2}\alpha_{012})\,\Gamma(1-\frac{1}{2}\alpha_{024})\,\Gamma(1-\frac{1}{2}\alpha_{015})\,\Gamma(1-\frac{1}{2}\alpha_{045})},$$

* *Loc. cit.*, Equation (12).

† *Loc. cit.*, Equation (28).

and it follows that

$$(15.73)\quad {}_6F_5\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ \frac{1}{2}+a-e+y,\ 1+a-2e,\ e,\ \frac{1}{2}+a-e-y; \\ \frac{1}{2}a,\ \frac{1}{2}+e-y,\ 2e,\ 1+a-e,\ \frac{1}{2}+e+y \end{matrix}\ -1\right]$$

$$= \frac{\pi^{\frac{1}{2}}\,\Gamma(e+\frac{1}{2})\,\Gamma(1+a-e)\,\Gamma(\frac{1}{2}+e-y)\,\Gamma(\frac{1}{2}+e+y)}{\Gamma(a+1)\,\Gamma(\frac{3}{4}+\frac{1}{2}a-\frac{1}{2}e+\frac{1}{2}y)\,\Gamma(\frac{3}{4}+\frac{1}{2}a-\frac{1}{2}e-\frac{1}{2}y)\,\Gamma(\frac{1}{4}-\frac{1}{2}a+\frac{3}{2}e+\frac{1}{2}y)\,\Gamma(\frac{1}{4}-\frac{1}{2}a+\frac{3}{2}e-\frac{1}{2}y)}$$

The example which has been given already, as (14 . 1), corresponds with the conditions

$$(15.8)\qquad r_0+r_1 = r_2+r_3 = r_4+r_5 = 0.$$

16. *A Generalization of Saalschütz's Formula.*—The equation (7 . 7) was obtained as a means of reducing the well-poised series ${}_7F_6$ to the form ${}_4F_3$. It may be used, however, for the opposite purpose. Making the substitutions

$$t = 1+a-c-d,\quad x = e,\quad y = f,\quad z = g,$$

$$u = e+f+g-a,\quad v = 1+a-c,\quad w = 1+a-d,$$

we find

$$a = v+w-t-1 = x+y+z-u,$$

$$c = w-t,\quad d = v-t,\quad e = x,\quad f = y,\quad g = z,$$

and (7 . 7) takes the form

$$(16.1)\quad {}_4F_3\left[\begin{matrix} t,\ x,\ y,\ z \\ u,\ v,\ w \end{matrix}\right]$$

$$= \frac{\Gamma(v+w-t)\,\Gamma(1+x-u)\,\Gamma(1+y-u)\,\Gamma(1+z-u)}{\Gamma(1+y+z-u)\,\Gamma(1+z+x-u)\,\Gamma(1+x+y-u)\,\Gamma(1-u)}$$

$$\times {}_7F_6\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ w-t,\ v-t,\ x,\ y,\ z \\ \frac{1}{2}a,\ v,\ w,\ 1+y+z-u,\ 1+z+x-u,\ 1+x+y-u \end{matrix}\right],$$

where $a = x+y+z-u$, and the parameters of ${}_4F_3$ are such that $u+v+w-t-x-y-z = 1$, whilst one of the four t, x, y, z is a negative integer. This result is a generalization of Saalschütz's formula (8 . 1 above), as is seen at once on writing $v = t$.

It is clear that from (16 . 1) relations between groups of ${}_4F_3$'s or groups of well-poised ${}_7F_6$'s could be worked out. That task has not yet been undertaken.

Note added June 6, 1925. Since this paper was set up in type, a proof of 7 . 7 by Dougall's method has been obtained. It is hoped that this proof will be published shortly.

A CONTRIBUTION TO THE THEORY OF BOUNDED MATRICES AND QUADRATIC FORMS WITH AN INFINITY OF VARIABLES

By JAMES HYSLOP.

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Introductory.

The Bounded Quadratic Form, with an infinite number of variables, was introduced by Hilbert in connexion with the solution of integral equations. The theory connected with it is a generalization of the theory of infinite determinants, first used by Hill,* and rigorously discussed by von Koch† and others. The literature of the subject is already considerable. In connexion with the present paper the following works have been extensively used. They are quoted in the text by the names of the authors alone.

Hilbert: *Grundzüge einer allgemeinen Theorie der Integralgleichungen* (Leipzig, 1912).

Hellinger and Toeplitz: *Math. Annalen*, 69 (1910), 289–330.

Hellinger: *Journal für Math.*, 136 (1909), 210-271.

Hahn: *Monatshefte für Math. u. Phys.*, 23 (1912), 161–224.

Schmidt: *Rend. di Palermo* (1908), 53–77, or, better, the account of Schmidt's methods in—

Kowalewski: *Einführung in die Determinantentheorie* (Leipzig, 1909), 407–455.

F. Riesz: *Systèmes d'Equations Linéaires* (Borel Tract, Paris, 1913).

Weyl: *Math. Annalen*, 66 (1909), 273-324 (or *Göttingen Thesis*).

Plancherel: *Rend. di Palermo*, 30 (1910), 289–335.

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* Cf. *Acta Math.*, 8 (1886), 1.

† Cf. *Comptes Rendus* (Congrès Stockholm, 1909).

The main object of the paper is to prove the analogue of a well known theorem of Weierstrass.*

This theorem may be stated thus :—

If $A(x,x)$, $B(x,x)$ denote two quadratic forms

$$\sum_{r,s=1}^{\infty} a_{rs}x_r x_s, \qquad \sum_{r,s=1}^{\infty} b_{rs}x_r x_s,$$

which are bounded in Hilbert's sense of the term, and if two members of the family $(\rho A+\sigma B)(x, x)$ are definite, and one of these is also complete, then we may write

$$A(x, x) = \sum_r \lambda_r (y^{(r)}, x)^2 + \sum_l \sum_{m,n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x)$$

$$B(x, x) = \sum_r \mu_r (y^{(r)}, x)^2 + \sum_l \sum_{m,n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \mu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x)$$

where $\phi_{lm}(g)$, $[m = 1, 2, \ldots]$ form, for each value of l, a complete set of orthogonal functions in the interval (α_l, β_l).

The numbers λ_r, μ_r and the functions $\lambda_l(g)$, $\mu_l(g)$, are bounded, and the functions have also certain monotonic properties.

The vectors $(y^{(r)})$, $(y^{(lm)})$, arranged in a simple sequence, are vectors of a complete bounded matrix.

It will be seen that Weierstrass's theorem regarding the simultaneous reduction of two quadratic forms to the sums of squares is a particular case of the above.

Part I of the paper contains a recapitulation of the symbols employed, and of some of the fundamental properties. The notation is essentially that of Riesz. References are given, wherever it seems desirable, to proofs of statements made. In addition to these, Hellinger and Toeplitz may be consulted with advantage.

In Part II is introduced a short treatment of the bounded linear substitution, by consideration of its characteristic properties. This aspect has already found a place in the theory of finite matrices, but, so far as the writer is aware, it is here applied for the first time to infinite matrices. A theorem is proved which appears to have wide application, viz. :—

If B, C be two bounded matrices, a necessary and sufficient condition for the existence of a matrix A whose bound does not exceed M, and

* *Monatsberichte der Berliner Akademie* (1868); cf. Kowalewski, Kap. 13, especially § 115, p. 272 *et sqq.*

such that $AB = C$, is that, for every vector (x), $|C(x)| \leqslant M|B(x)|$. Here, as throughout the paper, $|C(x)|$ denotes the "length" of the vector $C(x)$.

In Part III use is made of the theorem of Part II to prove an expansion theorem for the bounded bilinear form analogous to Schmidt's expansion of the real completely continuous form.*

Part IV is a digression which further illustrates the value of the well known expansion of the quadratic or Hermitian form in obtaining other expansion theorems. The case of the unitary form is treated, and the theorem regarding orthogonal forms is proved, whose analogue in the finite case exhibits the orthogonal transformation as the result of simple mutually orthogonal rotations and reflections.

Part V contains the theorem mentioned above, together with a discussion of the properties of the expansion, and of the uniqueness of its essential features.

Part VI is an attempt to indicate briefly the applications of the theorem to physics, and in particular to dynamical systems where Lagrange's equations are applicable, but where the configuration requires for its expression an infinite set of coordinates. A short discussion is given of Wirtinger's problem of the infinitely long stretched string.

I desire to express my indebtedness and thanks to Prof. E. W. Hobson, who suggested to me the subject of the simultaneous reduction of positive quadratic forms as a topic of research, and has acted as my supervisor of studies in the University of Cambridge during the course of the work.

I. PRELIMINARY DEFINITIONS, NOTATION, AND FUNDAMENTAL PROPERTIES.

§ 1. VECTORS.

Vector: Any system of numbers, real or complex, $x_1, x_2, \ldots, ad\ infinitum$ has been called by Schmidt† a Vector, if the sum $\sum_{r=1}^{\infty} |x_r|^2$ converges.

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* This expansion is given in functional form in Schmidt, *Math. Annalen*, 63 (1907), 464, § 16.

† Kowalewski, p. 408, also § 179, p. 453.

Components, Length: The quantities $x_1, x_2, \ldots$ are the components of the vector; its length is the positive value of $\sqrt{\left\{\sum_{r=1}^{\infty} |x_r|^2\right\}}$.

Notation: Vectors are denoted by bracketed italics as (x), (y), ..., their lengths by $|x|$, $|y|$, ..., and their components by $x_1, x_2, \ldots$; $y_1, y_2, \ldots$; ... respectively.

Zero and Unit Vectors: The zero vector (0) has all its components zero. A unit vector is any vector of unit length.

Notation: Throughout the paper barred letters denote complexes conjugate to the corresponding unbarred letters. Thus $\bar{x}_r$ denotes the complex conjugate to x_r, $(\bar{x})$ denotes the vector whose components are the complexes conjugate to the components of (x), and so on.

Greek small letters, unless otherwise defined, denote constants. $\lambda(x)$ denotes the vector having components $\lambda x_1, \lambda x_2, \ldots$.

Sum, Product of Vectors: The sum of the vectors $\lambda(x)$, $\mu(y)$ is the vector $\lambda(x)+\mu(y)$ also written $(\lambda x+\mu y)$, whose components are $\lambda x_1+\mu y_1$, $\lambda x_2+\mu y_2, \ldots$.

The product of (x), (y) is the sum of the absolutely convergent† series $\sum_{r=1}^{\infty} x_r y_r$, and is written (x, y).

Orthogonal, Normal: (x), (y) *are orthogonal* if $(x, y) = 0$, *normal* if $(x, \bar{y}) = (\bar{x}, y) = 0$. For real vectors the properties are obviously coincident.

§ 2. Bounded Matrices.

A matrix is a doubly infinite array of real or complex numbers or "elements."

$$\left.\begin{matrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix}\right\}$$

Bounded: a matrix is bounded* if a fixed constant M^* exists, such that, for every pair of unit vectors (x), (y), and for every positive integral value of n,

$$\left| \sum_{r,s=1}^{n} a_{rs} x_s y_r \right| \leqslant M^*. \qquad (1)$$

* This is Hilbert's criterion, cf. Riesz, 78, 85.

† Crelle. 1910. 140. p. 293

This condition is equivalent to the following,* which, however, are not independent :—

(i) a_{1s}, a_{2s}, a_{3s}, ... are, for every value of s, components of a vector. These vectors I call the "Vectors of the Matrix," and denote by

$$(a^{(s)}) \quad [s = 1, 2, \ldots.]$$

(ii) If $\sum_{s=1}^{\infty} a_{rs}x_s = z_r$, then for every vector (x), and every positive integral value of r, z_r exists as a finite number; and z_1, z_2, ... are components of a vector (z) such that

$$|z| \leqslant M^*|x|.$$

(iii) For every pair of unit vectors (x), (y) the Pringsheim sum $\sum_{r,s=1}^{\infty} a_{rs}y_r x_s$ exists, and does not exceed M^* in absolute value.

Bound: If M be the least possible value of M^* which satisfies relation (1), M is the *bound* of the matrix.

§ 3. SUBSTITUTION, BILINEAR FORM.

Corresponding to the matrix, condition (ii), together with considerations of symmetry, enables us to define the two *linear substitutions*.

$$\sum_{s=1}^{\infty} a_{rs}x_s = z_r \quad [r = 1, 2, \ldots] \tag{2}$$

$$\sum_{r=1}^{\infty} a_{rs}y_r = w_s \quad [s = 1, 2, \ldots]. \tag{3}$$

Bound: The *bound* of the substitution (2) is the upper bound of $|z|$ when (x) is restricted to unit vector values. The bounds of the substitutions (2), (3) are both equal to M, the bound of the matrix.† In the general case we have $|z| \leqslant M|x|$, $|w| \leqslant M|y|$.

The double sum of (iii) is a *bilinear form*, and exists for *all* vectors (x), (y). Clearly also

$$\left|\sum_{r,s=1}^{\infty} a_{rs}x_s y_r\right| \leqslant M|x||y|.$$

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* Riesz, *loc. cit.*

† *Ibid.*

Notation: We denote the sum

$$\sum_{r,s=1}^{\infty} a_{rs}x_s y_r \text{ by } A(x, y), \qquad \sum_{r,s=1}^{n} a_{rs}x_s y_r \text{ by } A_n(x, y).$$

The substitutions (2), (3) are written

$$A(x) = (z), \quad A'(y) = (w).$$

Matrices are denoted by A, B, ... Such letters will also be used in discussing general relations of matrices, forms, or substitutions, where these are the same for all.

Transponent: The substitution (3) is the *transponent* of (2). The matrix A' associated with it is formed from A by an interchange of rows and columns. In general dashed letters will be used to denote transponents.

§ 4. Series of Vectors; Convergence.

If we have a series of vectors $(x^{(1)})$, $(x^{(2)})$, $(x^{(3)})$, ... and a vector (x) exists such that

$$\lim_{n\to\infty} x_r^{(n)} = x_r \quad [r = 1, 2, \ldots],$$

then the series is said to *converge*,* or *converge weakly* to (x), and we write:

$$\lim_{n\to\infty} (x^{(n)}) = (x).$$

If, however, $\lim_{n\to\infty} |x - x^{(n)}| = 0$, the series *converges strongly*, and we write $\underset{n\to\infty}{\mathrm{LIM}}\,(x^{(n)}) = (x)$.

A series which converges strongly to (x) must also converge weakly to (x). And clearly limit vectors of a series are uniquely defined by the series.

§ 5. Special Kinds of Matrices.

The Identity: The identical substitution E has a matrix whose elements are given by:

$$e_{rs} = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{if } r \neq s. \end{cases}$$

Its vectors are the fundamental unit vectors $(e^{(s)})$.

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* It should be noted that in Kowalewski, *convergence* always means *strong convergence*.

Symmetric, Hermitian: A matrix A is *symmetric* if $A' = A$, *Hermitian if* $\bar{A}' = A$.

Quadratic and Hermitian Forms: If A is real and symmetric, the double sum $\sum_{r,s=1}^{\infty} a_{rs}x_r x_s$ is a *quadratic form*, and is denoted by $A(x, x)$. In the theory of quadratic forms, the vector (x) is always understood to be real. If A is Hermitian, $\sum_{r,s=1}^{\infty} a_{rs}\bar{x}_r x_s$ is a *Hermitian form*, written $A(x, \bar{x})$.

Definite: A quadratic or Hermitian form is *definite* if its sign is the same for every vector (x). Forms capable of zero values are not excluded.

Completeness: A set of vectors is *complete* if no non-zero vector exists orthogonal to all of them. The set of vectors $(e^{(s)})$ is complete. A matrix, substitution, or form A is complete if the vectors of A' form a complete set. The completeness of sets of vectors has been studied by Schmidt.

Sum: The *sum* of two matrices is formed in the same way as the sum of two vectors, viz. by adding the corresponding elements. Thus $\lambda A + \mu B$ denotes the matrix whose (r, s)-th element is $\lambda a_{rs} + \mu b_{rs}$.

Characteristic Vector: A *vector invariant* or *characteristic vector* of a substitution A is a vector (x) such that

$$A(x) = \lambda(x).$$

Characteristic vectors of A will only exist for such values of λ as make $(A - \lambda E)$ incomplete. These values of λ are the *characteristic numbers* of the substitution.

NOTE.—Some authors use the term *characteristic numbers* to denote the reciprocals of the quantities so defined above.

II. FUNDAMENTAL PROPERTIES OF THE SUBSTITUTION.

§ 6. THE CHARACTERISTIC PROPERTIES.

The substitution $A(x) = (y)$ may be considered as a transformation which transforms the vector (x) into another vector (y). Considered in

this way it has the following fundamental properties :—

I. To every vector (x) corresponds one and only one vector (y).

II. The transformation is additive, *i.e.* if to $(x^{(1)})$, $(x^{(2)})$ correspond $(y^{(1)})$, $(y^{(2)})$, then to $(\lambda x^{(1)}+\mu x^{(2)})$ corresponds $(\lambda y^{(1)}+\mu y^{(2)})$.

III. It is bounded, *i.e.* $|y| \leqslant M|x|$.

These properties follow at once from definition. They are also characteristic* of the bounded substitution. That is to say, any vector transformation having these properties can be exhibited as a substitution. The proof is simple :—

Denote the given transformation, having properties I, II, III, by $T(x) = (y)$. By I, the transforms $(a^{(s)})$ of the vectors $(e^{(s)})$ $[s = 1, 2, \ldots]$ exist, and are uniquely defined.

Let A be the matrix whose vectors are $(a^{(s)})$. Then we have to show that A is bounded, and that $A(x) = T(x)$ for every vector (x).

Now, by definition of A, $A(x) = T(x)$ if $(x) = (e^{(s)})$. Hence, by property II, $A(x) = T(x)$ if (x) has only a finite number of non-zero components. (1)

It follows readily, by use of property III, that $A_n(x, y) \leqslant M|x||y|$ for every finite value of n. Thus A is bounded.

Now take any vector (x), and let $(x^{(n)})$, be the vector whose first n components are equal to those of (x), the remaining components of $(x^{(n)})$ being zero.

Then $$\lim_{n\to\infty} |x - x^{(n)}| = 0.$$

Therefore, by III, $$\lim_{n\to\infty} |T(x - x^{(n)})| = 0.$$

Whence, using II, $$\operatorname{LIM}_{n\to\infty} T(x^{(n)}) = T(x).$$

Similarly, since A is bounded,

$$\operatorname{LIM}_{n\to\infty} A(x^{(n)}) = A(x).$$

But, by equation (1), $$A(x^{(n)}) = T(x^{(n)}).$$

Therefore $$A(x) = T(x).$$

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* This fact has been assumed implicitly by several authors (cf. Plancherel or Riesz, 168–9), but so far as the writer is aware, it is here explicitly stated for the first time, and the theory built on it is, to the best of his knowledge, entirely new.

§ 7. THE NORMAL SUBSTITUTION.

A normal substitution A is one such that, if $A(x) = (y)$, $|x| = |y|$. It will be at once clear, by considering the transforms of $(e^{(s)})$, $(e^{(r)}+\lambda e^{(s)})$, that a necessary and sufficient condition that A be normal is that the vectors $(a^{(s)})$ be all unit vectors and normal to one another.

If A, A' are both normal, A is *unitary*. It will be clear from Schmidt's work* that, if A is normal and A' complete, A is unitary.

§ 8. PRODUCTS.

Let $(y) = A(x)$, $(z) = B(y)$, where A, B are any two bounded substitutions. Then the operation which transforms (x) into (z) has the properties of a bounded substitution. On putting $(e^{(1)})$, $(e^{(2)})$, ... in succession for (x) we obtain the following equations for its matrix C :

$$c_{rs} = \sum_{t=1}^{\infty} b_{rt} a_{ts}, \tag{2}$$

C is the *product* of B, A, and we may write symbolically

$$BA = C.$$

The form of the matrix C, taken along with the definition of C, also shows that

$$\sum_{t=1}^{\infty} \left\{ \sum_{s=1}^{\infty} b_{rt} a_{ts} x_s \right\} = \sum_{s=1}^{\infty} \left\{ \sum_{t=1}^{\infty} b_{rt} a_{ts} x_s \right\},$$

or symbolically

$$B \,.\, \{A(x)\} = (BA) \,.\, (x),$$

§ 9. A FUNDAMENTAL THEOREM.

Suppose we regard B, C as known, then it is clear that, in order that a bounded substitution A exist satisfying the equation $AB = C$, it is *necessary* that, for every vector (x),

$$|C(x)| \leqslant M\,|B(x)|$$

where M is a constant independent of (x); in fact, M is the bound of A. This condition is also sufficient for the existence of A, as the following proof shows :—

Case (i).—Suppose B' is complete. Normalize† the system of vectors $(b^{(1)})$, $(b^{(2)})$, ..., and let $(v^{(1)})$, $(v^{(2)})$, ... be the equivalent normal set, which is, of course, complete.

* Kowalewski, 421 and 453, §179.

† Kowalewski, 423, §168; 418, §166; 453, §179.

If (y) be any vector, and we write

$$(y^{(n)}) = \sum_{r=1}^{n} (y, \bar{v}^{(r)})(v^{(r)}),$$

then

$$\text{LIM}_{n\to\infty} (y^{(n)}) = (y). \tag{3}$$

Clearly $(y^{(n)})$ is of the form $B(x^{(n)})$, where $(x^{(n)})$ is a vector with only a finite number of non-zero components.

Let $$C(x^{(n)}) = (z^{(n)}).$$

Then $$B(x^{(m)}-x^{(n)}) = (y^{(m)}-y^{(n)}); \quad C(x^{(m)}-x^{(n)}) = (z^{(m)}-z^{(n)}).$$

Hence, by hypothesis and equation (3),

$$\lim_{m,\, n\to\infty} |z^{(m)}-z^{(n)}| = 0.$$

It follows* that $\text{LIM}_{n\to\infty} (z^{(n)})$ exists as a vector (z), say, and

$$|z| = \lim_{n\to\infty} |z^{(n)}| \leqslant \lim_{n\to\infty} |y^{(n)}| . M \leqslant M|y|.$$

Consider now the transformation which replaces (y) by (z). It is readily seen to have the characteristic properties of a bounded substitution, and must therefore be such. Let its matrix be A. We now show $AB = C$.

Suppose (x) is a vector, only the first m of whose components may be non-zero, and let $B(x) = (y)$. Then, on carrying out the above process to determine $A(y)$, it will be seen that, if $n \geqslant m$, $(y^{(n)}) = (y)$, *i.e.* $B(x-x^{(n)}) = 0$. Hence, by hypothesis, $C(x-x^{(n)}) = 0$. Therefore

$$C(x) = C(x^{(n)}) \ [n \geqslant m] = \lim C(x^{(n)}) = \lim (z^{(n)}) = (z) = A(y).$$

In particular if $(x) = (e^{(s)})$, we find $A(b^{(s)}) = (c^{(s)})$ $[s = 1, 2, \ldots]$.

Hence, by equation (2) above, $AB = C$.

Case (ii).—If B' is not complete, let $(u^{(1)})$, $(u^{(2)})$, ... be a set of unit vectors normal to one another and to all the vectors of the substitution B, and forming along with these vectors a complete set.† The substitution $\mathcal{B}$ is defined to be the substitution whose vectors in order are

$$(b^{(1)}), (u^{(1)}), (b^{(2)}), (u^{(2)}), \ldots.$$

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* Cf. Kowalewski, 413.

† Kowalewski, 426, § 169.

$\mathcal{B}$ is bounded, and has an upper bound not exceeding that of B by more than unity. And $\mathcal{B}'$ is complete.

Let $\mathcal{C}$ be the substitution, also bounded, whose vectors are

$$(c^{(1)}),\ (0),\ (c^{(2)}),\ (0),\ \ldots.$$

Then it is readily seen that, for every vector (x),

$$|\mathcal{C}(x)| \leqslant M|\mathcal{B}(x)|.$$

Therefore we may write $A\mathcal{B} = \mathcal{C}$ as in Case (i). But the form of the matrices shows that, this being so, we must also have $AB = C$.

We have thus proved the following theorem.

THEOREM 2.1.—*If we have two bounded substitutions B, C, then in order that a substitution A exist, having bound not exceeding M, and such that $AB = C$, it is necessary and sufficient that, for every vector (x),*

$$|C(x)| \leqslant M|B(x)|.$$

Two remarks may be added concerning the theorem :—

It is sufficient for the truth of the result that $|C(x)| \leqslant M|B(x)|$ for all vectors (x) which have only a finite number of non-zero components. In fact, this is all that has been assumed in the course of the proof. The condition originally given is also readily found to be equivalent to this.

Again, if B' is complete, A is unique. The proof of this fact follows immediately from the equation $B'A' = C'$.

Several particular cases of the theorem are of special interest or importance, and may be given as corollaries.

§ 10. SIMPLE CONSEQUENCES OF THE THEOREM.

If we put $C = E$, we obtain a theorem previously proved by several authors.*

COR. 2.11.—*In order that, for a given B, a substitution A exist such that $AB = E$, it is necessary and sufficient that a constant M exist, such that, for every vector (x),*

$$|x| \leqslant M|B(x)|.$$

* Riesz, 85, 86 *et seq.*

Reciprocal: If substitutions A, B are such that

$$AB = BA = E,$$

then AB are *reciprocal substitutions*. The reciprocal of A, if such exist, is denoted by A^{-1}. If A is unitary, $\bar{A}'$ is its reciprocal.

COR. 2.12.—*A necessary and sufficient condition that B have a reciprocal is that constants M_1, M_2 exist such that, for every (x),*

$$|x| \leqslant M_1 |B(x)|, \quad |x| \leqslant M_2 |B'(x)|.$$

The proof is immediate from Cor. 2.11. In fact, if the conditions are satisfied, we may write:

$$A_1 B = E = A_2' B' = B A_2.$$

Hence $$A_1 = A_1(BA_2) = (A_1 B) A_2 = A_2 = B^{-1}.$$

COR. 2.13.—*If A', B' are complete, and if for every (x)*

$$|A(x)| = |B(x)|,$$

then there is a Unitary Substitution U such that $UA = B$.

We may write at once

$$U_1 A = B, \quad U_2 B = A,$$

where U_1, U_2 have bounds not exceeding unity.

Hence $$U_2 U_1 A = A.$$

Therefore, since A' is complete, $U_2 U_1 = E$.

Therefore $$|x| \equiv |U_2 U_1(x)| \leqslant |U_1(x)| \leqslant |x|.$$

This can only be so if $|U_1(x)| = |x|$ for every (x), *i.e.* U_1 is normal. But $B' = A' U_1'$ and B' is complete. Therefore so is U_1', and U_1 is unitary.

The completeness of A', B' is essential to the truth of the corollary.

COR. 2.14.—*In order that, for every vector (x), a vector (y) exist such that $A(y) = C(x)$, it is sufficient that, for every vector (z),*

$$|C'(z)| \leqslant M |A'(z)|.$$

In fact, if the condition is fulfilled, we may write $B'A' = C'$ or $AB = C$. The result follows with $(y) = B(x)$. If we further require that $|y| \leqslant M|x|$, the condition is necessary as well as sufficient. In fact, if we choose that particular value of (y), in case several exist,

which is normal to all vectors transformed by A into the zero vector, then (y) so chosen is unique for given (x), and the transformation from (x) to (y) is a bounded substitution. Details of the proof are left to the reader.

COR. 2.15.*—*In order that, given a vector (c), a vector (x) exist such that $A(x) = (c)$, it is sufficient that a constant M exist, such that, for every vector (z), $|(z, c)| \leqslant M|A'(z)|$.*

Let C be the matrix with vectors (c), (0), (0); .... Then the condition of the corollary may be written $|C'(z)| \leqslant M|A'(z)|$. Hence we may write $B'A' = C'$, or $AB = C$; and so $A(b^{(1)}) = (c)$. And

$$|b^{(1)}| = |B(e^{(1)})| \leqslant M,$$

since the bound of B' and so also of B cannot exceed M.

The condition is also necessary. In fact, if $A(x) = (c)$, then

$$|(z, c)| = |A(x, z)| = |\{x, A'(z)\}| \leqslant |x||A'(z)|.$$

§ 11. RIESZ'S THEOREM.

Riesz (p. 61) has proved the following theorem :—

In order that a finite or enumerable system of equations,

$$\sum_{k=1}^{\infty} a_{ik}x_k = c_i \quad [i = 1, 2, \ldots],$$

whose coefficients a_{ik} are such that the series

$$\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} \quad [i = 1, 2, \ldots; \quad p > 1],$$

converge, should admit at least one solution (x) such that

$$\sum_{k=1}^{\infty} |x_k|^p \leqslant M^p,$$

it is necessary and sufficient that the inequality

$$\left|\sum_{i=1}^{n} \mu_i c_i\right| \leqslant M\left[\sum_{k=1}^{\infty}\left|\sum_{i=1}^{n} \mu_i a_{ik}\right|^{p/(p-1)}\right]^{(p-1)/p}$$

should hold for all values of n, and all sets or real or complex constants μ_i.

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* Cor. 2.15 was added at the suggestion of the London Mathematical Society's Referee who noted the connection with Riesz's theorem; to him are also due some improvements in the proof of Theorem 2.1.

Necessity of Riesz's condition is very simply proved, but Riesz's proof of its sufficiency is rather involved. For the case $p = 2$, with which only we are here concerned, the sufficiency of the condition is a direct consequence of Cor. 2.15.

In fact, if we restrict (z)—as we have seen to be legitimate*—to vector values with only a finite number of non-zero components, the corollary gives precisely Riesz's condition for the case when A is bounded and $\sum_{i=1}^{\infty} |c_i|^2$ converges. In the general case, $\sum_{k=1}^{\infty} |a_{ik}|^2$ is known to converge for all values of i; hence we may introduce a set of real positive constants $\lambda_1, \lambda_2, \ldots$, such that the matrix of the substitution

$$\sum_{k=1}^{\infty} a_{ik} x_k \lambda_i = c_i \lambda_i$$

is bounded, and at the same time $\sum_{i=1}^{\infty} |\lambda_i c_i|^2$ converges. The condition, proved sufficient for this modified case, clearly retains its validity when the λ's are removed. Thus Riesz's condition is proved sufficient with all generality for the case $p = 2$.

III. AN EXPANSION THEOREM.

§ 12. The Quadratic Form.

Hilbert, Hellinger, and Hahn have shown that a real bounded quadratic form $A(x, x)$ can be expanded thus†

$$A(x, x) = \sum_r \lambda_r \left[\sum_{s=1}^{\infty} p_{rs} x_s\right]^2 + \sum_l \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg \left[\sum_{s=1}^{\infty} q_{lms} x_s\right]\left[\sum_{s=1}^{\infty} q_{lns} x_s\right],$$

$$E(x, x) = \sum_r \left[\sum_{s=1}^{\infty} p_{rs} x_s\right]^2 + \sum_l \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \phi_{lm}(g)\, \phi_{ln}(g)\, dg \left[\sum_{s=1}^{\infty} q_{lms} x_s\right]\left[\sum_{s=1}^{\infty} q_{lns} x_s\right].$$

* See remarks at end of § 9 above.

† The form here given is essentially that of Hahn [206–244, along with 188, § 19, and 205, § 33.]

It is to be understood that the summations with respect to r, l may be over an infinite, finite, or null group of terms. Those with respect to m, n are over an infinite group, unless the l-summation is entirely absent. Further

I. $\phi_{lm}(g)$ $[m = 1, 2, \ldots]$ form, for each value of l, a complete set of real orthogonal functions of the variable g in the real interval (α_l, β_l).

II. λ_r are real numbers, whose arithmetic upper bound does not exceed M, the bound of A. Zero values of λ_r are not excluded.

III. $\lambda_l(g)$ are real strictly increasing functions of the variable g in (α_l, β_l), never exceeding M in absolute value. In order to make the function $\lambda_l(g)$ well defined, even at possible points of discontinuity, we may state $\lambda_l(\alpha_l) = \lambda_l(\alpha_l + 0)$ and $\lambda_l(g) = \lambda_l(g - 0)$ if $g \neq \alpha_l$.

IV. p_{rs}, q_{lms} $[s = 1, 2, \ldots]$ are components of real vectors $(p^{(r)})$, $(q^{(lm)})$ which, arranged in a simple sequence, form a complete set of real unit vectors orthogonal to one another. In other words, we can introduce a real unitary change of variables by which the quadratic form is reduced thus :

$$A(x, x) = \sum_r \lambda_r y_r^2 + \sum_l \left\{ \sum_{m,\, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, y_{lm}\, y_{ln} \right\}.$$

Each summation of this form is absolutely convergent. We shall use the first form of the expansion, writing as usual

$$\sum_{s=1}^{\infty} p_{rs} x_s = (p^{(r)}, x), \qquad \sum_{s=1}^{\infty} q_{lms} x_s = (q^{(lm)}, x).$$

§ 13. THE HERMITIAN FORM.

In exactly the same way as this expansion is obtained,* or by deduction from it,† it may be shown that the Hermitian form has an expansion thus :

$$H(x, \bar{x}) = \sum_r \lambda_r (p^{(r)}, x)(\bar{p}^{(r)}, \bar{x}) + \sum_l \sum_{m,\, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg \,.\, (q^{(lm)}, x)(\bar{q}^{(ln)}, \bar{x}).$$

Where now the vectors $(p^{(r)})$, (q^{lm}) are not necessarily real, and are *normal* to one another, the change of variables is thus still unitary.

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* See remark in Riesz near foot of 122.

† Cf. Hilbert, 162 *et seq.*

If (x), (y) be any two vectors, then we show

$$H(x,y) = \sum_r \lambda_r (p^{(r)}, x)(\bar{p}^{(r)}, y) + \sum_l \sum_{m,n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\phi_{ln}(g)\, dg\, .(q^{(lm)}, x)(\bar{q}^{(ln)}, y).$$

Denote the series on the right of the equation by $\Sigma(x,y)$, using this symbol only where the series is known to converge.

Then we have $$H(x, \bar{x}) = \Sigma(x, \bar{x}).$$

Putting $(x+y)$, $(x-y)$ in turn for (x), and subtracting the resulting equations, we have

$$H(x, \bar{y}) + H(y, \bar{x}) = \Sigma(y, \bar{x}) + \Sigma(x, \bar{y}),$$

the sum on the right being *term by term*, the series converging on account of the manner of its derivation.

Put $(y) = (y^{(1)})$, a real vector, then

$$H(x, y^{(1)}) + H(y^{(1)}, \bar{x}) = \Sigma(x, y^{(1)}) + \Sigma(y^{(1)}, \bar{x}).$$

Put $(y) = i(y^{(1)})$, we have

$$H(x, y^{(1)}) - H(y^{(1)}, \bar{x}) = \Sigma(x, y^{(1)}) - \Sigma(y^{(1)}, \bar{x}).$$

Whence, on adding,

$$H(x, y^{(1)}) = \Sigma(x, y^{(1)}).$$

The expansion is thus proved for (y) real; it follows at once that it is correct if (y) is purely imaginary. And hence, on addition, it holds for every value of (y).

§ 14. The General Case (Preliminary).

Square Root: Suppose for every (x), $H(x, \bar{x})$ is positive. Then the constants λ_r, and the functions $\lambda_l(g)$ are positive. Let $K(x,y)$ be obtained from $H(x,y)$ by writing $\lambda_r^{\frac{1}{2}}$, $[\lambda_l(g)]^{\frac{1}{2}}$ in place of λ_r, $\lambda_l(g)$ respectively. Then the matrix K has the properties:*

(1) $K(x, \bar{x})$ is a positive bounded Hermitian form.

(2) $KK \equiv K^2 = H$.

* This fact will be fairly obvious to those acquainted with the literature of the subject. Reference may be made to Riesz, 129 *seq.*, in particular equation (14), 137, where similar results are given for quadratic forms, the Stieltjes integral there appearing being later proved equivalent to the expansion given above.

We may call K the *square root* of H.

We proceed to prove the following theorem.

THEOREM 3.1.—*If A, A' be both complete, then we may write $A = UK$, where U is unitary, K a positive Hermitian.*

Let $H = \bar{A}' A$; then H is Hermitian, and

$$H(x, \bar{x}) = \left(A(x), \bar{A}(\bar{x})\right) = |A(x)|^2 > 0.$$

Let K be the square root of H.

Then $$H(x, \bar{x}) = \left(K(x), \bar{K}(\bar{x})\right) = |K(x)|^2.$$

Therefore $|A(x)| = |K(x)|$, and that for every (x).

But A is complete, therefore so is K, K'.

Hence, by Cor. 2.13, $A = UK$.

§ 15. THE GENERAL CASE (COMPLETE FORMS).

We now introduce the expansion above, writing throughout μ_r for $\lambda_r^{\frac{1}{2}}$, $\mu_l(g)$ for $[\lambda_l(g)]^{\frac{1}{2}}$.

Then

$$A(x, y) = UK(x, y) = K\left(x, U'(y)\right)$$

$$= \sum_r \mu_r(p^{(r)}, x)\left(\bar{p}^{(r)}, U'(y)\right) + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} \mu_l(g)\{\text{etc.}\}\, dg(q^{(lm)}, x)\left(\bar{q}^{(ln)}, U'(y)\right)$$

$$= \sum_r \mu_r(p^{(r)}, x)\left(U(\bar{p}^{(r)}), y\right) + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} \mu_l(g)\{\text{etc.}\}\, dg(q^{(lm)}, x)\left(U(\bar{q}^{(ln)}), y\right).$$

Since now $(p^{(r)})$, $(q^{(ln)})$ form a complete set of unit normal vectors, and U is unitary, $U(\bar{p}^{(r)})$, $U(\bar{q}^{(lm)})$ form another such set, and we have the following theorem :—

THEOREM 3.2.—

$$A(x, y) = \sum_r \mu_r(p^{(r)}, x)(u^{(r)}, y) + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} \mu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg(q^{(lm)}, x)(v^{(ln)}, y),$$

where $(p^{(r)})$, $(q^{(ln)})$ form a complete set of mutually normal unit vectors, and $(u^{(l)})$, $(v^{(ln)})$ form another such set, provided A, A' are complete.

§ 16. THE GENERAL CASE (INCOMPLETE FORMS).

As may be readily anticipated, the stipulation that A, A' be complete may be dispensed with, but in the more general case the sets of normal vectors appearing in the expansion are not necessarily complete sets, or one set may be complete and the other not so. This we proceed to show:

Case (i).—Suppose A contains only zero elements in the first row, but that the matrix A_1, obtained by omitting this row, is complete along with its transponent. Then $A(x, y)$ does not actually contain y_1; but, if it be considered as a bilinear form in the other variables only, its matrix, now A_1, is complete both ways, and the above expansion is therefore valid. The products $(u^{(r)}, y)$, $(v^{(lm)}, y)$, which appear in the expansion, contain naturally no term in y_1. For example,

$$(u^{(r)}, y) = \sum_{s=1}^{\infty} u_s^{(r)} y_{s+1}.$$

This may, however, with a slight change of notation, be written

$$\sum_{s=1}^{\infty} u_s^{(r)} y_s,$$

where $u_1^{(r)} = 0$. Inserting this expression for $(u^{(r)}, y)$, and the similar form of $(v^{(lm)}, y)$, in the expansion, we obtain an expansion of $A(x, y)$ which takes account of y_1. The vectors $(u^{(r)})$, $(v^{(lm)})$ have now zero first components, and so are all normal to $(e^{(1)})$, but they form along with $(e^{(1)})$ a complete set of mutually normal unit vectors.

Case (ii).—Similar reasoning holds if any finite or infinite set of variables $x_{r_1}, x_{r_2}, \ldots; y_{s_1}, y_{s_2}, \ldots$ be absent from $A(x, y)$, provided the matrix of $A(x, y)$, considered as a bilinear form only in the variables actually appearing, is complete both ways.

Case (iii).—Consider now the general case when A is incomplete. Normalize the vectors $(\bar{a}'^{(s)})$ $[s = 1, 2, \ldots]$, and let $(w^{(s)})$ $[s = 1, 2, \ldots]$ be the equivalent normal system. Let $(z^{(r)})$ $[r = 1, 2, \ldots]$ be a set of vectors which together with $(w^{(s)})$ form a complete set of mutually normal unit vectors. We assume that both sets $\{(w^{(s)})\}$, $\{(z^{(r)})\}$ are infinite. The modifications of the proof, necessary if this is not so, are obvious.

Let W be the unitary matrix having as its vectors

$$(w^{(1)}),\ (z^{(1)}),\ (w^{(2)}),\ (z^{(2)}),\ \ldots.$$

Since $(z^{(r)})$ is normal to $(w^{(s)})$, and therefore orthogonal to $(a'^{(s)})$ $[s = 1, 2, \ldots]$, the matrix AW has only zero elements in the $2r$-th column $[r = 1, 2, \ldots]$. But $\bar{W}'\,W\,(x) \equiv (x)$. Hence, if (x) is a vector whose $(2r-1)$-th components are not all zero, then $W(x)$ is not, for all values of s, orthogonal to $(\bar{w}^{(s)})$.

Therefore $W(x)$ is not, for all values of s, orthogonal to $(a'^{(s)})$, *i.e.* $AW(x) \neq (0)$.

Hence the matrix, formed from AW by omission of the columns of zeros, is complete. $W'A'$ is, of course, complete or not, according as A' is complete or not. If A' is not complete, we define, exactly as above, a unitary substitution Z such that $W'A'Z$ has zero elements in its $2r$-th rows* $[r = 1, 2, \ldots]$, and also in certain columns, but the matrix obtained by omitting these rows and columns is complete along with its transponent. $Z'AW(x, y)$ may be expanded as in Case (ii), and we obtain the expansion of $A(x, y)$ from

$$A(x, y) = \bar{Z}Z'AW\bar{W}'(x, y) = Z'AW\{\bar{W}'(x), \bar{Z}'(y)\}.$$

The expansion has the form previously stated.

In the particular case when all the elements of A are real, $H(x, x)$, $K(x, x)$ are quadratic forms. All the vectors appearing in the expansion are real, each of the two sets consisting of mutually orthogonal unit vectors.

§ 17. Applications of the Expansion.

The expansion just obtained gives us a further means of determining whether or not the equation

$$A(x) = (c)$$

is satisfied by any vector (x) when A, (c) are given. It also provides the solution where such exists.

The method will be obvious to those acquainted with the literature of the subject,† and need not be detailed here. By means of the Riesz-

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* This follows from the fact that $W'A'$ has zeros in these rows.

† Cf. Picard, *Rend. di Palermo*, 29 (1910), 79.

Fischer theorem we are enabled to apply the result to the solution of integral equations of the first kind, viz. :

$$\int_a^b K(x, y)\, \phi(y)\, dy = f(x),$$

where $\phi(y)$ is the unknown. For the restrictions to be placed on the various functions involved, the reader may be referred to Weyl's thesis or to Plancherel's paper.

If A is Hermitian, including the real symmetric case, the *spectrum* of A consists of the set of points λ_r, and the set of values of $\lambda_l(g)$, or more precisely the values of the functions $\lambda_l(g+0)$ for $g < \beta_l$ and of $\lambda_l(g-0)$ for $g > \alpha_l$.

IV. THE UNITARY FORM.

§ 18. Expansion of a Unitary Form.

As an interesting example of an expansion of another nature which is derivable from that of the Hermitian, we give the following case of Unitary Bilinear Form. The results of this section will not be required later.

Let U be the unitary matrix.

Then $\bar{U}'$ is also unitary, and $\bar{U}' = U^{-1}$.

Further $(U+\bar{U}')$, $i(U-\bar{U}')$ are Hermitian.

Let (z) be a characteristic vector of $(U+\bar{U}')$, so that

$$(U+\bar{U}')(z) = \mu(z). \tag{1}$$

Since $(U+\bar{U}')$ is Hermitian, μ is real.

Now $$|U(z)| = |\bar{U}'(z)| = |z|.$$

Hence $|(U+\bar{U}')(z)| \leqslant 2|z|$, equality being possible only if

$$U(z) = \bar{U}'(z). \tag{2}$$

Hence $|\mu| \leqslant 2$, and we may write $\mu = 2\cos\theta$, where θ is a real angle, and $-\pi < \theta \leqslant \pi$.

If $\theta = 0, \pi$, then equation (1) gives

$$(U+\bar{U}')(z) = \pm 2(z).$$

Hence, by (2), $U(z) = \bar{U}'(z) = \pm(z)$

and $i(U-\bar{U}')(z) = (0)$ } (3)

If $\theta \neq 0, \pi$, then $(U+\bar{U}')(z) = 2\cos\theta(z)$.

Therefore $(U^2+E)(z) - 2\cos\theta U(z) = (0)$,

i.e. $(U-e^{i\theta}E)(U-e^{-i\theta}E)(z) = (0)$. (4)

Let $(U-e^{-i\theta}E)(z) \equiv (e^{i\theta}-e^{-i\theta})(z^{(1)})$

and $(z) = (z^{(1)})+(z^{(2)})$. (5)

Then by (4) $(U-e^{i\theta}E)(z^{(1)}) = (0)$,

i.e. $U(z^{(1)}) = e^{i\theta}(z^{(1)})$, (6)

therefore $(U-e^{-i\theta}E)(z^{(1)}) = (e^{i\theta}-e^{-i\theta})(z^{(1)}) = (U-e^{-i\theta})(z)$,

therefore $(U-e^{-i\theta}E)(z^{(2)}) = (0)$,

*i.e.** $U(z^{(2)}) = e^{-i\theta}(z^{(2)})$. (7)

It follows that $\bar{U}'(z^{(1)}) = e^{-i\theta}(z^{(1)})$; $\bar{U}'(z^{(2)}) = e^{i\theta}(z^{(2)})$,

since $\bar{U}' = U^{-1}$.

Hence $i(U-\bar{U}')(z^{(1)}) = -2\sin\theta(z^{(1)})$,

$i(U-\bar{U}')(z^{(2)}) = 2\sin\theta(z^{(2)})$.

Therefore $(z^{(1)})$, $(z^{(2)})$ are normal to one another.

Also $(U+\bar{U}')(z^{(1)}) = 2\cos\theta(z^{(1)})$,

$(U+\bar{U}')(z^{(2)}) = 2\cos\theta(z^{(2)})$.

That is to say: Every characteristic vector of $(U+\bar{U}')$ is *either* a characteristic vector of $i(U-\bar{U}')$, *or* it is the sum of two mutually normal vectors, each of which is characteristic both of $(U+\bar{U}')$ and of $i(U-\bar{U}')$. In exactly the same way it may be shown that every characteristic vector of $i(U-\bar{U}')$ is the sum of at most two mutually normal characteristic vectors of $(U+\bar{U}')$.

It follows that, corresponding to each characteristic number $2\cos\theta$, we can arrange a set of unit mutually normal characteristic vectors of

* The result that from (4) follows (5), (6), (7) may be extended to any number of distinct "factors" of form $(U-\lambda E)$. It will recall similar results in other branches of mathematics, *e.g.*, the solution of linear differential equations with constant coefficients by symbolic factor methods.

$(U+\bar{U}')$, each vector of which is also characteristic of $i(U-\bar{U}')$, the corresponding characteristic number being by suitable choice of θ expressible as $-2\sin\theta$. Moreover, if we form such sets of vectors for every "characteristic" value of θ, we shall have a set of vectors to which no characteristic vector of $(U+\bar{U}')$ or of $i(U-\bar{U}')$ is normal.

These characteristic vectors are, of course, precisely the vectors which appear in the expansions of the Hermitians. The part of the expansion involving integrals may be similarly discussed by means of "characteristic differentials,"* and we are able finally to write :

$$(U+\bar{U}')(x, y) = \sum_r 2\cos\theta_r (p^{(r)}, x)(\bar{p}^{(r)}, y) + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} [2\cos\theta_l(g)]\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (q^{(lm)}, x)(\bar{q}^{(ln)}, y),$$

$$(U-\bar{U}')(x, y) = \sum_r 2i\sin\theta_r (p^{(r)}, x)(\bar{p}^{(r)}, y) + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} [2i\sin\theta_l(g)]\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (q^{(lm)}, x)(\bar{q}^{(ln)}, y),$$

where θ_r, $\theta_l(g)$ are real, and the other symbols have their usual meaning.

Hence, on adding

$$U(x, y) = \sum_r e^{i\theta_r}(p^{(r)}, x)(\bar{p}^{(r)}, y) + \sum_{l, m, n} \int e^{i\theta_l(g)}\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (q^{(lm)}, x)(\bar{q}^{(ln)}, y),$$

this is the expansion desired. It exhibits all the singularities of $(U-\lambda E)^{-1}$ considered as a function of λ. These consist of the points $e^{i\theta_r}$ and the values of the functions $e^{i\theta_l(g)}$ together with their limit points. As was to be expected, these values lie wholly on the unit circle.

§ 19. The Real Orthogonal Case.

If U is real orthogonal, the expansion is capable of further transformation :—

(1) If $\theta_r = 0, \pi$, then $(p^{(r)})$ may be chosen real.

(2) If $\theta_r \neq 0, \pi$, then $(p^{(r)})$ is complex, say,

$$(p^{(r)}) = \frac{1}{\sqrt{2}}[(u^{(r)}) + i(v^{(r)})],$$

where $(u^{(r)})$, $(v^{(r)})$ are real.

* See Hellinger or Hahn.

Now if $$U(\bar{p}^{(r)}) = e^{i\theta_r}(\bar{p}^{(r)}),$$

$$U(p^{(r)}) = e^{-i\theta_r}(p^{(r)}).$$

Hence we may group the terms of the expansion corresponding to conjugate values of the characteristic numbers thus :

$$e^{i\theta_r}(p^{(r)}, x)(\bar{p}^{(r)}, y)+e^{-i\theta_r}(\bar{p}^{(r)}, x)(p^{(r)}, y)$$

$$= \tfrac{1}{2}(\cos\theta_r+i\sin\theta_r)[(u^{(r)}, x)+i(v^{(r)}, x)][(u^{(r)}, y)-i(v^{(r)}, y)]$$

$$+\tfrac{1}{2}(\cos\theta_r-i\sin\theta_r)[(u^{(r)}, x)-i(v^{(r)}, x)][(u^{(r)}, y)+i(v^{(r)}, y)]$$

$$= \cos\theta_r[(u^{(r)}, x)(u^{(r)}, y)+(v^{(r)}, x)(v^{(r)}, y)]$$

$$+\sin\theta_r[(u^{(r)}, x)(v^{(r)}, y)-(v^{(r)}, x)(u^{(r)}, y)].$$

But $(p^{(r)})$, $(\bar{p}^{(r)})$ are unit vectors, and normal to one another since

$$i(U-\bar{U}')(p^{(r)}) = 2\sin\theta(p^{(r)}) \quad \text{and} \quad i(U-\bar{U}')(\bar{p}^{(r)}) = -2\sin\theta(\bar{p}^{(r)}).$$

Hence $$\tfrac{1}{2}[(u^{(r)}, u^{(r)})+(v^{(r)}, v^{(r)})] = 1,$$

$$\tfrac{1}{2}[(u^{(r)}, u^{(r)})-(v^{(r)}, v^{(r)})] = \tfrac{1}{2}[(u^{(r)}, v^{(r)})+(v^{(r)}, u^{(r)})] = 0.$$

Hence $(u^{(r)})$, $(v^{(r)})$ are mutually orthogonal, real, unit vectors, which are also readily seen to be orthogonal to $(u^{(s)})$, $(v^{(s)})$ if $r \neq s$.

Dealing similarly with the integral part, we see that we may write finally, as the general form of expansion of the Real Orthogonal Bilinear Form :

$$U(x, y) = \sum_r (w^{(r)}, x)(w^{(r)}, y)-\sum_r (z^{(r)}, x)(z^{(r)}, y)$$

$$+\sum_r \{\cos\theta_r[(u^{(r)}, x)(u^{(r)}, y)+(v^{(r)}, x)(v^{(r)}, y)]$$

$$+\sin\theta_r[(u^{(r)}, x)(v^{(r)}, y)-(v^{(r)}, x)(u^{(r)}, y)]\}$$

$$+\sum_l \Big[\sum_{m,\,n=1}^{\infty} \Big\{\int_{\alpha_l}^{\beta_l} \cos\theta_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg\,[(u^{(lm)}, x)(u^{(ln)}, y)+(v^{(lm)}, x)(v^{(ln)}, y)]$$

$$+\int_{\alpha_l}^{\beta_l} \sin\theta_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg\,[(u^{(lm)}, x)(v^{(ln)}, y)-(v^{(lm)}, x)(u^{(ln)}, y)]\Big\}\Big],$$

where it is to be understood that any of the first four summation signs may extend over a finite or enumerable sequence of terms, or the corresponding group may be entirely absent.

In this expansion, the vectors $(u^{(r)})$, $(v^{(r)})$, $(w^{(r)})$, $(z^{(r)})$, $(u^{(lm)})$, $(v^{(lm)})$ taken together in a single series form a complete set of real, mutually

orthogonal, unit vectors. The expansion gives the analogue, for the infinite case, of the theorem in space of a finite number of dimensions, that any real linear affine transformation which conserves distances can be exhibited as the resultant of a number of reflections involving one coordinate, and of mutually orthogonal rotations involving each two coordinates.

V. THE SIMULTANEOUS REDUCTION OF POSITIVE QUADRATIC FORMS.

§ 20. Simultaneous Expansion.

In the following section we deal with Real Quadratic Forms. The argument is equally applicable to the Hermitian forms.

Let $S(x,x)$ be a real positive quadratic form. And let L be the square root of S. Then

$$S(x,x) = |L(x)|^2.$$

Let $T(x,x)$ be a similar form, K the square root of T. Then

$$T(x,x) = |K(x)|^2.$$

Suppose further that, for every (x),

$$S(x,x) \leqslant M^2 T(x,x),$$

and that T is complete. Then

$$|L(x)| \leqslant M|K(x)|.$$

But this is precisely the condition that a bounded substitution C, in the present case also *real*, exist such that $L = CK = KC'$ by symmetry of L, K.

Therefore
$$S = L^2 = KC'CK,$$
$$= KDK,$$

where D is symmetric, and $D(x,x)$ a positive quadratic form. We may thus write, if $(v) = K(x)$:

$$S(x,\ x) = D(v,\ v).$$

$$T(x,\ x) = E(v,\ v).$$

And, in the usual way,

$$\left.\begin{aligned} D(v, v) &= \sum_r \nu_r (p^{(r)}, v)^2 + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} \nu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (q^{(lm)}, v)(q^{(ln)}, v) \\ E(v, v) &= \sum_r (p^{(r)}, v)^2 + \sum_{l, m, n} \int_{\alpha_l}^{\beta_l} \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (q^{(lm)}, v)(q^{(ln)}, v) \end{aligned}\right\} \quad (1)$$

This, on inserting the value of (v), gives the simultaneous reduction of $S(x, x)$, $T(x, x)$. We have, in fact, if $K(p^{(r)}) = (y^{(r)})$, $K(q^{(lm)}) = (y^{(lm)})$,

$$\left.\begin{aligned} S(x, x) &= \sum_r \nu_r (y^{(r)}, x)^2 + \sum_l \sum_{m, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \nu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x) \\ T(x, x) &= \sum_r (y^{(r)}, x)^2 + \sum_l \sum_{m, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x) \\ &= \sum_r (y^{(r)}, x)^2 + \sum_{l, m} (y^{(lm)}, x)^2 \end{aligned}\right\} \quad (2)$$

Suppose now $A(x, x)$, $B(x, x)$ are any two quadratic forms. Then in order that we may be able to write

$$\left.\begin{aligned} A(x, x) &= \sum_r \lambda_r (y^{(r)}, x)^2 + \sum_l \sum_{m, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x) \\ B(x, x) &= \sum_r \mu_r (y^{(r)}, x)^2 + \sum_l \sum_{m, n=1}^{\infty} \int_{\alpha_l}^{\beta_l} \mu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, (y^{(lm)}, x)(y^{(ln)}, x) \end{aligned}\right\} \quad (3)$$

it is only necessary to show that two independent members of the family $(\rho A + \sigma B)(x, x)$ can be so expressed.

Suppose two members of the family $R(x, x)$, $S(x, x)$ exist, such that $R(x, x) > 0$, $S(x, x) \geqslant 0$ for every vector (x). Then if $T = R + S$, $0 \leqslant S(x, x) < T(x, x)$, and we may expand $S(x, x)$, $T(x, x)$ as above. Under these circumstances the simultaneous expansion of $A(x, x)$, $B(x, x)$ in form (3) is always possible.

§ 21. PROPERTIES OF THE EXPANSION.

We proceed to examine the properties of the expansion, which may be compared with those of the orthogonal expansion of § 7. These properties are :—

I. The functions $\phi_{lm}(g)$, $[m = 1, 2, \ldots]$, form a complete set of orthogonal functions for the interval (α_l, β_l).

II. The functions $\lambda_l(g)$, $\mu_l(g)$, and the numbers $\mu_r\lambda_r$, are bounded.

III. The vectors $(y^{(r)})$, $(y^{(lm)})$, $[r, l, m = 1, 2, \ldots]$, arranged in a simple sequence, are vectors of a complete matrix.

IV. Suppose $(\rho_1 A+\sigma_1 B)(x, x)$, $(\rho_2 A+\sigma_2 B)(x, x)$ are any two members of the family $(\rho A+\sigma B)(x, x)$, and consider the function

$$\theta_l(g) \equiv \frac{\rho_2\lambda_l(g)+\sigma_2\mu_l(g)}{\rho_1\lambda_l(g)+\sigma_1\mu_l(g)}.$$

(1) If $(\rho_1 A+\sigma_1 B)(x, x)$ is definite, then for each value of l, $\theta_l(g)$ is strictly monotonic, that is to say, either it is strictly increasing, or it is strictly decreasing, or it is constant.

(2) If $(\rho_1 A+\sigma_1 B)(x, x)$ is capable of both signs, then we may divide (α_l, β_l) into (at most) two sub-intervals, in each of which $\theta_l(g)$ is strictly monotonic and in the same sense. Further, if $\theta_l(g)$ is not constant, it does not assume the same value for any two different values of g.

Properties I, II are obvious consequences of the mode of derivation of the expansion.

To prove III, we note that the vectors $(y^{(r)})$, $(y^{(lm)})$ are of the form $K(p^{(r)})$, $K(q^{(lm)})$, where K is a complete symmetric matrix, and $(p^{(r)})$, $(q^{(lm)})$, $[r, l, m = 1, 2, \ldots]$, arranged in any simple sequence whatever, are vectors of a real unitary substitution. Thus $(y^{(r)})$, $(y^{(lm)})$, arranged in a simple sequence, are vectors of a matrix of form KU, where U is unitary. This matrix is clearly bounded and complete.

To prove Property IV, we use the following lemma :—

LEMMA 5.11.—*Consider any expansion of form* (3) *which has Properties I, II, III. If* $(\rho_1 A+\sigma_1 B)(x, x)$ *is positive definite, then* $\rho_1\lambda_r+\sigma_1\mu_r \geqslant 0$, *and the set of points in which* $\rho_1\lambda_l(g)+\sigma_1\mu_l(g) < 0$, *is (at most) one of measure zero.*

Write $$(y^{(r)}, x) = \xi_r, \quad (y^{(lm)}, x) = \xi_{lm}.$$

Then ξ_r, ξ_{lm}, arranged in a simple sequence, are components of a vector (ξ); in fact, we may write, in the notation previously used, $(\xi) = U'K(x)$.

$$(\rho_1 A+\sigma_1 B)(x, x)$$
$$= \sum_r(\rho_1\lambda_r+\sigma_1\mu_r)\xi_r^2 + \sum_{l, m, n}\int_\alpha^{\beta_l}[\rho_1\lambda_l(g)+\sigma_1\mu_l(g)]\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg\,\xi_{lm}\,\xi_{ln}$$
$$= H(\xi, \xi), \quad \text{say.}$$

Using Property II, we readily see that H is bounded.

Since KU is complete, we can,* given any vector (ξ), define a series of vectors $(x^{(1)})$, $(x^{(2)})$, ...; such that, if

$$U'K(x^{(n)}) = (\xi^{(n)}),$$

then $$\mathrm{LIM}\,(\xi^{(n)}) = (\xi).$$

Hence

$$\lim_{n\to\infty} (\rho_1 A+\sigma_1 B)(x^{(n)}, x^{(n)}) \equiv \lim_{n\to\infty} H(\xi^{(n)}, \xi^{(n)}) = H(\xi, \xi).$$

Since, for every (x), $\quad (\rho_1 A+\sigma_1 B)(x, x) \geqslant 0.$

Therefore for every vector (ξ),

$$H(\xi, \xi) \geqslant 0. \tag{4}$$

Suppose, if possible, that, for some value of r, $\rho_1\lambda_r+\sigma_1\mu_r < 0$. Let

$$\xi_r = 1; \quad \xi_s = 0 \text{ if } s \neq r; \quad \xi_{lm} = 0.$$

Then $H(\xi, \xi) = \rho_1\lambda_r+\sigma_1\mu_r < 0$, which is impossible by (4). Again suppose $\rho_1\lambda_l(g)+\sigma_1\mu_l(g) < 0$ in D, a set of non-zero measure.

Let $\psi(g) = 1$ in D, and zero outside D. Let

$$\xi_r = 0; \quad \xi_{lm} = \int_{\alpha_l}^{\beta_l} \psi(g)\,\phi_{lm}(g)\,dg; \quad \xi_{im} = 0 \text{ if } i \neq l.$$

Then $$H(\xi, \xi) = \int_{\alpha_l}^{\beta_l} [\rho_1\lambda_l(g)+\sigma_1\mu_l(g)][\psi(g)]^2\,dg\,\dagger$$

$$= \int_D [\rho_1\lambda_l(g)+\sigma_1\mu_l(g)]\,dg < 0,$$

which is impossible.

This completes the proof of the lemma.

Returning to the proof of Property IV, we note that, in the case now considered, $\lambda_l(g)$, $\mu_l(g)$ are both of the form $\gamma+\delta\nu_l(g)$ where γ, δ are real constants, and $\nu_l(g)$ is the function so named in § 20.

We may therefore write :

$$\rho_1\lambda_l(g)+\sigma_1\mu_l(g) = \gamma_1+\delta_1\nu_l(g); \quad \rho_2\lambda_l(g)+\sigma_2\mu_l(g) = \gamma_2+\delta_2\nu_l(g).$$

* See proof of Theorem 2.1, Case (i).

† We here use the theory of complete sets of orthogonal functions, cf. Riesz, p. 152.

In Case (1) it is sufficient to establish the property when $(\rho_1 A+\sigma_1 B)(x, x)$ is positive.

By Lemma 5.11, $\gamma_1+\delta_1\nu_l(g) \geqq 0$ almost everywhere. But $\nu_l(g)$ is strictly increasing, and its values at α_l, β_l are $\nu_l(\alpha_l+0)$, $\nu_l(\beta_l-0)$ respectively. Hence, if $\gamma_1+\delta_1\nu_l(g)$ is negative at any point, it must be so over an entire interval.

We conclude that $\gamma_1+\delta_1\nu_l(g) \geqq 0$ for all values of g. Now

$$\theta_l(g) \equiv \frac{\rho_2\lambda_l(g)+\sigma_2\mu_l(g)}{\rho_1\lambda_l(g)+\sigma_1\mu_l(g)} = \frac{\gamma_2+\delta_2\nu_l(g)}{\gamma_1+\delta_1\nu_l(g)}.$$

Therefore $$\theta_l(g_2)-\theta_l(g_1) = \frac{(\gamma_1\delta_2-\delta_1\gamma_2)\{\nu_l(g_2)-\nu_l(g_1)\}}{\{\gamma_1+\delta_1\nu_l(g_2)\}\{\gamma_1+\delta_1\nu_l(g_1)\}}.$$

Bearing in mind that the denominator of this fraction is positive,* and that $\nu_l(g)$ is strictly increasing, we find that $\theta_l(g)$ is strictly increasing, strictly decreasing, or constant, according as $(\gamma_1\delta_2-\delta_1\gamma_2)$ is positive, negative, or zero.

In Case (2), $\gamma_1+\delta_1\nu_l(g)$ may be capable of both signs. If so, we can divide (α_l, β_l) into two sub-intervals, in each of which $\gamma_1+\delta_1\nu_l(g)$ is of constant sign. In each of these sub-intervals the reasoning of Case (1) is applicable. It is also clear that, if $g_1 \neq g_2$ then $\theta_l(g_1) \neq \theta_l(g_2)$ unless $\gamma_1\delta_2-\delta_1\gamma_2 = 0$.

Thus Property IV is proved.

LEMMA 5.12.—*In order that an expansion of form* (3) *which possesses Properties I, II, III, should also have Property IV, it is sufficient that two forms* $(\rho A+\sigma B)(x, x)$, $(\rho' A+\sigma' B)(x, x)$ *exist such that* $(\rho' A+\sigma' B)(x, x)$ *is definite, and*

$$\theta_l'(g) \equiv \frac{\rho\lambda_l(g)+\sigma\mu_l(g)}{\rho'\lambda_l(g)+\sigma'\mu_l(g)}$$

is strictly increasing and is continuous at α_l, β_l.

It may be noted that, if $\theta_l'(g)$ be monotonic, but not continuous at α_l, β_l, we can alter its values at these points, so as to make it continuous there, without altering the value of the integral in the expansion.

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* $\gamma_1+\delta_1\nu_l(g)$ may be zero at a single point, either α_l or β_l. The case when g_1 or g_2 coincides with this point requires special treatment, but will conform to the rule, if the value of $\theta_l(g)$ be defined by continuity at such a point.

To prove the lemma, we note that the forms $\rho A+\sigma B$, $\rho' A+\sigma' B$ must be independent. Hence we can write :

$$\rho_1\lambda_l(g)+\sigma_1\mu_l(g) = \gamma_1'[\rho'\lambda_l(g)+\sigma'\mu_l(g)]+\delta_1'[\rho\lambda_l(g)+\sigma\mu_l(g)], \text{ etc.}$$

The proof is now practically identical with that given above, $\theta_l'(g)$ being put in place of $\nu_l(g)$.

§ 22. THE PRINCIPAL THEOREM.

We may summarize the main results of §§ 20, 21 in the following theorem :—

THEOREM 5 . 1.—*If two independent members of the family of forms* $(\rho A+\sigma B)(x, x)$ *are definite, and one of these is also complete, then we may write*

$$A(x, x) = \sum_r \lambda_r(y^{(r)}, x)^2+\sum_l\Big[\sum_{m, n=1}^{\infty}\int_{\alpha_l}^{\beta_l}\lambda_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg(y^{(lm)}, x)(y^{(ln)}, x)\Big],$$

$$B(x, x) = \sum_r \mu_r(y^{(r)}, x)^2+\sum_l\Big[\sum_{m, n=1}^{\infty}\int_{\alpha_l}^{\beta_l}\mu_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg(y^{(lm)}, x)(y^{(ln)}, x)\Big],$$

and the expansion has Properties I, II, III, IV given above.

This theorem is the analogue of the theorem for finite quadratic forms, that a reduction of two quadratic forms simultaneously to the sum of squares is possible if *one* member of the family is definite and complete. If, in the finite case, $A(x, x) > 0$ then for sufficiently small values of λ, $(A+\lambda B)(x, x) > 0$. In the infinite case this is not necessarily so, and we must insist that *two* members of the family be definite.

§ 23. THE SPECTRUM OF A WITH REGARD TO B.

The Spectrum : The spectrum of A with respect to B is defined by means of a simultaneous expansion of form (3) having Properties I, II, III, IV. The point spectrum is the set of points $\kappa_r \equiv \lambda_r/\mu_r$, and the continuous spectrum is the set of values of the functions $\lambda_l(g)/\mu_l(g)$, or more precisely the set of values of the pairs of functions $\kappa_l(g+0)$ for $\alpha_l \leqslant g < \beta_l$ and $\kappa_l(g-0)$ for $\alpha_l < g \leqslant \beta_l$, where

$$\kappa_l(g) \equiv \frac{\lambda_l(g)}{\mu_l(g)}.$$

Bounded Forms : If $|A(x, x)| \leqslant M.B(x, x)$ then A is said to be bounded with respect to B.

The ordinary spectrum and ordinary boundedness thus constitute a special case in which $B(x, x)$ is replaced by $E(x, x)$.

Simple and Multiple Spectra : A point spectrum is simple if the values of κ_r are all distinct for different values of r. If for m different values of r, $\kappa_r = k$ (say), then k is a point of multiplicity m in the point spectrum. Clearly m must be finite or enumerably infinite. In the same way if for m different values of l, $\kappa_l(g)$ assumes, over a set of non-zero measure, the same set of values, then this set of values is of multiplicity m in the continuous spectrum.

NOTE.—In accordance with Property IV, $\kappa_l(g)$ does not assume the same set of values more than once for any single value of l.

THEOREM 5 . 2.—*If B is definite, a necessary and sufficient condition that A be bounded with respect to B is that the spectrum of A with respect to B lie in a finite interval.*

Proof : Suppose $|A(x, x)| \leqslant MB(x, x)$. Then $MB(x, x) \pm A(x, x)$ are positive quadratic forms. Hence, by Lemma 5 . 11,

$$M\mu_r \pm \lambda_r \geqslant 0, \quad M\mu_l(g) \pm \lambda_l(g) \geqslant 0,$$

i.e.

$$\frac{|\lambda_r|}{\mu_r} \leqslant M, \quad \frac{|\lambda_l(g)|}{\mu_l(g)} \leqslant M,$$

except possibly at a set of measure zero, which cannot belong to the spectrum. The steps being reversible, the condition is also sufficient.

§ 24. UNIQUENESS OF THE SPECTRUM.

In order that the above definitions of the spectrum and its multiplicities be admissible, it is necessary to show that, if we have a number of expansions of form (3) obtained (say) by different choice of S, T, then the spectrum is uniquely defined, and independent of the particular expansion from which it is obtained. We therefore prove the theorem:

THEOREM 5 . 3.—*If $A(x, x)$, $B(x, x)$ be two quadratic forms which satisfy the hypothesis of Theorem 5 . 1, then the following quantities are the same in all expansions of Form (3) which possess Properties I, II, III, IV, viz.* :

(*a*) *The spectrum of A with respect to B.*

(*b*) *The multiplicities of its parts.*

(*c*) *The sum of the terms in the expansion of either form connected with any point or set of points of the spectrum.*

Now the quantities (a), (b), (c) in the expansion of A, B are derivable from the similar quantities in the expansion of S, T, the notation being the same as in § 20. Hence it is sufficient to prove the theorem for the forms S, T where, it will be remembered,

$$0 \leqslant S(x, x) < T(x, x). \tag{5}$$

Let any expansion, having Properties I, II, III, IV, be

$$\left.\begin{aligned} S(x, x) &= \sum_r h_r (u^{(r)}, x)^2 + \sum_{l, m, n} \int_{\gamma_l}^{\delta_l} h_l(g)\, \psi_{lm}(g)\, \psi_{ln}(g)\, dg\, (u^{(lm)}, x)(u^{(ln)}, x) \\ T(x, x) &= \sum_r k_r (u^{(r)}, x)^2 + \sum_{l, m, n} \int_{\gamma_l}^{\delta_l} k_l(g)\, \psi_{lm}(g)\, \psi_{ln}(g)\, dg\, (u^{(lm)}, x)(u^{(ln)}, x) \end{aligned}\right\} \tag{6}$$

We may outline the remainder of the proof thus :—

(1) We find an expansion of S, T [Equation (9) below] which has the form (2) above, that is to say, the expanded form of T consists of the sum of squares without multiplying factors as k_r and without terms involving integrals. Further, in (9) the quantities (a), (b), (c) are the same as in (6).

(2) We then reverse the process of § 20 by showing that expansion (9) is expressible as an orthogonal expansion of $E(v, v)$, $D(v, v)$, the notation being that of § 20. When these steps have been established, the result follows at once. It is known that, in the orthogonal expansion of $E(v, v)$, $D(v, v)$, the spectrum of D with respect to E, its multiplicities, and the sums of corresponding terms are unique. Since, however, the quantities (a), (b), (c), as derived from expansion (6), are identical with the corresponding quantities in the orthogonal expansion, it follows that (a), (b), (c) are uniquely determinate. We have only now to establish the two steps above :—

By Lemma 5 . 11 and Theorem 5 . 2,

$$0 \leqslant h_r \leqslant k_r, \quad 0 \leqslant h_l(g) \leqslant k_l(g).$$

Suppose for any value of r, $k_r = 0$, then $h_r = 0$, and the corresponding term in the expansion of all members of the family $(\rho + A\sigma B)(x, x)$ vanishes. Such a term may be rejected as inadmissible. Hence we may assume $k_r \neq 0$, and in the same way $k_l(g) \not\equiv 0$.

Write $$(u^{(r)}, x) = \eta_r, \quad (u^{(lm)}, x) = \eta_{lm}.$$

Then, since $(u^{(r)})$, $(u^{(lm)})$ arranged in a simple sequence, are the vectors

of a complete bounded matrix L', say, η_r, η_{lm}, arranged in sequence, are components of a vector (η) and

$$(\eta) = L(x).$$

$$\left.\begin{aligned} T(x, x) &= \sum_r k_r \eta_r^2 + \sum_{l, m, n} \int k_l(g)\, \psi_{lm}(g)\, \psi_{ln}(g)\, dg\, \eta_{lm}\, \eta_{ln} \\ &= G(\eta, \eta), \quad \text{say.} \\ \text{And} \qquad S(x, x) &= \sum_r h_r \eta_r^2 + \sum_{l, m, n} \int h_l(g)\, \psi_{lm}(g)\, \psi_{ln}(g)\, dg\, \eta_{lm}\, \eta_{ln} \\ &= H(\eta, \eta), \quad \text{say.} \end{aligned}\right\} \tag{7}$$

On account of Property II, $G(\eta, \eta)$, $H(\eta, \eta)$ are bounded quadratic forms. Now let

$$\left.\begin{aligned} y_r &= k_r^{\frac{1}{2}} \eta_r \\ y_{lm} &= \sum_{n=1}^{\infty} \int [k_l(g)]^{\frac{1}{2}}\, \psi_{lm}(g)\, \psi_{ln}(g)\, dg\, \eta_{ln} \end{aligned}\right\}. \tag{8}$$

Then the y's arranged in simple sequence are components of a vector (y). In fact, $(y) = J(\eta)$, where J is the square root of G, and is, by its form, symmetric and complete.

$$\left.\begin{aligned} T(x, x) &= G(\eta, \eta) = |J(\eta)|^2 = \sum_r y_r^2 + \sum_{l, m} y_{lm}^2 \\ S(x, x) &= H(\eta, \eta) = \sum_r \frac{h_r}{k_r}\, y_r^2 + \sum_{l, m, n} \int \frac{h_l(g)}{k_l(g)}\, \phi_{lm}(g)\, \psi_{ln}(g)\, dg\, y_{lm}\, y_{ln} \\ &= M(y, y), \quad \text{say.} \end{aligned}\right\} \tag{9}$$

as is readily verified by inserting the value of (y) from (8) in (7).* Since J, L' are complete, so is $L'J$, and $JL(x) = (y)$. By (5), M is a bounded form.

Step (1) is thus established.

Now let K be the square root of T, $(v) = K(x)$. Then

$$T(x, x) = |JL(x)|^2 = |K(x)|^2$$

and K, $L'J$ are complete. Hence, by Cor. 2.13, $JL = OK$ where O is unitary.

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* See also footnote, p. 279, *supra*.

Therefore $(y) = JL(x) = OK(x) = O(v)$.

That is to say, $y_r = (o^{(r)}, v)$, $y_{lm} = (o^{(lm)}, v)$ where $(o^{(r)})$, $(o^{(lm)})$, the vectors of O', form a complete set of mutually orthogonal vectors.

Substituting this value in (9), we find

$$\left.\begin{aligned} T(x, x) = E(v, v) &= \sum_r (o^{(r)}, v)^2 + \sum_{l, m, n} \int \psi_{lm}(g)\psi_{ln}(g)\, dg\, (o^{(lm)}, v)(o^{(ln)}, v) \\ S(x, x) &= \sum_r \frac{h_r}{k_r} (o^{(r)}, v)^2 + \sum_{l, m, n} \int \frac{h_l(g)}{k_l(g)} \psi_{lm}(g)\psi_{ln}(g)\, dg\, (o^{(lm)}, v)(o^{(ln)}, v) \\ &= F(v, v), \quad \text{say,} \end{aligned}\right\} \quad (10)$$

where F is also bounded.

It only remains to be shown that F is identical with D.

Now $F(v, v) = S(x, x) = D(v, v)$; $K(x) = (v)$.

Therefore $KFK = S = KDK$,

or $K(F-D)K \equiv 0$.

But K is complete, hence $(F-D)K \equiv 0$. And so, on taking transponents, $K(F-D) \equiv 0$.

Therefore $F \equiv D$.

Thus the second step is proved, and theorem is completed.

§ 25. CANONICAL FORM.

Canonical Expansion: If $A(x, x)$, $B(x, x)$ be two quadratic forms of a family which contains two independent definite members, of which one is also complete, then any simultaneous expansion of form (3) which possesses Properties I, II, III, IV is defined to be a *canonical expansion* of the forms.

Properties I, IV add to the convenience of the expansion. Property IV also ensures that the terms of the expansion of any two independent forms of the family, connected with multiple parts of the spectrum of one form with respect to the other, are properly separated out.

Properties II, III are essential to the argument, as may be seen from the proof of Theorem 5 . 3. An expansion which has not Property III

must be considered trivial, being analogous to the trivial expansion of quadratic forms in n variables as sums of more than n squares. Property II is necessary to lend precision to III.

If, for example, we have

$$\left.\begin{aligned} T(x, x) &= \sum_r k_r (p^{(r)}, x)^2 = \sum_r [k_r^{\frac{1}{2}} (p^{(r)}, x)]^2 \\ S(x, x) &= \sum_r h_r (p^{(r)}, x)^2 = \sum_r \frac{h_r}{k_r} [k_r^{\frac{1}{2}} (p^{(r)}, x)]^2 \end{aligned}\right\}, \tag{11}$$

where $$\lim_{r \to \infty} k_r = \infty,$$

then it is possible that the matrix whose vectors are $(p^{(r)})$ is bounded and complete, but that the matrix with vectors $k_r^{\frac{1}{2}}(p^{(r)})$ lacks either or both of these properties. Thus of the two forms of expansion (11), the first may have Property III, while the second and more fundamental form has not. The reader will readily convince himself that no such confusion is possible if only expansions having Property II are considered. Since, by Theorem 5 . 1, the canonical expansion is always possible, and by Theorem 5 . 3 it is, in its essential features, unique, we conclude that it is the natural and correct analogy to the Weierstrassian expansion for the class of forms under consideration.

VI. APPLICATION TO DYNAMICS, &c.

§ 26. Applications to Physics.

It is well known that the solutions of many of the problems of physics can be made dependent on the simultaneous reduction of two quadratic forms. The typical example is the case of a dynamical system to which Lagrange's method is applicable, in which the kinetic and potential energies are expressible as quadratic forms involving generalized "velocities" and "displacements" respectively. The case when the system is not completely determined by a finite number of coordinates, so that the quadratic forms involve an infinite set of variables, has already been used in a simple form by Lord Rayleigh* in discussing the vibrations of a stretched string. The method is applicable to non-rigid systems, including all kinds of vibratory and wave motions. That it

* *Theory of Sound*, 1 (London, 1894), ch. 6.

has not been more extensively employed may be due to the lack of a theory of infinite quadratic forms, such as appears in the foregoing pages.

The method is extremely interesting, in that it seems to point to a possibility at least of the expression in a form to which Lagrangian methods are applicable of the mechanism of light.* The simultaneous reduction shows not only a series of squares, to which might correspond lines in the spectrum of a light ray, but a series of integral terms, which may be interpreted as band spectra or a continuous spectrum.

§ 27. LAGRANGE EQUATIONS.

Suppose, then, that a dynamical system is such that :

(1) The configuration can be uniquely expressed by means of an infinite number of independent coordinates $(x_1, x_2, \ldots)$.

(2) The motion is holonomic, so that the coordinates may vary independently.

(3) The forces acting are conservative.

(4) The kinetic and potential energies T, V of the system are expressible throughout the motion as positive, bounded, complete quadratic forms in the variables $(x_1', x_2', \ldots)$, $(x_1, x_2, \ldots)$ respectively, where Σx_r^2, $\Sigma x_r'^2$, $\Sigma x_r''^2$ converge, and do not surpass fixed bounds. †

(5) If we introduce the variables u_r, u_{lm} $[r, l, m = 1, 2, \ldots]$, whose existence has been proved in Part V, such that

$$\left.\begin{aligned} T &= \sum_r \lambda_r u_r'^2 + \sum_l \sum_{m,n} \int_{\alpha_l}^{\beta_l} \lambda_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, u_{lm}' u_{ln}' \\ V &= \sum_r \mu_r u_r^2 + \sum_l \sum_{m,n} \int_{\alpha_l}^{\beta_l} \mu_l(g)\, \phi_{lm}(g)\, \phi_{ln}(g)\, dg\, u_{lm} u_{ln} \end{aligned}\right\}, \tag{1}$$

then independent variation of the u's is also possible.

It will be clear that conditions (1) to (4) are still satisfied, if we replace the x's by the u's as determining coordinates of the system.

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* It is not intended to suggest that the equations in this case have their ordinary dynamical meaning.

† From this point the discussion is purely formal. The solution can, however, be made mathematically rigorous, on the assumptions made.

We then have

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial u_r'}+\frac{\partial V}{\partial u_r} &= 0 \\ \frac{d}{dt}\frac{\partial T}{\partial u_{lm}'}+\frac{\partial V}{\partial u_{lm}} &= 0 \end{aligned}\right\}. \tag{2}$$

The first of these sets of equations gives

$$\lambda_r u_r''+\mu_r u_r = 0,$$

or, since by hypothesis (4), λ_r, μ_r are positive,

$$u_r = a_r \cos\left(\sqrt{\frac{\mu_r}{\lambda_r}}\, t\right)+b_r \sin\left(\sqrt{\frac{\mu_r}{\lambda_r}}\, t\right). \tag{3}$$

§ 28. SOLUTION OF THE EQUATIONS (CONTINUOUS SPECTRUM).

Now since Σx_r^2, $\Sigma x_r'^2$, $\Sigma x_r''^2$ are bounded, so are

$$\sum_m u_{lm}^2, \quad \sum_m u_{lm}'^2, \quad \sum_m u_{lm}''^2.$$

Hence we may write

$$\sum_{m=1}^{\infty} u_{lm}\phi_{lm}(g) \sim \Phi_l(t, g); \quad \sum_{m-1}^{\infty} u_{lm}''\phi_{lm}(g) \sim \Psi_l(t, g),$$

where $\Phi_l(t, g)$, $\Psi_l(t, g)$ are functions of integrable square with respect to g, and

$$\frac{d^2}{dt^2}\Phi_l(t, g) = \Psi_l(t, g). \tag{4}$$

The second set of equations (2) gives, for each value of l, n,

$$\frac{d}{dt}\left[\sum_m u_{lm}' \int_{\alpha_l}^{\beta_l} \lambda_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg\right]+\sum_m u_{lm}\int_{\alpha_l}^{\beta_l}\mu_l(g)\,\phi_{lm}(g)\,\phi_{ln}(g)\,dg = 0,$$

or

$$\int_{\alpha_l}^{\beta_l}[\lambda_l(g)\,\Psi_l(t, g)+\mu_l(g)\Phi_l(t, g)]\,\phi_{ln}(g)\,dg = 0. \tag{5}$$

The functions $\phi_{ln}(g)$ $[n = 1, 2, \ldots]$ form a complete set of orthogonal

functions for the interval of integration, and the other factor of the integrand in (5) is a function of integrable square.

Therefore $$\lambda_l(g)\,\Psi_l(t, g)+\mu_l(g)\,\Phi_l(t, g) = 0, \tag{6}$$

or, using (4),

$$\frac{d^2}{dt^2}\,\Phi_l(t, g)+\frac{\mu_l(g)}{\lambda_l(g)}\,\Phi_l(t, g) = 0. \tag{7}$$

Since T, V are positive, so are $\mu_l(g)$, $\lambda_l(g)$ and

$$\Phi_l(t, g) = f_1^{(l)}(g)\cos\left\{\sqrt{\frac{\mu_l(g)}{\lambda_l(g)}}\,t\right\}+f_2^{(l)}(g)\sin\left\{\sqrt{\frac{\mu_l(g)}{\lambda_l(g)}}\,t\right\}. \tag{8}$$

Equations (3), (8) give the complete solution, for

$$u_{lm} = \int_{\alpha_l}^{\beta_l}\Phi_l(t, g)\,\phi_{lm}(g)\,dg. \tag{9}$$

The constants a_r, b_r and the functions $f_1^{(l)}(g)$, $f_2^{(l)}(g)$ are determined by initial or boundary conditions.

§ 29. WIRTINGER'S PROBLEM (PRELIMINARY).

Examples of the method in which the spectrum of the pair of quadratic forms has no continuous part are to be found in Rayleigh. The following is one of the simplest cases in which a continuous spectrum occurs.

Consider the very small transverse plane vibrations of a stretched string, after being disturbed in the vicinity of a fixed end. The other end is supposed infinitely remote along a fixed line. We denote by $\rho(x)$ the density at a distance x from the fixed end, and suppose the configuration given by $y(x, t)$ a function of position and time measured at right angles to the equilibrium position of the string. The variations in the tension P are assumed to be very slight, and we have

$$V = P\int_0^\infty\left[\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}-1\right]dx = \frac{P}{2}\int_0^\infty\left(\frac{dy}{dx}\right)^2 dx \quad \text{(approx.)},$$

$$T = \tfrac{1}{2}\int_0^\infty\rho(x)\left(\frac{dy}{dt}\right)^2 dx.$$

We now state the postulates made in the course of the proof :

(i) $$\int_0^\infty y^2\,dx, \quad \int_0^\infty \left(\frac{dy}{dt}\right)^2 dx, \quad \int_0^\infty \left(\frac{d^2y}{dt^2}\right)^2 dx$$

remain bounded throughout the motion.

(ii) y, y', y'' must vanish at the origin for all time, and vary smoothly along the length of the string. We may therefore write

$$y = \int_0^x \frac{dy}{dx}\,dx; \quad y' = \int_0^x \frac{dy'}{dx}\,dx; \quad y'' = \int_0^x \frac{dy''}{dx}\,dx.$$

(iii) We also postulate the boundedness of

$$\int_0^\infty \left(\frac{dy}{dx}\right)^2 dx, \quad \int_0^\infty \left(\frac{dy'}{dx}\right)^2 dx, \quad \int_0^\infty \left(\frac{dy''}{dx}\right)^2 dx.$$

Taken along with (i) this ensures that y, y', y'' are always vanishingly small at a large distance from the origin. It seems reasonable to suppose that these integrals, initially small, do not tend to increase as the movement becomes more evenly distributed.

(iv) $\rho(x)$ is bounded, and has a non-zero lower bound.

§ 30. SOLUTION OF THE PROBLEM.

Under the conditions stated we may write*

$$\left.\begin{aligned} y &= \int_0^\infty \frac{\sin \alpha x}{\alpha} f(\alpha, t)\,d\alpha; \quad y' = \int_0^\infty \frac{\sin \alpha x}{\alpha} \frac{df(\alpha, t)}{dt}\,d\alpha \\ \frac{dy}{dx} &= \int_0^\infty \cos \alpha x\, f(\alpha, t)\,d\alpha \end{aligned}\right\} \qquad (10)$$

and so on, the functions dy'/dx, y'', dy''/dx being obtained by corresponding differentiations under the integral sign.

Since $$\int_0^\infty y^2\,dx, \quad \int_0^\infty \left(\frac{dy}{dx}\right)^2 dx$$

are bounded, so are†

$$\int_0^\infty \frac{1}{\alpha^2}[f(\alpha, t)]^2\,d\alpha, \quad \int_0^\infty [f(\alpha, t)]^2\,d\alpha.$$

* Here again the working is formal, but capable of being made rigorous.

† See Plancherel.

These conditions may be summed up in the inequality

$$\left.\begin{aligned} &\int_0^\infty \frac{a^2+1}{a^2}\,[f(a,t)]^2\,da \leqslant M. \\ \text{Similarly}\quad &\int_0^\infty \frac{a^2+1}{a^2}\left[\frac{d}{dt}f(a,t)\right]^2 da, \quad \int_0^\infty \frac{a^2+1}{a^2}\left[\frac{d^2}{dt^2}f(a,t)\right]^2 da \leqslant M. \end{aligned}\right\} \tag{11}$$

Let $\psi_r(x)$ $[r=1, 2, \ldots]$ be a complete set of orthogonal functions for the interval $(0, \infty)$, and let

$$\int_0^\infty \frac{\sqrt{(a^2+1)}}{a} f(a,t)\,\psi_r(a)\,da = c_r.$$

Then c_r', c_r'' are obtained by differentiation under the integral sign, and*

$$\Sigma c_r^2,\ \Sigma c_r'^2,\ \Sigma_r''^2 \leqslant M. \tag{12}$$

Now

$$T = \tfrac{1}{2}\int_0^\infty \rho(x)\left(\frac{dy}{dt}\right)^2 dx = \tfrac{1}{2}\int_0^\infty \rho(x)\left[\int_0^\infty \frac{\sin ax}{a}\left\{\frac{d}{dt}f(a,t)\right\} da\right]^2 dx$$

$$= \tfrac{1}{2}\int_0^\infty \rho(x)\left[\sum_{r=1}^{\infty} c_r' \int_0^\infty \frac{\sin ax}{\sqrt{(a^2+1)}}\,\psi_r(a)\,da\right]^2 dx$$

$$= \tfrac{1}{2}\sum_{r,s=1}^{\infty}\int_0^\infty \rho(x)\,p_r(x)\,p_s(x)\,dx\,c_r', c_s', \tag{13}$$

where

$$p_r(x) = \int_0^\infty \frac{\sin ax}{\sqrt{(a^2+1)}}\,\psi_r(a)\,da.$$

$$V = \frac{P}{2}\int_0^\infty \left(\frac{dy}{dx}\right)^2 dx = \frac{P}{2}\int_0^\infty dx\left[\int_0^\infty \cos ax\, f(a,t)\,da\right]^2$$

$$= \frac{P\pi}{4}\int_0^\infty [f(a,t)]^2\,da$$

$$= \frac{P\pi}{4}\sum_{r,s=1}^{\infty} c_r c_s \int_0^\infty \frac{a^2}{a^2+1}\,\psi_r(a)\,\psi_s(a)\,da. \tag{14}$$

If (11), or its equivalent (12), is fulfilled, T, V cannot surpass fixed bounds. That is to say, (13), (14) are bounded quadratic forms. Further, the coordinates chosen are capable of independent variation without transgressing the conditions of restraint, viz. the fixed position of the ends, and the continuity of the string.

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* Cf. Riesz, 152.

Suppose $\rho(x) = k$, a constant. Then T reduces to T_1 where

$$T_1 \equiv \frac{k}{2} \int_0^\infty \left[\sum_{r=1}^{\infty} c_r' \int_0^\infty \frac{\sin ax}{\sqrt{(a^2+1)}} \psi_r(a)\, da \right]^2 dx$$

$$= \frac{k\pi}{4} \sum_{r,\, s=1}^{\infty} c_r' c_s' \int_0^\infty \frac{1}{a^2+1} \psi_r(a)\, \psi_s(a)\, da. \tag{15}$$

Writing $$A_1(c', c') = \frac{4T_1}{k\pi}, \quad B(c, c) = \frac{4V}{P\pi},$$

we find $$A_1(x, x) + B(x, x) = E(x, x). \tag{16}$$

If now $\rho(x)$ varies, with k as lower bound, $T \geqslant T_1$. Hence, if A, B be the corresponding quadratic forms,

$$A(x, x) + B(x, x) \geqslant E(x, x). \tag{17}$$

Hence the square root of $(A+B)$ has in all cases considered a reciprocal (see Cor. 2 . 12), and the simultaneous reduction of A, B can be so carried out that the new coordinates are still independent. The theory of §§ 27, 28 is therefore applicable, and the solution has the form there stated.

§ 31. Simple Case Completely Solved.

The solution may readily be given analytic form for the simple case when $\rho(x) = k$. The quadratic forms (14), (15) are already in the canonical form of equation (1), and the solution is given by

$$c_r = \int_0^\infty \psi_r(a) \left\{ f_1(a) \cos\left(\sqrt{\frac{P}{k}} \cdot at\right) + f_2(a) \sin\left(\sqrt{\frac{P}{k}} \cdot at\right) \right\} da. \tag{18}$$

So that, with a slight change of notation,

$$y = \int_0^\infty \sin ax \left\{ g_1(a) \cos\left(\sqrt{\frac{P}{k}}\, at\right) + g_2(a) \sin\left(\sqrt{\frac{P}{k}}\, at\right) \right\} da. \tag{19}$$

If initially $y = h_1(x)$, $y' = h_2(x)$, then

$$g_1(a) = \frac{2}{\pi} \int_0^\infty \sin ax\, h_1(x)\, dx, \quad g_2(a) = \frac{2}{\pi a} \sqrt{\frac{k}{P}} \int_0^\infty \sin ax\, h_2(x)\, dx.$$

It is clear that the solution found satisfies the appropriate differential equation

$$\rho(x)\frac{d^2y}{dt^2} = P\frac{d^2y}{dx^2}$$

the differentiation under the integral sign being permissible.

The problem of the infinite stretched string here given was first propounded by Wirtinger,* who foresaw that, with certain distributions of density at least, a continuous spectrum must result. In the case treated in this paragraph the spectrum is continuous with frequency varying from zero to infinity. The solution may be interpreted as the resultant of two wave-configurations travelling in opposite directions along the string with the constant velocity $\sqrt{(P/k)}$. The problem is, of course, a one-dimensional form of the problem of propagation of waves in the atmosphere or ether.

* *Math. Annalen*, 48 (1897), 387.

ON A TYPE OF PLANE RATIONAL QUARTIC CURVE

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1. It was shown in *Proc. London Math. Soc.* (2), 20 (1920), 93–100, that there is just one type of plane curve of degree n whose $2n$ intersections with a conic are consecutive, the n-ic having a multiple point of order $n-1$. The discussion of the case in which the curve is any rational (unicursal) curve of degree n seems impracticable, but the case in which $n = 4$ can be worked out; and the results are stated in this paper, the details of straightforward analysis being suppressed. We also consider rational quartics which meet a conic in eight points, all coincident, though not consecutive on the same branch of the curve.

2. Suppose a conic Σ has eight-point contact with a rational quartic at C, which is not a singular point of the quartic. Suppose the tangent at C to Σ meets the quartic again at M and N. Suppose also that M and N have not coincident parameters, a case dealt with in § 5. Let (AC, MN) be a harmonic range, and let AB be the second tangent from A to Σ. If we take the triangle ABC as triangle of reference with suitable homogeneous coordinates, and choose a parameter t such that $t = \infty$ at C and $t^2 \pm 1 = 0$ at M and N, then the equation of Σ is

$$x^2 = yz, \tag{i}$$

while the quartic takes the form

$$x : y : z = (t^3 + at^2 + bt + c) : (t^2 \pm 1) : (t^4 + pt^3 + qt^2 + rt + s).$$

If now we use the facts that the quartic divides AC harmonically

and meets (i) only where $t = \infty$, we get readily four types :—

$$\text{I.} \quad x : y : z = t(t^2+m+1) : (t^2+1) : \{t^4+(2m+1)t^2+m^2\},$$

$$\text{II.} \quad x : y : z = t(t^2-m-1) : (t^2-1) : \{t^4-(2m+1)t^2+m^2\},$$

$$\text{III.} \quad x : y : z = (t^3+t-m) : (t^2+1) : (t^4+t^2-2mt),$$

$$\text{IV.} \quad x : y : z = (t^3-t+m) : (t^2-1) : (t^4-t^2+2mt),$$

where m is some constant.

The points M and N are unreal for types I and III, and real for types II and IV.

The types I and II, III and IV respectively can be projected into one another, but not by a real projection. In fact, we get II by putting $-ix$, $-y$, it, m for x, y, t, m in I; and we get IV by putting $-ix$, $-y$, it, im for x, y, t, m in III.

One conic having four-point contact with Σ at C touches the quartic elsewhere. The point of contact D is given by $t = 0$. For types I and II, D lies on CB; for types III and IV, D lies on AB.

3. Type I given by

$$x : y : z = t(t^2+m+1) : (t^2+1) : \{t^4+(2m+1)t^2+m^2\} \qquad \text{(i)}$$

has the equation

$$(yz-x^2)\{(m-2)^2x^2+(2m^2+1)y^2-(m^2+2)yz+z^2\} = m^2y^4. \qquad \text{(ii)}$$

As only even powers of x occur in (ii), the curve can be projected into one having an axis of symmetry.

One double point (0, 1, 1) is given by

$$t^2 = -(m+1).$$

The parameters of the other double points are given by

$$t^4+2t^2+(m-1)^2 = 0, \qquad \text{(iii)}$$

and these points are the intersections of

$$z = (2m-1)y \quad \text{with} \quad (\tfrac{1}{2}m-1)x^2 = (m-1)^2y^2.$$

There are therefore three real acnodes if $m > 2$, one real acnode if $2 > m > -1$, and one real crunode if $-1 > m$.

If $m = -1$, (0, 1, 1) is a cusp, and the other two double points are unreal. If $m = 1$, (0, 1, 1) is a triple point. If $m = 2$, (0, 1, 1) is an

acnode, and (1, 0, 0) is a tacnode with equal and opposite unreal curvatures. If $m = 0$, the curve is degenerate.

The bitangents are

$$z = \{2m-1 \pm 2(m^2-2m)^{\frac{1}{2}}\}y, \tag{iv}$$

$$z \pm 2(2m)^{\frac{1}{2}}x + (2m+1)y = 0, \tag{v}$$

The bitangents (iv), the line CA, and the line joining the double points given by (iii) form a harmonic pencil with vertex at A.

The points of contact of the bitangents (v) are always unreal; of the bitangents (iv), the second has unreal points of contact, and the first has real points of contact if $(m-1)^2 > 2$.

The inflexions are given by

$$(t^2-m+1)^3 + 6m(t^2-m+1) + 4m^2 = 0;$$

two coalesce at a point of undulation if $m = 1 \pm \sqrt{2}$. The conic through the inflexions meets the curve again where

$$(m+4)t^2 = m^2-5m-4.$$

If a conic through the nodes and C meets CA again in Q, and meets at P the second tangent from Q to the conic

$$(m-2)^2x^2 + 4(m-1)y(z-m^2y) = 0$$

touching the quartic at C, D and two other points, then P lies on the quartic.

We shall see that a similar geometrical generation of the quartic exists for the other types.

Writing

$$\xi = 2(\tfrac{1}{2}m-1)^{\frac{1}{2}}x - y + z, \quad \eta = -2(\tfrac{1}{2}m-1)^{\frac{1}{2}}x - y + z, \quad \zeta = z-(2m-1)y,$$

we obtain

$$m(m-1)(\xi^2+\eta^2)\zeta^2 + 4(m-1)(m-2)\xi^2\eta^2 - 4m(m-2)\xi\eta\zeta(\xi+\eta) + 2(m^2-m-4)\xi\eta\zeta^2 = 0,$$

which is the equation of the quartic referred to the nodal triangle, ξ, η, ζ being current coordinates. From this equation the conics connected with the quartic can be obtained. For instance, the conic through the intersections of the tangents from a node with the line joining the other two is

$$8(m-2)x^2 + (4m^2-4m-5)y^2 + 2(2m+3)yz - 5z^2 = 0.$$

We can deal similarly with type II. The double point $(0, -1, 1)$ is a crunode, cusp, or acnode according as $m > -1$, $= -1$, < -1. The other two double points are unreal if $m > 2$, coalesce at a tacnode with real branches if $m = 2$, are crunodes if $2 > m > 0$, and are acnodes if $0 > m$.

4. Type III given by

$$x : y : z = (t^3+t-m) : (t^2+1) : (t^4+t^2-2mt) \tag{i}$$

has the equation

$$(x^2-yz)(4x^2+y^2+z^2-2yz+4mxy) = m^2y^4. \tag{ii}$$

If the parameter of a point in which the line joining C to a double point meets the quartic again is $-p$, then

$$2p^3+2p+m = 0, \tag{iii}$$

and the parameters of the double point are given by

$$t^2-2pt+2p^2+1 = 0. \tag{iv}$$

The parameters of the points of contact of the tangents from this double point are given by

$$t^2+2pt+2p^2+1 = 0. \tag{v}$$

The coordinates of the double point are $(p, 1, 2p^2+1)$, and the coordinates of the remaining intersection of the line joining it to C with the quartic are $(p, 1, -3p^2)$.

The inflexions are given by

$$t^6+3t^4+6mt^3+3t^2+6mt+(2m^2+1) = 0. \tag{vi}$$

The bitangents are

$$2wx+(w^2+1-2m/w)y+z = 0, \tag{vii}$$

where

$$w^4-2mw^3-m^2 = 0. \tag{viii}$$

If w_1, w_2, w_3, w_4 are the roots of (viii), the roots of (iii) are

$$(w_2w_3+w_1w_4)/2m, \text{ etc.}$$

If in (ii) we put

$$\left.\begin{aligned} \xi &= 2p_1x+(2p_1^2+1)y+z \\ \eta &= 2p_2x+(2p_2^2+1)y+z \\ \zeta &= 2p_3x+(2p_3^2+1)y+z \end{aligned}\right\}, \tag{ix}$$

where p_1, p_2, p_3 are the roots of (iii), we get

$$(1+p_1^2)\{(4+3p_1^2)\eta^2\xi^2-4(1+3p_1^2)\xi^2\eta\xi\}+\ldots+\ldots=0, \qquad \text{(x)}$$

which is the equation of the quartic referred to the triangle formed by its nodes, with ξ, η, ζ as current coordinates. It has eight-point contact with the conic

$$(1+p_1^2)\{(4+3p_1^2)\xi^2-4(1+3p_1^2)\eta\zeta\}+\ldots+\ldots=0 \qquad \text{(xi)}$$

at (1, 1, 1). The equations (x) and (xi) are transformed into each other, if we replace ξ, η, ζ by $1/\xi, 1/\eta, 1/\zeta$.

By means of the transformation (ix), or otherwise, the equations of the conics connected with the quartic may be obtained. They are—*

Conics through the nodes :

$$4x^2+(m^2+1)y^2+z^2-2yz+mzx+3mxy=0, \qquad \text{(U)}$$

$$my^2+x(y+z)=0. \qquad \text{(T)}$$

Conic through the points of contact of the bitangents :

$$(11m^2+1)y^2+z^2+2yz+2mzx+10mxy=0. \qquad \text{(I)}$$

Conic through the intersections of the tangents at any node with the line joining the other two nodes :

$$32(9m^2+8)x^2+(81m^4+231m^2+64)y^2+(135m^2+64)z^2$$
$$-2(153m^2+64)yz+162m^3zx+2m(81m^2+128)xy=0. \qquad \text{(II)}$$

Conic touching the tangents at the nodes :

$$16x^2-(m^2-5)y^2+5z^2-6yz-10mzx+6mxy=0. \qquad \text{(III)}$$

Conic through the intersections of the tangents from any node with the line joining the other nodes :

$$16x^2+5(3m^2+1)y^2+5z^2-6yz+6mzx+22mxy=0. \qquad \text{(IV)}$$

Conic touching the six tangents from the nodes :

$$(m^2-1)y^2-z^2-2yz+2mzx+2mxy=0. \qquad \text{(V)}$$

Conic touching the inflexional tangents :

$$16x^2+(3m^2+1)y^2+z^2-14yz-2mzx+14mxy=0. \qquad \text{(VI)}$$

* The reference symbols are those used in Hilton's *Plane Algebraic Curves*, ch. 17, § 4.

Conic through the inflexions :

$$4x^2+(9m^2+1)y^2+z^2-2yz+mzx+11mxy = 0. \qquad \text{(VII)}$$

Conic through the points where the tangents at the nodes meet the curve again :

$$4(77m^2+64)x^2+(53m^4+253m^2+64)y^2+(125m^2+64)z^2$$
$$-2(157m^2+64)yz+125m^3zx+m(111m^2+256)xy = 0. \qquad \text{(VIII)}$$

Conic through the points of contact of tangents from the nodes :

$$4x^2+(5m^2+1)y^2+z^2-2yz+mzx+7mxy = 0. \qquad \text{(IX)}$$

The following points of geometrical interest may be noted.

Of the four common points of (U), (VII), (IX), two are the intersections of CD with the quartic other than C and D ($t = \infty$ and $t = 0$), and the other two lie on CA.

These conics meet CB in an involution, of which C is one double point. We shall denote the other (0, 1, 1) by H, and by K the point (0, 1, -1) such that (CB, HK) is a harmonic range. The polar cubic of C with respect to the quartic goes through H.

The conics (II) and (IV) are known to have double contact. The chord of contact is

$$6mx+(9m^2+2)y+2z = 0.$$

The conics (III), (V), (VI) are known to have double contact. The pole of the chord of contact is H. The lines CA, CB are conjugate for each of these conics. The conics become a point-pair, if $m^2 = 16/5$, 0, 16 respectively. The conic touching them at their points of contact, and touching AB, touches CA at C.

The conics (I) and (V) pass through A, while (V) has K and CA for pole and polar. The conic through C and the intersections of (V) with (I) touches CB at C.

Two conics pass through the intersections of (IV) and (VI), one going through B and C, the other touching AK at A.

A conic touching the common tangents of (VII) and (IX) touches (U) at the points of contact of tangents from C.

The polars of any point with respect to (VII) and (U), (IX) and the line-pair CA, CD form a harmonic pencil.

The conic touching the four common tangents of (III) and (IV), and the lines joining each pair of nodes, passes through A.

The following results are worth mention in addition to those just given.

(1) A conic has four-point contact with the quartic at its intersections M, N with CA, and touches CB, CD. The chord of contact with these lines is AH.

The conic is

$$4x^2+y^2+z^2-2yz+4mxy=0, \qquad \text{(xii)}$$

as is evident from (ii). A conic through the intersections of (U) and (xii) touches AH, CB at A and C.

(2) The conic through the nodes and M, N touches CB at H. It is

$$4x^2+y^2+z^2-2yz+2mxy=0.$$

(3) The conic through the nodes and through C and A touches CB, AK, and passes through D. The conic is (T).

(4) The conic through the nodes, touching the quartic at C, passes through H, divides AB harmonically, and meets Σ at its intersections with AK. It is

$$2x^2+y^2=yz,$$

and is the locus of the nodes of all quartics of the family obtained by varying m in (i).

(5) The points where the lines joining C to the nodes cut the quartic again lie on a conic having double contact with Σ at C and B. It is

$$3x^2+yz=0.$$

(6) If the polars of two nodes E, F with respect to Σ meet in G', and G is the third node, GG' passes through A, and is divided harmonically by A and CB.

If CG meets Σ again in G_1, and (CG_1, GG_2) is a harmonic range, G_2 lies on AK; and so for E and F.

(7) If any conic through the nodes and C meets CA in Q, the conic meets the second tangent from Q to Σ again in a point P of the quartic.

The converse of this may also be proved in the form :—

"C is a point on a conic Σ, and a line l meets the tangent at C to Σ in A. A conic touches l at A, and the polar of A with respect to Σ at C. The three points E, F, G are taken on this conic such that the polar of

E with respect to Σ meets CE on l, and so for F and G. Then, if any conic through E, F, G, C meets CA again in Q, and meets again in P the second tangent from Q to Σ, the locus of P is a quartic with nodes at E, F, G having eight-point contact with Σ at C."

(8) If in (7) the line joining two positions of P passes through a node G, the two positions of PQ meet on the line joining the other two nodes E, F.

If the two positions of P lie on a conic through E, F touching the quartic at C, the two positions of PQ meet on GH.

(9) The points of contact of the six tangents from H to the quartic lie on a conic. It is

$$4x^2+(3m^2+1)y^2+z^2-2yz+mzx+5mxy = 0. \qquad \text{(xiii)}$$

It passes through the intersections of (U), (VII), (IX) with CA and CD. The polars of any point with respect to (IX) and (U), (xiii) and the line-pair CA, CD form a harmonic pencil.

Similar results hold, of course, for type IV, which is projectively equivalent to type III.

The equations of types III and IV given in § 2 are not altered if we replace m, t, x by $-m$, $-t$, $-x$. Hence we may suppose m positive without loss of generality.

For type III two bitangents are real, and one of them has real points of contact. One node is real, and is always an acnode lying inside the conic Σ.

For type IV no bitangent is real if $27m^2 < 16$, while there are two crunodes and an acnode. If $27m^2 > 16$, there are two real bitangents and one real acnode. If $27m^2 = 16$, the curve has an acnode and a tacnode with real branches. If $2m^2 = 1$, the curve has a flecnode given by $t = 0$.

5. We have so far assumed that the tangent to the quartic at the point of eight-point contact was not a bitangent of the quartic, and did not pass through a cusp. The latter alternative is readily seen to be impossible. If we suppose it to be a bitangent with real points of contact, we obtain a quartic whose equation is most conveniently put in the form :

Type V.— $$x : y : z = (\sqrt{2}t^3+1) : (t^4+\sqrt{2}t) : t^2. \qquad \text{(i)}$$

Eliminating t we have

$$(x^2-2yz)(y^2-2xz) = z^4. \qquad \text{(ii)}$$

The quartic has *two* conics of eight-point contact at the points A, B where it touches the bitangent $z = 0$. The remaining sides of the triangle of reference ABC are the tangents at A and B to these conics.

The other bitangents are

$$2x+2y+5z = 0, \quad 2\omega^2 x+2\omega y+5z = 0, \quad 2\omega x+2\omega^2 y+5z = 0,$$

where $$\omega^2+\omega+1 = 0.$$

The nodes are

$$(1, 1, 1), \quad (\omega, \omega^2, 1), \quad (\omega^2, \omega, 1),$$

the first being an acnode.

Of the inflexions, given by

$$t^6-3\sqrt{2}t^3+1 = 0,$$

two are real.

If a conic through the nodes and C meets CA in C and P, the tangents from P to the conic having eight-point contact at A meet it again in its remaining intersections with the quartic.

If a conic through the nodes and A meets CA in A and P, the second tangent from P to the conic having eight-point contact at B meets it again on the quartic. Of course, a similar result holds for a conic through the nodes and B.

Replacing x, y, z in type V by

$$\omega x+\omega^2 y+z, \quad \omega^2 x+\omega y+z, \quad x+y+z,$$

we get

Type VI.— $$y^2z^2+z^2x^2+x^2y^2 = 4xyz(x+y+z),$$

which has two unreal conics of eight-point contact where it touches

$$x+y+z = 0.$$

The other bitangents are

$$3x+y+z = 0, \quad x+3y+z = 0, \quad x+y+3z = 0.$$

They touch in real points. The quartic has three crunodes and no real inflexions. Its properties are obvious by symmetry.

6. The shape of the quartic will be evident from the diagram. The curves shown are the following :—

First column.—

Type I: $m = 3,\ \sqrt{2}+1,\ 2,\ 1,\ -\sqrt{2}+1,\ -\frac{3}{4},\ -1,\ -3.$

Second column.—Type II: $m = 3,\ \frac{9}{4},\ 2,\ \frac{7}{4},\ 1,\ \frac{3}{4},\ -\frac{1}{4},\ -1.$

Third column.—Type II: $m = -2.$ Type III: $m = 4,\ \frac{5}{4}.$ Type V.

Fourth column.—Type IV: $m = 2,\ \frac{5}{4},\ 1,\ \frac{4}{9}\sqrt{3},\ \frac{1}{2}.$

The curves in each column are given in descending order from top to bottom. The curves have been projected* so that the line CA is at infinity, C being in the vertical direction. The conic Σ is an asymptotic parabola with Cartesian equation $x^2 = y$.

7. A brief enumeration must suffice for those rational quartics which meet a conic in eight points which coincide at a double or triple point of the quartic. Their geometrical properties are not so interesting for the most part as those of the quartics previously considered.

(1) *Quartic with a triple point at which one branch has six-point contact with a conic.*

There are three types.

(A) $$(x^2-yz)x^2+my^4 = 0.$$

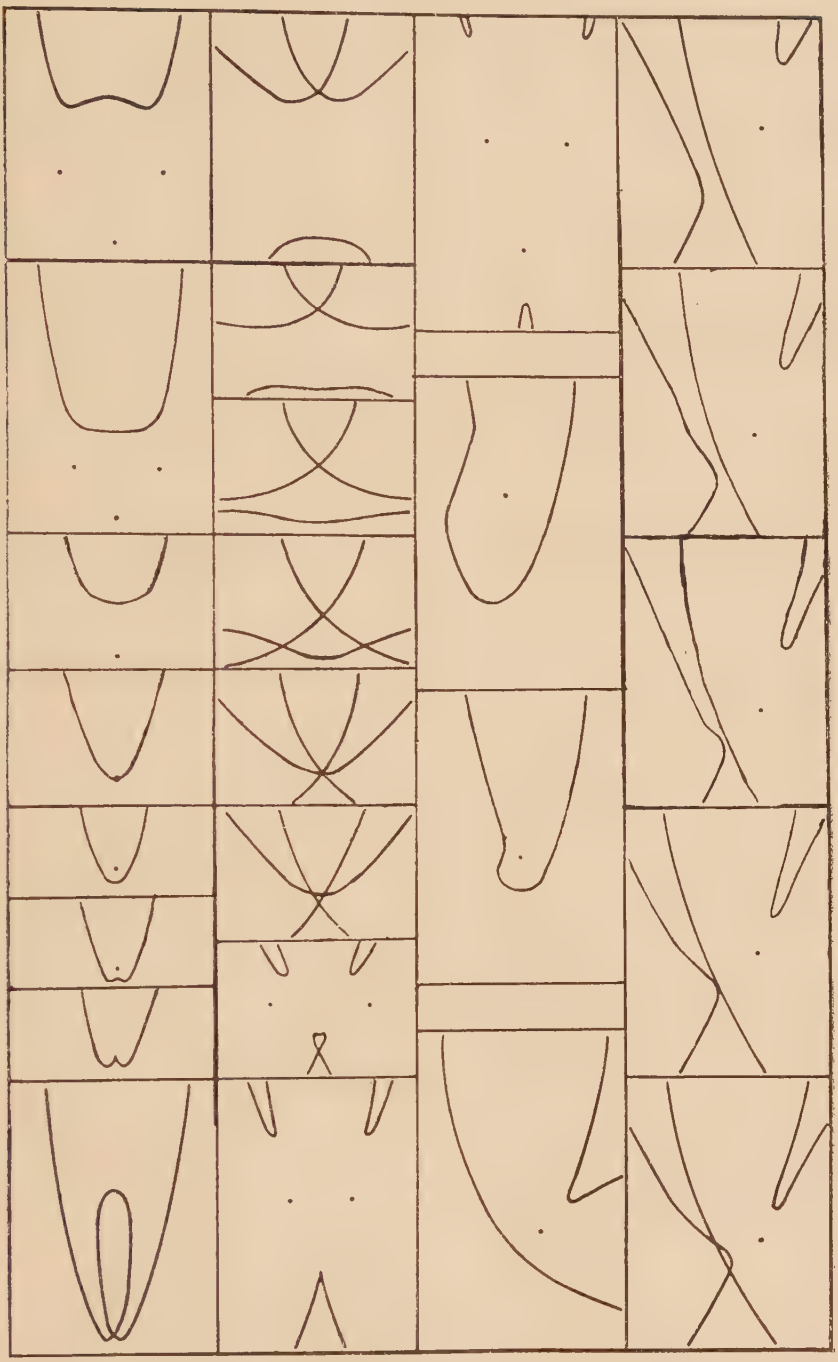
The curve has a bitangent with real points of contact if $m > 0$, and two real inflexions if $m < 0$. The lines joining $(0, 0, 1)$ to the intersections of the quartic with any tangent to $x^2 = yz$ trace out two fixed involution pencils.

(B) $$(x^2-yz)(x^2+y^2)+my^4 = 0.$$

The curve has bitangents

$$z+(1\pm 2m^{\frac{1}{2}})y = 0, \quad z\pm 2(-m)^{\frac{1}{2}}x+(1-m)y = 0,$$

* Except for Type V, where C is the origin and A, B the points at infinity in the directions $y = 0$, $x = 0$.



whose points of contact lie on

$$2mx^2+(1-3m)y^2+z^2+(2-m)yz = 0,$$

which is a line-pair if $m = -8$.

There is a point of undulation if $m = 1$ or -4, and two real inflexions if $m > 1$ or $m < -4$.

Similar remarks apply to

(C) $$(x^2-yz)(x^2-y^2)+my^4 = 0,$$

whose bitangents are

$$z-(1\pm 2m^{\frac{1}{2}})y = 0, \quad z\pm 2m^{\frac{1}{2}}x-(1-m)y = 0.$$

(2) *Quartic with an oscnode.*

Either conic having five-point contact with a branch at the oscnode meets the quartic eight times there (see Hilton's *Plane Algebraic Curves*, 296).

(3) *Rational quartic with a tacnode at which one branch has six-point contact with a conic.*

(A) $$mx^2(yz-x^2) = (my^2+yz-x^2)^2;$$

$$x : y : z = t(t^2+1) : t^2 : \{(t^2+1)^2+m\}.$$

There is an acnode given by $t^2+1 = 0$. The conic through the inflexions is

$$2(m+1)x^2+2(3m+8)y^2+z^2-(3m+10)yz = 0;$$

and a conic through the points of contact of the bitangents touches the quartic at the tacnode, namely

$$2(m+1)x^2-2my^2-(m+2)yz = 0.$$

Two inflexions are real if $m > -1$, and four if $m < -1$. The curve degenerates if $m = 0$ or -1.

Similar remarks apply to

(B) $$mx^2(yz-x^2) = (-my^2+yz-x^2)^2;$$

$$x : y : z = t(t^2-1) : t^2 : \{(t^2-1)^2+m\}.$$

It has a crunode, and has two real inflexions if $m < -1$.

The quartics $$z^2(x^2 \pm y^2) \pm x^4 = 0$$

have eight intersections with the conic of closest contact with *either* branch at the tacnode. Two of them are given by putting $m = -2$ in (A) and (B).

(4) *Rational quartic meeting a conic eight times at a crunode.*

Suppose that the conic Σ has seven-point contact with one branch of a rational quartic at a crunode C. Let the tangent to the other branch at C meet Σ again in B, and let the tangents at B and C to Σ meet at A.

Take suitable homogeneous coordinates with ABC as triangle of reference, $t = \infty$ and $t = 0$ being the parameters of the two branches at C. Then Σ becomes $x^2 = yz$, and the quartic becomes

$$x : y : z = \{t^3 + (m-1)t^2\} : (t^2 - t) : \{t^4 + (2m-1)t^3 + m^2t^2 + m^2t + m^2\} \qquad \text{(i)}$$

leading to

$$m^2y^4 + (x^2 - yz)\{(3m+2)x^2 - (3m^2+m+1)xy + m^2(m-1)y^2 - zx\} = 0. \qquad \text{(ii)}$$

The double points, other than C, are given by

$$2t^2 - 3t + 2m + 1 = \pm(4m+1)^{\frac{1}{2}}(t-1),$$

$$x : y : z = \{2m+1 \pm (4m+1)^{\frac{1}{2}}\} : 2 : 2\{m^2 + 2m \pm (m+1)(4m+1)^{\frac{1}{2}}\}.$$

They are unreal if $m < -\frac{1}{4}$, crunodes if $0 > m > -\frac{1}{4}$, an acnode and a crunode if $\frac{4}{9} > m > 0$, and two acnodes if $m > \frac{4}{9}$.

If $m = -\frac{1}{4}$, the quartic has a tacnode. If $m = \frac{4}{9}$, the quartic has a cusp and an acnode besides the crunode at C.

The conic through the points of contact of the bitangents

$$z - mx + my + (1 \pm m^{\frac{1}{2}})^2(my - x) = 0,$$

$$z + (2m-1)x + (5m^2 - 2m)y \pm 2(4m^2+m)^{\frac{1}{2}}(x + my) = 0$$

is

$$(12m^2 + 4m - 1)x^2 + m^2(3m^2 - 4m + 4)y^2 - z^2$$

$$-6m^2yz + 2zx - 2m^2(4m+1)xy = 0.$$

If a conic through the nodes touches Σ at C and meets CB again at Q, while the tangent from Q to the fixed conic

$$(m^2-8m-4)x^2+(m+1)^2y^2+z^2+2(m+1)yz-2mzx+2m(m-1)xy = 0,$$

other than CB, meets it again at P, then the locus of P is the quartic.

This follows at once from the fact that

$$t\{yz-x^2-xy+(m+1)y^2\}+\{x^2-(2m+1)xy+m^2y^2\} = 0$$

and

$$\{t^2+mt+(2m+1)\}x-\{(m+1)t+m^2\}y-tz = 0$$

intersect on $x = 0$, and pass through the point (i). The fixed conic touches the quartic in the four points where

$$t^4+(m-1)t^3-(2m+1)t^2-(m^2-m-1)t-m^2 = 0.$$

ON BOREL INTERVALS

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This investigation had its origin in an attempt to determine approximately what part of the unit length is covered by the intervals

$$\left(\frac{p}{q}-\frac{1}{q^3},\ \frac{p}{q}+\frac{1}{q^3}\right),$$

to which Borel had drawn attention in his *Leçons sur la Théorie des Fonctions*, p. 39.

Borel showed that the total length covered did not exceed

$$2\sum_{q=2}^{\infty}(q^{-2}-q^{-3}) = 0{\cdot}885\ 754\ 327\ 2\ldots,$$

but this includes an infinite number of lengths counted many times over.

It was therefore desirable to find some method of rejecting intervals already overlapped without making a separate examination of each. In doing this it was found that the overlapping of intervals was subject to four laws, the interest of which overshadowed that of the original quest. These laws are stated at the commencement of Arts. 1, 2, 4, 5 below.

Subsequently it was found that these laws also held good for the intervals

$$\left(\frac{p}{q}-\frac{\alpha}{\beta}\,\frac{1}{q^n},\ \frac{p}{q}+\frac{\alpha}{\beta}\,\frac{1}{q^n}\right),$$

where n is a positive integer not less than 3, where α, β are positive integers prime to one another, such that $\beta^{n-1} > \alpha^n$ (which involves $\beta > \alpha$); and the case $\alpha = \beta = 1$ is also included.

The paper concludes with a brief outline of the argument employed to determine that portion of the unit length which is actually covered by the intervals first mentioned. The result is that the covered portion exceeds 0·620 705 785 ..., but does not exceed 0·623 475 738 ....

1. The First Law.—*If the centre of an interval be inside another interval (at any point but its centre), then the whole of the first interval lies inside that half of the second interval in which the centre of the first interval is situated, i.e. it does not extend as far as the centre nor as far as an end-point of the second interval.*

It being supposed that p/q is in its lowest terms, let the interval

$$\left(\frac{p}{q}-\frac{\alpha}{\beta q^n},\quad \frac{p}{q}+\frac{\alpha}{\beta q^n}\right)$$

be denoted by $\{p/q\}$.

If r/s lie inside $\{p/q\}$ suppose first that

$$\frac{p}{q}<\frac{r}{s}<\frac{p}{q}+\frac{\alpha}{\beta}\frac{1}{q^n}, \tag{I}$$

so that

$$0<qr-ps<\frac{\alpha}{\beta}\frac{s}{q^{n-1}}. \tag{II}$$

Since $qr-ps$ is positive and integral it follows that

$$\alpha s>\beta q^{n-1}. \tag{III}$$

The following cases must be considered separately:—

(*a*) $\alpha s=k\beta q^{n-1}$, where k is a positive integer greater than unity;

(*b*) $\alpha s=k\beta q^{n-1}+b$, where k is any positive integer including unity and where $1\leqslant b\leqslant \beta q^{n-1}-1$.

In case (*a*)

$$\frac{\alpha q}{\beta s^{n-1}}=\left(\frac{\alpha}{\beta}\right)^n\frac{q}{(kq^{n-1})^{n-1}},$$

which is less than unity both when $\alpha=\beta=1$ and $\alpha<\beta$.

In case (*b*)

$$\frac{\alpha q}{\beta s^{n-1}}=\left(\frac{\alpha}{\beta}\right)^n\frac{q}{(kq^{n-1}+b/\beta)^{n-1}},$$

which is less than unity both when $\alpha=\beta=1$ and $\alpha<\beta$.

Hence in both cases (*a*) and (*b*) the proper fraction

$$\frac{\alpha q}{\beta s^{n-1}}<qr-ps,$$

so that

$$\frac{p}{q}<\frac{r}{s}-\frac{\alpha}{\beta}\frac{1}{s^n}. \tag{IV}$$

This shows that the interval $\{r/s\}$ does not extend as far as p/q.

The next step is to show that

$$\frac{r}{s}+\frac{\alpha}{\beta}\frac{1}{s^n}<\frac{p}{q}+\frac{\alpha}{\beta}\frac{1}{q^n}, \tag{V}$$

i.e.

$$0<qr-ps<\frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}-\frac{q}{s^{n-1}}\right). \tag{VI}$$

But

$$0<qr-ps<\frac{\alpha}{\beta}\frac{s}{q^{n-1}}. \tag{II}$$

Hence it must be shown that no integer can lie between

$$\frac{\alpha}{\beta}\frac{s}{q^{n-1}} \quad \text{and} \quad \frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}-\frac{q}{s^{n-1}}\right).$$

The two cases marked (a) and (b) above have to be considered.

In case (a) it was shown that $\alpha q/\beta s^{n-1}$ is a positive proper fraction. But since in this case $\alpha s/\beta q^{n-1}=k$ it follows that $(\alpha/\beta)(s/q^{n-1}-q/s^{n-1})$ lies between k and $k-1$, so that what is required is proved.

In case (b) $\alpha s/\beta q^{n-1}=k+b/\beta q^{n-1}$, which lies between k and $k+1$, because $1\leqslant b\leqslant \beta q^{n-1}-1$.

Also

$$\frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}-\frac{q}{s^{n-1}}\right)=k+\frac{b}{\beta q^{n-1}}-\left(\frac{\alpha}{\beta}\right)^n\frac{q}{(kq^{n-1}+b/\beta)^{n-1}},$$

which is less than $k+b/\beta q^{n-1}$, which is less than $(k+1)$.

On the other hand $(\alpha/\beta)(s/q^{n-1}-q/s^{n-1})$ will be greater than k if

$$\frac{b}{\beta q^{n-1}}>\left(\frac{\alpha}{\beta}\right)^n\frac{q}{(kq^{n-1}+b/\beta)^{n-1}},$$

i.e. if

$$\frac{\beta^{n-1}}{\alpha^n}>\frac{q^n}{b\,(kq^{n-1}+b/\beta)^{n-1}}. \tag{VII}$$

Now the right-hand side of (VII) is less than unity, since b, k, q are all positive integers, and since $n<(n-1)^2$, because $n\geqslant 3$.

So (VII) will be satisfied both when $\beta^{n-1}>\alpha^n$ and when $\alpha=\beta=1$.

These conditions being satisfied, it follows that $(\alpha/\beta)(s/q^{n-1}-q/s^{n-1})$ lies between k and $(k+1)$. So there is no integer between it and $\alpha s/\beta q^{n-1}$ in either of the cases (a) and (b).

Hence, if the positive integer $qr-ps$ is such that

$$qr-ps < \frac{\alpha}{\beta}\frac{s}{q^{n-1}}, \tag{II}$$

then also

$$qr-ps < \frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}} - \frac{q}{s^{n-1}}\right), \tag{VI}$$

so that

$$\frac{r}{s} + \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} \tag{V}.$$

Hence if r/s lies between p/q and $(p\beta q^{n-1}+\alpha)/\beta q^n$, then the interval $\{r/s\}$ lies between the same values, and the two intervals $\{r/s\}$ and $\{p/q\}$ have no common end-point (or, as it may be expressed, do not abut *internally*) and $\{r/s\}$ does not extend as far as p/q.

It has now been shown that, if

$$\frac{p}{q} < \frac{r}{s} < \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n}, \tag{I}$$

then

$$\frac{p}{q} < \frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{r}{s} < \frac{r}{s} + \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n}. \tag{VIII}$$

From these it may be deduced that, if

$$\frac{q-p}{q} > \frac{s-r}{s} > \frac{q-p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n},$$

then

$$\frac{q-p}{q} > \frac{s-r}{s} + \frac{\alpha}{\beta}\frac{1}{s^n} > \frac{s-r}{s} > \frac{s-r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} > \frac{q-p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n}.$$

Replacing $q-p$ by p and $s-r$ by s, we see that, if

$$\frac{p}{q} > \frac{r}{s} > \frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n}, \tag{IX}$$

then

$$\frac{p}{q} > \frac{r}{s} + \frac{\alpha}{\beta}\frac{1}{s^n} > \frac{r}{s} > \frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} > \frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n}. \tag{X}$$

Putting together (I), (VIII), (IX), and (X) we deduce that if r/s lie in one of the halves of $\{p/q\}$, then the whole interval $\{r/s\}$ lies inside that

half of $\{p/q\}$, but does not extend as far as p/q, nor as far as an end-point of $\{p/q\}$. This is the first law.

2. THE SECOND LAW.—*If the centre of one interval is at an end-point of another, then that half of the first interval which lies inside the second interval does not extend as far as its centre.*

Suppose that r/s is at an end-point of $\{p/q\}$. Let $r/s = (p\beta q^{n-1}+\alpha)/\beta q^n$, it being understood that r/s is in its lowest terms.

It is necessary to reduce $(\beta p q^{n-1}+\alpha)/\beta q^n$ to its lowest terms, and to show that when so reduced its denominator will be greater than q; for, if the denominator were less than q, then $\{r/s\}$ would contain p/q.

If $\alpha = \beta = 1$, then $r/s = (pq^{n-1}+1)/q^n$, which is in its lowest terms, and therefore $s = q^n$, which is greater than q.

Leaving out this case, then, since α is prime to β, it follows that the greatest common measure of $(\beta q^{n-1}p+\alpha)$ and βq^n is a divisor of q^n, which will be called g. It may happen that g divides q^{n-1}. If so, let $q^{n-1} = gh$. Then the denominator $\beta q^n = \beta q\, gh$; and after g has been divided out from the numerator and denominator there is left in the denominator βqh, which is greater than q, since β is greater than unity.

Next suppose that g does not divide q^{n-1}. It must however divide q^n. Hence it is the product of two factors, say k and l, of which one, k, divides q^{n-1}, and the other, l, divides q. Let $q^{n-1} = km$, $q = lt$. The denominator $\beta q^n = \beta kmlt$ becomes, after division by g, *i.e.* by kl, equal to βmt. Now since k divides q^{n-1}, and since k, being a factor of g, must divide $\beta q^{n-1}p+\alpha$, it must also divide α, so that $k \leqslant \alpha$, but $\alpha < \beta$. Hence $k < \beta$. Consequently

$$\beta mt > kmt,$$

$$kmt = q^{n-1}t,$$

$$\beta mt > q^{n-1}t > q.$$

Hence $(\beta q^{n-1}p+\alpha)/\beta q^n$, when reduced to its lowest terms, has a denominator greater than q. This proves the second law.

3. PRELIMINARY TO THE THIRD AND FOURTH LAWS.—*Take next the case in which r/s is outside the interval $\{p/q\}$.*

It is sufficient to take here

$$\frac{r}{s} > \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} \qquad \text{(XI).}$$

For if $$\frac{r}{s} < \frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n},$$

then $$\frac{s-r}{s} > \frac{q-p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n}.$$

Replacing $q-p$ by p and $s-r$ by s, this becomes the same as (XI).

When (XI) holds there are several alternatives, according to the position of the point $(r\beta s^{n-1}-\alpha)/\beta s^n$ with regard to the interval $\{p/q\}$.

(i) If $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n},$$

and also $$\frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s},$$

then the interval $\{p/q\}$ lies wholly inside one half of $\{r/s\}$. This is possible, as is shown in Art. 1.

(ii) If $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} = \frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n},$$

and also $$\frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s},$$

then p/q would lie inside one half of $\{r/s\}$, and then by Art. 1 it is known that $\{p/q\}$ is wholly inside this half of $\{r/s\}$, and these two intervals cannot have a common end-point. So this case is impossible.

(iii) If $$\frac{p}{q} - \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q},$$

and $$\frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s},$$

then p/q is inside $\{r/s\}$, and therefore by Art 1 the whole of the interval $\{p/q\}$ must be inside $\{r/s\}$, and so $(p\beta q^{n-1}-\alpha)/\beta q^n$ cannot be less than $(r\beta s^{n-1}-\alpha)/\beta s^n$. So this case is impossible.

(iv) If $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} = \frac{p}{q}$$

and $$\frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s},$$

then p/q is an end-point of $\{r/s\}$, and so by Art. 2 one half of $\{p/q\}$ is inside $\{r/s\}$ but does not reach as far as r/s. The other half of $\{p/q\}$ is outside $\{r/s\}$. This case is possible.

(v) If next $$\frac{p}{q} < \frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s},$$

it will be shown in Art. 4 that this is impossible, and this will prove the third law.

(vi) If next $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} = \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n},$$

then the two intervals $\{p/q\}$ and $\{r/s\}$ would have a common end-point, but no other point in common. They would abut externally. It will be shown in Art. 5 that this is impossible, and this will prove the fourth law.

(vii) Lastly if $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} > \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n},$$

then the intervals $\{p/q\}$ and $\{r/s\}$ would be entirely outside one another. This is possible.

4. THE THIRD LAW.—*If the centre of one interval is outside a second interval, then either the first interval is wholly outside the second interval, or else the first interval contains the second interval in one of its halves.*

This is proved by showing that the case (v) of Art. 3 is impossible. Suppose, if possible, that

$$\frac{p}{q} < \frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} < \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n} < \frac{r}{s}. \qquad \text{(XII)}$$

In this case the point $$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} = \frac{\beta r s^{n-1} - \alpha}{\beta s^n}$$

is inside $\{p/q\}$.

Applying the condition (III) it follows that

$$\alpha s^n > q^{n-1}. \qquad \text{(XIII)}$$

Again the point $$\frac{p}{q}+\frac{\alpha}{\beta}\frac{1}{q^n}=\frac{\beta pq^{n-1}+\alpha}{\beta q^n}$$

is inside $\{r/s\}$.

If we apply (III) again, it follows that

$$\alpha q^n > s^{n-1}. \tag{XIV}$$

Now from (XII) it follows that

$$\left.\begin{aligned} &\frac{\alpha}{\beta}\frac{q}{s^{n-1}} < qr-ps < \frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right) \\ \text{and}\quad &\frac{\alpha}{\beta}\frac{s}{q^{n-1}} < qr-ps \end{aligned}\right\}. \tag{XV}$$

Now supposing that either $\alpha=\beta=1$ or $\beta^{n-1}>\alpha^n$, we shall show that, if (XIII) and (XIV) hold, then

$$\frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right)$$

is a positive proper fraction, so $\alpha s/\beta q^{n-1}$ and $\alpha q/\beta s^{n-1}$ must also be proper fractions. Then (XV) would require that the positive integer $qr-ps$ should lie between two positive proper fractions, which is impossible. This will demonstrate the impossibility of (XII), and so will prove the third law.

Now $$\frac{\alpha}{\beta}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right)=\frac{\alpha^{n/(n-1)}}{\beta}\cdot\frac{1}{\alpha^{1/(n-1)}}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right). \tag{XVI}$$

When $\alpha=\beta=1$ the factor $\alpha^{n/(n-1)}/\beta=1$, but when $\beta^{n-1}>\alpha^n$ this factor is less than 1. It remains to show that

$$\frac{1}{\alpha^{1/(n-1)}}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right)$$

is less than unity. This expression itself and (XIII) and (XIV) show that the conditions satisfied by q and s are symmetrical. Now if $s=q$ there is no possibility of an overlap because $2\alpha/\beta q^n<1/q$. Dismissing this case, suppose $s>q$, and put $s=\mu q$, so that $\mu>1$.

From (XIII) and (XIV)

$$a^{-1/n} q^{(n-1)/n} < s < a^{1/(n-1)} q^{n/(n-1)}, \qquad \text{(XVII)}$$

$$a^{-1/n} q^{-1/n} < \mu < a^{1/(n-1)} q^{1/(n-1)}. \qquad \text{(XVIII)}$$

Hence $$\frac{1}{a^{1/(n-1)}}\left(\frac{s}{q^{n-1}} + \frac{q}{s^{n-1}}\right) = \frac{1}{a^{1/(n-1)}} \frac{1}{q^{n-2}} \left(\mu + \frac{1}{\mu^{n-1}}\right),$$

but from (XVIII) $$\mu / a^{1/(n-1)} < q^{1/(n-1)}.$$

Consequently $$\frac{1}{a^{1/(n-1)}}\left(\frac{s}{q^{n-1}} + \frac{q}{s^{n-1}}\right) < \left(1 + \frac{1}{\mu^n}\right) \frac{1}{q^{n-2-1/(n-1)}}.$$

The exponent of q, viz.: $[n-2-1/(n-1)]$, increases as n increases.

If $n \geqslant 4$, it is greater than or equal to $\frac{5}{3}$, and since the least possible value of q is 2, the least possible value of $q^{n-2-1/(n-1)}$ is greater than 2. And, since $\mu > 1$,

$$1 + 1/\mu^n < 2.$$

Consequently, when $n \geqslant 4$,

$$\frac{1}{a^{1/(n-1)}}\left(\frac{s}{q^{n-1}} + \frac{q}{s^{n-1}}\right) < 1.$$

If, however, $n = 3$,

$$\frac{1}{a^{1/(n-1)}}\left(\frac{s}{q^{n-1}} + \frac{q}{s^{n-1}}\right) < \left(1 + \frac{1}{\mu^3}\right) \frac{1}{q^{\frac{1}{2}}},$$

and so will be less than unity if $q \geqslant 4$, since $\mu > 1$.

It remains to be seen what happens when q is 2 or 3, n being 3.

If $q = 2$, s is 3 at least, and so μ is at least $\frac{3}{2}$, and

$$\left(1 + \frac{1}{\mu^3}\right) \frac{1}{q^{\frac{1}{2}}} \leqslant \frac{1}{2^{\frac{1}{2}}} \left(1 + \tfrac{8}{27}\right),$$

which is less than unity.

If $q = 3$, s is 4 at least, and μ is at least $\frac{4}{3}$, and

$$\left(1 + \frac{1}{\mu^3}\right) \frac{1}{q^{\frac{1}{2}}} \leqslant \frac{1}{3^{\frac{1}{2}}} \left(1 + \tfrac{27}{64}\right),$$

which is less than unity.

Hence in all cases $\frac{1}{\alpha^{1/(n-1)}}\left(\frac{s}{q^{n-1}}+\frac{q}{s^{n-1}}\right)<1,$

and since $\alpha^{n/(n-1)}/\beta \leqslant 1$, it follows that $(\alpha/\beta)(s/q^{n-1}+q/s^{n-1})$ is less than unity.

Hence (XV) and consequently (XII) cannot be satisfied.

This proves the third law.

5. The Fourth Law.—*Two intervals cannot have a common end-point.*

It was proved in § 1 that they could not abut *internally.*

It remains to prove that they cannot abut *externally.*

This is the case (vi) of §3, in which

$$\frac{r}{s}-\frac{\alpha}{\beta}\frac{1}{s^n}=\frac{p}{q}+\frac{\alpha}{\beta}\frac{1}{q^n},$$

$$\beta(qr-ps)q^{n-1}s^{n-1}=\alpha(q^n+s^n). \qquad \text{(XIX)}$$

Let M be the Greatest Common Measure of q and s.

Let $q=MQ$, $s=MS$. Then (XIX) becomes

$$\beta(rQ-pS)M^{n-1}Q^{n-1}S^{n-1}=\alpha(Q^n+S^n). \qquad \text{(XX)}$$

If $\alpha=\beta=1$, this equation can be written

$$(rQ-pS)M^{n-1}Q^{n-1}S^{n-1}-Q^n=S^n,$$

the left-hand side of which is divisible by Q, whilst the right-hand side is not, since S is prime to Q. In the case $Q=1$, S is not equal to 1, for this would make $q=s$, but in that case, as already remarked, abutment is impossible, for $2\alpha/\beta q^n<1/q$. If $Q=1$ the equation can be written

$$(r-pS)M^{n-1}S^{n-1}-S^n=1,$$

which is impossible, since the left-hand side is divisible by S, whilst the right-hand side is not.

Hence equation (XX) is impossible if $\alpha=\beta=1$.

Passing now to the case $\beta^{n-1}>\alpha^n$, where α, β are prime to one another, the equation (XX) requires the existence of an integer k such that

$$M^{n-1}Q^{n-1}S^{n-1}(rQ-pS)=k\alpha, \qquad \text{(XXI)}$$

$$Q^n+S^n=k\beta. \qquad \text{(XXII)}$$

Now if k had a factor in common with Q, then by (XXII) that factor must also divide S. Therefore Q and S would have a common factor, which is contrary to the hypothesis that M is the greatest common measure of q and s.

Hence k cannot have a factor in common with Q, and in like manner it cannot have a factor in common with S.

Therefore, by (XXI), k must be a factor of $M^{n-1}(rQ-pS)$.

Consequently $$k \leqslant M^{n-1}(rQ-pS). \qquad \text{(XXIII)}$$

Now from (XXI) and (XXII) it follows that

$$\frac{(k\beta)^{n-1}}{(k\alpha)^n} = (Q^n+S^n)^{n-1}/[M^{n-1}S^{n-1}Q^{n-1}(rQ-pS)]^n.$$

Hence, putting $$T = k/\{M^{n-1}(rQ-pS)\},$$

$$U = 1/\{M^{(n-1)(n-2)}(rQ-pS)^{n-1}\},$$

$$W = [(S^n+Q^n)/\{MS^nQ^n\}]^{n-1},$$

it follows that $$\frac{\beta^{n-1}}{\alpha^n} = TUW. \qquad \text{(XXIV)}$$

By (XXIII), T cannot exceed unity.

In general U is less than unity, but may be equal to unity if $M=1$ and $rQ-pS=1$.

It remains to show that $(S^n+Q^n)/\{MS^nQ^n\}$ is less than unity.

Since Q is not equal to S, let us suppose $S>Q$. If $Q=1$, then $M=2$ at least, because $q=2$ at least. In this case

$$\frac{S^n+Q^n}{MS^nQ^n} \leqslant \frac{S^n+1}{2S^n} < 1.$$

If neither S nor Q is equal to unity, then

$$(S^n-1)(Q^n-1) > 1,$$

$$S^nQ^n > S^n+Q^n,$$

$$\frac{S^n+Q^n}{MS^nQ^n} < 1,$$

and therefore W is less than unity.

Hence the right-hand side of (XXIV) is less than unity.

Hence if equation (XIX) were possible, it follows from (XXIV) that $\beta^{n-1} < \alpha^n$, which is contrary to the hypothesis $\beta^{n-1} > \alpha^n$.

Hence (XIX) is impossible. Consequently it is impossible that

$$\frac{r}{s} - \frac{\alpha}{\beta}\frac{1}{s^n} = \frac{p}{q} + \frac{\alpha}{\beta}\frac{1}{q^n}$$

in the cases $\alpha = \beta = 1$ and $\beta^{n-1} > \alpha^n$.

This proves the fourth law.

If then $\alpha = \beta = 1$ or $\beta^{n-1} > \alpha^n$ all the four laws hold good.

These restrictions on α, β are thus shown to be *sufficient* for the validity of the four laws.

6. When the centre of one interval is inside one of the halves of another interval, then by the first law the first interval cannot extend as far as the centre of the second interval.

From this fact the following deduction can be made.

Suppose that at each rational point of the unit interval an ordinate is drawn to represent the length of the interval of which that point is the centre, then these ordinates alter discontinuously. Take the intervals $(p/q - 1/q^3,\ p/q + 1/q^3)$.

Then the ordinate drawn at the point p/q is $2/q^3$; but the ordinate drawn at the point $(mp+1)/mq$ is $2/m^3q^3$, so that in passing along the unit length a distance $1/mq$, the ordinate changes from $2/q^3$ to $2/m^2q^3$.

By increasing m the difference in the abscissae tends to zero, whilst the difference in the ordinates tends to $2/q^3$. There are therefore discontinuities in the lengths of the intervals at every rational point.

7. There is also another kind of discontinuity to which attention may be drawn.

Consider the intervals $\left(\frac{p}{q} - \frac{\epsilon}{q^3},\ \frac{p}{q} + \frac{\epsilon}{q^3}\right)$.

As the number ϵ passes through rational values from 1 to 0, there will be abrupt changes from values for which the four laws are satisfied to values for which they are not satisfied.

It will be noticed that the conditions $\alpha = \beta = 1$ and $\beta^{n-1} > \alpha^n$ are *sufficient* to ensure that the four laws are satisfied.

We have not determined the *necessary* conditions. But it will be evident from the examination of a particular case that such abrupt changes take place. To see this let us find a value of ϵ which satisfies the inequalities

$$\frac{25}{51} - \frac{\epsilon}{51^3} < \frac{1}{2} - \frac{\epsilon}{2^3} < \frac{25}{51} < \frac{1}{2}.$$

Then ϵ must lie between $\frac{10404}{132651}$, which is $\frac{4}{51}$, and $\frac{10404}{132643}$.

Thus ϵ may be taken equal to $\frac{867}{11054}$, which is $\frac{10404}{132648}$ and lies between $\frac{1}{12}$ and $\frac{1}{13}$. But n being 3 in this case, both $\frac{1}{12}$ and $\frac{1}{13}$ satisfy the condition $\beta^2 > \alpha^3$. On the other hand, if $\alpha = 867$, $\beta = 11054$, for which $\beta^2 < \alpha^3$, then, taking $\epsilon = \frac{867}{11054}$, the intervals whose centres are $\frac{1}{2}$ and $\frac{25}{51}$ do not satisfy the first law, because $\frac{25}{51}$ is inside the interval corresponding to $\frac{1}{2}$, but the interval corresponding to $\frac{25}{51}$ is not inside the interval corresponding to $\frac{1}{2}$.

8. It remains only to state in outline the method employed to determine what portion of the unit length is covered by the intervals $(p/q-1/q^3,\ p/q+1/q^3)$. This may be arrived at either

(1) from below by adding together as many of the non-overlapped intervals as is practicable; or

(2) from above by adding together as many of the overlapped intervals as is practicable, and subtracting the sum from the known upper bound

$$2\sum_{q=2}^{\infty}(q^{-2}-q^{-3}).$$

In the first method the calculation was facilitated by the theorem that if s range from (r^2-r+2) to r^2 inclusive,* and from (r^2+2) to (r^2+r+1) inclusive,† then the intervals whose centres are at $1/s$, $2/s$, $3/s \ldots r/s$; and at $(s-1)/s$, $(s-2)/s$, $(s-3)/s$, $\ldots$ $(s-r)/s$ are not overlapped if these fractions are in their lowest terms.

If $s = r^2+1$,‡ then the intervals whose centres are at r/s and $(s-r)/s$ must be excluded from the above list of non-overlapped intervals.

Adding the corresponding intervals together a series was obtained, of

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* For which $\frac{r}{s} < \frac{1}{r-1} - \frac{1}{(r-1)^3} < \frac{r+1}{s}$ if $r > 3$.

† For which $\frac{r}{s} < \frac{1}{r} - \frac{1}{r^3} < \frac{r+1}{s}$.

‡ For which $\frac{r-1}{s} < \frac{1}{r} - \frac{1}{r^3} < \frac{r}{s} < \frac{1}{r} + \frac{1}{r^3}$.

which the first six terms were as follows:—

$$4[S-T_1-\tfrac{1}{2}.2^{-3}]$$
$$+4[(1-2^{-3})S-T_3+2^{-3}T_1-5^{-3}]$$
$$+4[(1-3^{-3})S-T_7+3^{-3}T_2-10^{-3}-\tfrac{1}{2}.8^{-3}]$$
$$+4[(1-2^{-3})S-T_{13}+2^{-3}T_6-17^{-3}]$$
$$+4[(1-5^{-3})S-T_{21}+5^{-3}T_4-26^{-3}]$$
$$+4[(1-2^{-3})(1-3^{-3})S-T_{31}+2^{-3}T_{15}+3^{-3}T_{10}-6^{-3}T_5-37^{-3}],$$

where $$S = 1+2^{-3}+3^{-3}+4^{-3}+ \ldots \text{ to infinity},$$

and $$T_n = 1+2^{-3}+3^{-3}+4^{-3}+\ldots+n^{-3}.$$

Of this series 16 terms were calculated. Their sum was

0·611 282 543 ....

To this sum the lengths of the non-overlapped intervals corresponding to those fractions with denominators not exceeding 250, *which had not been already included in the previous summation,* were added.

This further addition showed that the total portion of the unit length, which is covered by non-overlapping Borel Intervals, certainly exceeds

0·620 705 785 ....

9. In the second method all the overlapped intervals with denominator not greater than 250 are first added together. This part of the work was facilitated by the following theorem:—

To determine whether $\{r/s\}$ lies inside $\{p/q\}$, write $s = qt+b$, $r = pt \pm a$, where $0 \leqslant b \leqslant q-1$, t is a positive integer, and a is a positive integer or zero, then $\{(pt \pm a)/(qt+b)\}$ lies inside $\{p/q\}$ for all values of t, such that if $b > 0$, then $t \geqslant |\, q^2a \mp pqb \,|$, but if $b = 0$, then $t \geqslant q^2a+1$.

The calculation shows that 0·257 395 029 6 ... of the unit length is overlapped by these intervals.

Coming now to the fractions with denominators greater than 250, it was shown first that the number of the Borel Intervals corresponding to denominator s, whose centres lie inside $\{p/q\}$, is either the integral part of $2s/q^3$, or exceeds that number by unity. Consequently it is certainly greater than $(2s/q^3-1)$. So that the length covered by intervals

corresponding to fractions with denominator s, which are overlapped by $\{p/q\}$, exceeds $(2/s^3)(2s/q^3-1)$.

Taking in succession $q = 2, 3, 4, 5$, and 6, it follows that the length covered by intervals corresponding to fractions with denominator s, which are overlapped by the *nine* intervals

$$\{\tfrac{1}{2}\},\ \{\tfrac{1}{3}\},\ \{\tfrac{2}{3}\},\ \{\tfrac{1}{4}\},\ \{\tfrac{3}{4}\},\ \{\tfrac{1}{5}\},\ \{\tfrac{4}{5}\},\ \{\tfrac{1}{6}\},\ \text{and}\ \{\tfrac{5}{6}\},$$

exceeds

$$\frac{2}{s^3}.s.\left\{\frac{2}{2^3}+\frac{4}{3^3}+\frac{4}{4^3}+\frac{4}{5^3}+\frac{4}{6^3}\right\}-9\left(\frac{2}{s^3}\right)=\frac{2}{s^2}(0{\cdot}51116\dot{})-9\left(\frac{2}{s^3}\right). \quad \text{(XXV)}$$

It is next shown that if s be greater than 250, there will always be

(i) *six* intervals corresponding to fractions with denominator s, which are overlapped by the intervals $\{\frac{1}{7}\}$, $\{\frac{2}{7}\}$, $\{\frac{5}{7}\}$, $\{\frac{6}{7}\}$, or by intervals corresponding to fractions whose denominators are of the form $(s/7)$.

(ii) *two* intervals corresponding to fractions with denominator s, which are overlapped by the intervals $\{\frac{1}{8}\}$, $\{\frac{7}{8}\}$, or by intervals corresponding to fractions whose denominators are of the form $(s/4)$ or $(s/8)$.

(iii) *two* intervals corresponding to fractions with denominator s, which are overlapped by the intervals $\{\frac{1}{9}\}$, $\{\frac{2}{9}\}$, $\{\frac{7}{9}\}$, $\{\frac{8}{9}\}$, or by intervals corresponding to fractions with denominators of the form $(s/9)$.

So that there may be added to the count in (XXV) *ten* more lengths of intervals of extent $(2/s^3)$, thus making it exceed $(2/s^2)(\cdot51116\dot{})$ for all values of s greater than 250.

Hence the portion of the unit length covered by non-overlapping Borel Intervals is less than

$$2\sum_{s=2}^{\infty}(s^{-2}-s^{-3})-(0{\cdot}257\ 395\ 029\ 6)-(0{\cdot}51116\dot{})\sum_{251}^{\infty}(2s^{-2})$$

$$=(0{\cdot}885\ 754\ 327\ 2)-(0{\cdot}257\ 395\ 029\ 6)-(0{\cdot}51116\dot{})(0{\cdot}007\ 984\ 0214)$$

$$=(0{\cdot}885\ 754\ 327\ 2)-(0{\cdot}257\ 395\ 029\ 6)-(0{\cdot}004\ 883\ 559\ 6)$$

$$=0{\cdot}623\ 475\ 738.$$

And it was shown in the preceding article that the length covered exceeded

$$0{\cdot}620\,705\,785.$$

[*Note.*—The condition that $\beta^{n-1} > \alpha^n$ was introduced into the argument by Mr. Cooke. This made possible a wide extension of the field of application of the four laws stated in §§ 1, 2, 4, and 5. The argument and calculations in §§ 4, 6, and 8 are also due to him, whilst those in §§ 7 and 9 are due to Miss Wood.

Tables showing the overlapped and non-overlapped intervals corresponding to fractions whose denominators do not exceed 250 have been constructed. These, together with the details of the calculations referred to in §§ 8 and 9, have been deposited in the Department of Pure Mathematics at University College, London.—M. J. M. H.]

CONTRAVARIANT ENVELOPES OF THE PLANE QUARTIC CURVE.

By WILLIAM P. MILNE.

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1. *Introduction.*

In a paper, "The Inflexional Tangents of Plane Quartic Curves,"* Jolliffe establishes the following beautiful result :—*The twenty-four inflexional tangents and the twelve bitangents of a Steiner's Complex touch a class sextic.* From this result follows at once the following very remarkable theorem, which, so far as I have been able to find out, is new, viz :—

There exists at least one configuration of sixty-three conics, each of which touches the harmonic envelope of the quartic curve at six distinct points.

Jolliffe's method of proof consists of rather intricate analysis, and the object of the present paper is to study the problem from the standpoint of synthetic geometry and invariants, thereby rendering the whole configuration more easily capable of visualization, and to obtain further properties.

2. *Contravariant Envelopes.*

It is a known theorem that the general quartic curve Γ can be regarded as the envelope of the polar-conics with respect to a cubic curve U of points on a conic S', and it is also known that the quartic curve can be regarded as generated in this manner in $2^6-1 \equiv 63$ different ways corresponding to the sixty-three systems of quadritangent conics. Also, the quartic curve Γ may be regarded as the locus of points whose polar-lines with respect to U touch S'. The polar-conics of the six points of intersection of S' with the Hessian of U degenerate into six pairs of

* A. E. Jolliffe, *Proc. London Math. Soc.* (2), 23 (1924), 250–278

straight lines, which constitute a Steinerian Complex of bitangents. It is also known that the locus of points whose polar-conics with respect to U touch a given straight line p is the "polo-conic" S of p, and S may also be regarded as the envelope of the polar-lines with respect to U of points on p.

If we wish to find the four points of intersection of p with the quartic Γ, we have merely to find the four points of p whose polar-lines with respect to U constitute the four common tangents of the conics S and S'. According as S and S' have two or three coincident tangents, p will be an ordinary tangent or an inflexional tangent of Γ respectively. If S and S' have double contact, p will be a bitangent.

If p cut Γ in four points constituting a harmonic or equiharmonic range, the tetrad of points at which S is touched by its four common tangents with S' will have an equal cross-ratio. These results can be conveniently expressed in terms of the invariants of S and S', and since S is the polo-conic of p, its coefficients will involve the line-coordinates of p in the second degree. We shall therefore describe for convenience some important contravariant envelopes which we shall require later on.

The tangential equation to the quartic curve Γ *is*

$$(9\Delta\Delta' - \Theta\Theta')^2 = 4(\Theta'^2 - 3\Delta'\Theta)(\Theta^2 - 3\Delta\Theta').$$

It is of class 12, *and its inflexional tangents are tangents of the three curves*, $\Theta'^2 - 3\Delta'\Theta = 0$, $\Theta^2 - 3\Delta\Theta' = 0$, $9\Delta\Delta' - \Theta\Theta' = 0$.

The envelope of a line p, *whose polo-conic* S (*regarded as a locus*) *is apolar to* S' (*regarded as an envelope*) *is the conic* Θ'.

The envelope of a line p, *whose polo-conic* S (*regarded as an envelope*) *is apolar to* S' (*regarded as a locus*) *is the class-quartic* $\Theta = 0$.

The envelope of a line p, *whose polo-conic* S *degenerates into two straight lines, is* $\Delta = 0$, *which is of the sixth degree in the coefficients of* p, *being the Cayleyan of* U *taken twice over.*

The envelope of a line p, *whose polo-conic* S *is such that the four common tangents of* S *and* S' *touch* S *at four points whose cross-ratio is equianharmonic, is the "Equianharmonic Envelope"* $I \equiv \Theta'^2 - 3\Delta'\Theta = 0$. *The class-quartic* Θ *has eight points of contact with* I, *and the eight common tangents at these points of contact each touch the conic* Θ'. *They are the eight lines whose polo-conics* S *are each doubly apolar to* S', *i.e.* S *regarded either as a locus or as an envelope is apolar to* S' *regarded either as an envelope or as a locus respectively.*

The envelope of a line p, whose polo-conic S is such that the four common tangents of S and S' touch S' at four points whose cross-ratio is equianharmonic, is the class octavic $H \equiv \Theta^2 - 3\Delta\Theta' = 0$. The conic Θ' touches H at eight points, and the eight common tangents at their points of contact touch the class-quartic Θ, i.e. they are the lines whose polo-conics S are doubly apolar to S'. The twelve common tangents of Θ and the Cayleyan of U are bitangents of H (since Δ is algebraically the square of the Cayleyan), i.e. the twelve bitangents of the Steinerian Complex are bitangents of H. H also touches the inflexional tangents of Γ.

We thus obtain another theorem akin to Jolliffe's in connexion with the inflexional tangents of a quartic curve and a Steinerian Complex.

A class-octavic exists touching each of the twenty-four inflexional tangents of a quartic curve, and having the twelve lines of a Steiner Complex each as a bitangent.

The class-quartic I and the class-octavic H have $4 \times 8 \equiv 32$ common tangents, consisting of the twenty-four inflexional tangents together with the eight lines whose polo-conics S are each doubly apolar to S'.

3. *Jolliffe's Class-Sextic.*

We shall now consider the class-sextic $K \equiv 9\Delta\Delta' - \Theta\Theta' = 0$. It is easy to show that the vanishing of the invariant $9\Delta\Delta' - \Theta\Theta'$ means that a conic through the four common points of S and S' exists such that all three pairs of opposite vertices of the complete quadrilateral formed by the four common tangents of S and S' are conjugate points. It is plain at once, therefore, that, if S and S' have three coincident common tangents, this invariant condition is satisfied. Also, if the polo-conic S degenerate into two straight lines giving $\Delta = 0$, the invariant condition is satisfied if S' passes through the point of intersection of these two lines giving $\Theta = 0$, *i.e.* if p be one of the twelve lines of the Steinerian Complex. The class-sextic K is also plainly satisfied by $\Theta' = 0$ and $\Delta = 0$, *i.e.* by the six tangents to the Cayleyan of U whose polo-conics are conjugate lines with respect to S'. We note also that, since Δ is the square of the Cayleyan in algebraic language, the conic Θ' touches K at six distinct points, and the class-quartic Θ touches K at twelve distinct points, the tangents at these eighteen points of contact being the common tangents of K and the Cayleyan of U. Thus the class-sextic K possesses a conic touching it at six distinct points. It is there-

fore the reciprocal of the enveloping cone drawn from the singular point of a nodal quartic surface to the surface itself, and belongs to the type of sextics studied by Rohn.*

The class-sextic $K \equiv 9\Delta\Delta' - \Theta\Theta' = 0$ is therefore Jolliffe's class-sextic, and we may therefore enunciate its fundamental properties as follows :—

The envelope of a line p whose polo-conic S with respect to the cubic U is so related to the conic S' that a conic can be drawn through the four common points of S and S', having the three pairs of opposite vertices of the complete quadrilateral formed by the four common tangents of S and S' as conjugate points, is a class-sextic K, touching the twenty-four inflexional tangents of the quartic Γ and the twelve bitangents of a Steiner Complex. The class-sextic K is touched at six distinct points by the conic Θ'. The six common tangents of Θ' and K touch the Cayleyan of U.

4. *The Harmonic Envelope J.*

Consider next the envelope of lines p cutting the quartic Γ in four points whose cross-ratio is harmonic. The equation to this envelope is :

$$J \equiv 27\Delta'^2\Delta - 9\Delta'\Theta\Theta' + 2\Theta'^3 \equiv 3\Delta' K + 2\Theta' I = 0.$$

Plainly the conic Θ' touches J at six distinct points, viz. the points where it touches K.

Thus the sixty-three conics Θ' each touch the harmonic envelope J at six distinct points. Also Jolliffe's class-sextic K touches J at the same six points. The six common tangents of J, K, and Θ' at their six points of contact all touch the Cayleyan of U, i.e. the class-cubic which touches each of the twelve bitangents of the Steiner Complex.

The nature of J in the light of its having at least one configuration of sixty-three contact conics as described above requires further elucidation.

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* K. Rohn, "Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte und ihrer Gestaltung", *Preisschrift der Jablonowskischen Gesellschaft*, *Leipzig*, 26 (1886).

ON THE FACTORIZATION OF CERTAIN FUNCTIONS RATIONAL AND INTEGRAL IN ONE OF SEVERAL VARIABLES.

By S. BEATTY.

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Introduction.

It is well known* that certain functions which are rational and integral in one variable, and have coefficients expansible in series of powers related to the remaining variables, are products of functions of the same general type, but linear in that one variable. It is usual to demonstrate the existence of one such linear factor of a function of this kind, and to infer the existence of a representation of the function as a product of linear factors. This procedure requires the choice of certain constants entering to be made in such a way as to ensure that some given one of the n linear factors, supposed all different, will be obtained. In the present paper, the existence of the product representation is established directly, which not only avoids the necessity of exercising choice as above, but is applicable should some of the linear factors be repeated. Moreover, the method for effecting the factorization in the case of $N+1$ additional variables flows naturally from and employs the result of that for N.

1. It will be supposed that $f(z,u)$ denotes

$$u^n+f_1u^{n-1}+\ldots+f_n,$$

in which each f_t is expansible formally as a series in integral powers of z, of which at most only a finite number may be negative. In case u^n does not exhaust $f(z,u)$, there is a least exponent,† hereafter denoted

* Newton, *Opuscula collegit Castillioneus*, 1 (1744), 39, 367. Puiseux, *Journal de Math.*, 15 (1850), 365.

† Hensel und Landsberg, *Theorie der Alg. Funktionen einer Var.*, 49.

by λ, appearing in the expansions of

$$(f_t)^{1/t} \quad (t = 1, 2, \ldots, n).$$

The expansions of f_t are, therefore,

$$f_t = a_t z^{t\lambda} + \text{higher powers of } z \quad (t = 1, 2, \ldots, n),$$

in which at least one of $a_1, a_2, \ldots, a_n$ is different from zero. The form of $f(z, u)$ is, therefore,

$$(1) \quad f(z, u) = u^n + z^\lambda [(a_1 + \ldots) u^{n-1} + z^\lambda (a_2 + \ldots) u^{n-2} + \ldots + z^{(n-1)\lambda}(a_n + \ldots)].$$

The part of (1) supplied by

$$u^n + a_1 z^\lambda u^{n-1} + a_2 z^{2\lambda} u^{n-2} + \ldots + a_n z^{n\lambda}$$

may be written as a product of factors in the form

$$(2) \qquad \prod_{s=1}^{r} (u - a_s z^\lambda)^{p_s},$$

in which $a_1, a_2, \ldots, a_r$ are all different. There is a least positive integer ρ such that $\rho\lambda$ is integral. If in particular $r = 1$, then $\rho = 1$. In case (2) does not exhaust (1), a representation of $f(z, u)$ is

$$(3) \quad f(z, u) = \text{the expression (2)} + z^{\lambda_1} [(b_1 + \ldots) u^{n-1} + z^\lambda (b_2 + \ldots) u^{n-2} + \ldots + z^{(n-1)\lambda}(b_n + \ldots)],$$

in which λ_1 is determined, and that uniquely, so that at least one of $b_1, b_2, \ldots, b_n$ is different from zero. The exponent λ_1 exceeds λ, and $\rho\lambda_1$ is integral.

The n numbers A_{st} determined through the identity in θ

$$(4) \qquad (b_1\theta^{n-1} + b_2\theta^{n-2} + \ldots + b_n) \prod_{s=1}^{r} \left(\frac{1}{\theta - a_s}\right)^{p_s} = \sum_{s=1}^{r} \sum_{t=1}^{p_s} \frac{A_{st}}{(\theta - a_s)^t}$$

are such that the expression

$$(5) \qquad \prod_{s=1}^{r} [(u - a_s z^\lambda)^{p_s} + A_{s1} z^{\lambda_1} (u - a_s z^\lambda)^{p_s - 1} + \ldots + A_{sp_s} z^{\lambda_1 + (p_s - 1)\lambda}],$$

on being arranged after the manner of (3), may differ from (3) only in coefficients of terms following $b_1, b_2, \ldots, b_n$. In case (3) and (5) differ, a representation of $f(z, u)$ is

$$(6) \quad f(z, u) = \text{the expression (5)} + z^{\lambda_2} [(c_1 + \ldots) u^{n-1} + z^\lambda (c_2 + \ldots) u^{n-2} + \ldots + z^{(n-1)\lambda}(c_n + \ldots)],$$

in which λ_2 is determined, and that uniquely, so that at least one of $c_1, c_2, \ldots, c_n$ is different from zero. The exponent λ_2 exceeds λ_1, and $\rho\lambda_2$ is integral.

The type of process described in the preceding paragraph can be continued. The next application of it, for example, is to determine the n numbers B_{st} from the identity in θ

$$(c_1\theta^{n-1}+c_2\theta^{n-2}+\ldots+c_n)\prod_{s=1}^{r}\left(\frac{1}{\theta-a_s}\right)^{p_s}=\sum_{s=1}^{r}\sum_{t=1}^{p_s}\frac{B_{st}}{(\theta-a_s)^t},$$

corresponding to (4), to construct the expression

$$(7)\quad \prod_{s=1}^{r}\left[(u-a_s z^{\lambda})^{p_s}+(A_{s1}z^{\lambda_1}+B_{s1}z^{\lambda_2})(u-a_s z^{\lambda})^{p_s-1}\right.$$
$$\left.+\ldots+(A_{sp_s}z^{\lambda_1+(p_s-1)\lambda}+B_{sp_s}z^{\lambda_2+(p_s-1)\lambda})\right],$$

corresponding to (5), to compare (7) with (6), just as formerly (5) was compared with (3), and in case they differ to obtain a representation of $f(z,u)$ in the form

$$(8)\quad f(z,u)=\text{the expression (7)}$$
$$+z^{\lambda_3}[d_1+\ldots)u^{n-1}+z^{\lambda}(d_2+\ldots)u^{n-2}+\ldots+z^{(n-1)\lambda}(d_n+\ldots)],$$

in which λ_3 is determined, and that uniquely, so that at least one of $d_1, d_2, \ldots, d_n$ is different from zero. The exponent λ_3 exceeds λ_2, and $\rho\lambda_3$ is integral. The order of construction of number and number groups is, therefore,

$$\lambda,\ a_t;\quad a_s,\ \lambda_1,\ b_t;\quad A_{st},\ \lambda_2,\ c_t;\quad B_{st},\ \lambda_3,\ d_t;\quad \ldots.$$

Forms (6), (8) are representations of $f(z,u)$ obtained from applying the process once, twice respectively; the processes furnishing forms (1), (3) may be regarded as preliminary. If there is no limit to the number of times the process can be applied, there is a corresponding representation of $f(z,u)$ associated with each finite number of applications of the process. In the second half of such representation, the least power of z within the bracket is never less than a certain finite number, and the power of z without the bracket increases indefinitely with the number of applications of the process. Therefore, as a result of the indefinite application of the process, $f(z,u)$ is represented formally as a product of factors

$$(9)\qquad f(z,u)=\prod_{s=1}^{r}\left[(u-a_s z^{\lambda})^{p_s}+f_{s_1}(u-a_s z^{\lambda})^{p_s-1}+\ldots+f_{sp_s}\right],$$

in which f_{st} denotes

$$A_{st}z^{\lambda_1+(t-1)\lambda}+B_{st}z^{\lambda_2+(t-1)\lambda}+\ldots.$$

If there is a limit to the number of times the process can be applied, a representation of $f(z,u)$ in the form (9) results from making as many applications as possible. In this case, each f_{st} terminates. The form (9) is the form (2), when and only when there is no number λ_1. This may be called the method of formal factorization.

It happens usually that $r=n$, that is, that the factors of $f(z,u)$ as supplied by (9) are all linear in u. If one of them,

$$(u-a_sz^{\lambda})^{p_s}+f_{s1}(u-a_sz^{\lambda})^{p_s-1}+\ldots+f_{sp_s},$$

is of degree in u greater than the first, the method of formal factorization can be applied to it as formerly to $f(z,u)$, the former rôle of u being taken by $u-a_sz^{\lambda}$ and that of z by $z^{1/\rho}$. The method of formal factorization, on being applied to $f(z,u)$ and all subsequent non-linear factors so obtained, either furnishes, after a finite number of applications, a representation of $f(z,u)$ as a product of linear factors, or does not avail in any finite number of applications, however large, to factor at least one non-linear factor. In this latter case, however, the non-linear factor is itself a power of a linear factor. For, it may be supposed that

$$g(\xi,\nu)=\nu^m+g_1\nu^{m-1}+\ldots+g_m, \tag{10}$$

in which each g_t involves only integral powers of ξ, of which at most only a finite number may be negative, is a non-linear factor, which does not break up formally into a product of factors, however frequently the method of formal factorization is applied to it. All powers of ξ resulting from the application of the method are integral. Throughout q successive applications of the method, the numbers corresponding to λ may be denoted by $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q)}$ respectively, each at least *one* greater than the preceding, and the numbers corresponding to a_1 by $a^{(1)}, a^{(2)}, \ldots, a^{(q)}$ respectively. The number corresponding to r is *one* in each case. Where $\nu^{(q)}$ denotes

$$\nu-a^{(1)}\xi^{\lambda^{(1)}}-a^{(2)}\xi^{\lambda^{(2)}}-\ldots-a^{(q)}\xi^{\lambda^{(q)}},$$

$$g(\xi,\nu)=(\nu^{(q)})^m+g_1^{(q)}(\nu^{(q)})^{m-1}+\ldots+g_m^{(q)},$$

and the least power of ξ in $g_t^{(q)}$ is not less than $t(\lambda^{(1)}+q)$. Hence $g_t^{(q)}(v^{(q)})^{m-t}$ contains no part of (10) in which the power of ξ is less than the smaller of $t\lambda^{(1)}+tq$, $m\lambda^{(1)}+tq$. This being true for $t=1, 2, \ldots, m$,

it follows from taking q larger and larger that $g(\xi, \nu)$ is represented by

$$g(\xi, \nu) = (\nu - \alpha^{(1)}\xi^{\lambda^{(1)}} - \alpha^{(2)}\xi^{\lambda^{(2)}} - \ldots)^m.$$

Therefore, by application of the method of formal factorization, repeated if necessary, $f(z, u)$ is represented as a product of linear factors in the form

$$f(z, u) = (u - F_1)(u - F_2)\ldots(u - F_n). \tag{11}$$

It is to be noted that the expansions $F_1, F_2, \ldots, F_n$ proceed according to ascending integral powers of $z^{1/\sigma}$, where σ is a positive integer not greater than n. Moreover, $f(z, F_t)$ vanishes identically for $t = 1, 2, \ldots, n$, and the factorization in (11) is unique. If the purpose is to obtain merely the first few terms in the expansions $F_1, F_2, \ldots, F_n$, it suffices to carry out the first few processes in the first few applications of the method leading to formal linear factorization.

Under certain additional limitations, the linear factorization rests on a basis of value as well as form. Suppose, for example, that $f_1, f_2, \ldots, f_n$ are all convergent in a circle about the origin with the possible exception of the origin itself. Suppose $u - F$ is a factor of $f(z, u)$ and to the power $p+1$ exactly. The standard method of Cauchy, applied to $f(z, u)$ if $p = 0$, and to the p-th partial derivative of $f(z, u)$ with regard to u if $p > 0$, leads to the conclusion that F is convergent in a circle about the origin with the possible exception of the origin itself.

Important special cases arise from taking $f_1, f_2, \ldots, f_n$ rational or, more generally, algebraic functions of z. There is, of course, no loss of generality in dealing with the origin as centre of expansion.

2. It will be supposed that $e(y, z)$ denotes

$$z^k + e_1 z^{k-1} + \ldots + e_k,$$

in which each e_t is expansible formally as a series in integral powers of y, of which at most only a finite number may be negative; it follows that

$$e(y, z) = (z - E_1)(z - E_2)\ldots(z - E_k),$$

in which each E_t proceeds according to ascending integral powers of $y^{1/\delta}$, of which at most only a finite number may be negative, and where δ is a positive integer not greater than k. A particular one of the expansions $E_1, E_2, \ldots, E_k$ may be selected and denoted by E. It will be supposed that $f(y, z, u)$ denotes

$$u^n + f_1 u^{n-1} + \ldots + f_n,$$

in which each f_t is expansible formally as a double series, each term being the product of $y^{1/\delta}$ to an integral power, and $z-E$ likewise to an integral power. It will be supposed that there is a finite number less than any of the powers of $z-E$, and also a finite number less than any of the powers of y in those terms where the powers of $z-E$ do not exceed any number assigned in advance. It is clear that there need not be a least power of y if all the terms are considered. Each f_t may be considered as a power series in $z-E$ with coefficients which are power series in $y^{1/\delta}$.

In case u^n does not exhaust $f(y, z, u)$, corresponding to (1), there is the form for $f(y, z, u)$

$$(12)\quad f(y, z, u) = u^n+(z-E)^\lambda[(a_1+\ldots)u^{n-1}+(z-E)^\lambda(a_2+\ldots)u^{n-2} +\ldots+(z-E)^{(n-1)\lambda}(a_n+\ldots)],$$

in which at least one of $a_1, a_2, \ldots, a_n$, which are power series in $y^{1/\delta}$, does not vanish identically. The number λ is determined as in § 1, and ρ denotes the least positive integer such that $\rho\lambda$ is integral. Corresponding to (2), there is the form

$$(13)\quad \prod_{s=1}^{r}[u-a_s(z-E)^\lambda]^{p_s},$$

in which $a_1, a_2, \ldots, a_r$ are formal power series all different and determined as provided for in § 1, so that

$$u^n+a_1u^{n-1}+\ldots+a_n = \prod_{s=1}^{r}(u-a_s)^{p_s}.$$

Each series a_s proceeds according to ascending integral powers of $y^{1/\delta'}$, of which at most only a finite number may be negative, and δ'/δ is an integer not greater than n. In the special case where $r=1$, it appears that $\rho=1$, $\delta'=\delta$. In case (13) does not exhaust (12), a representation of $f(y, z, u)$, corresponding to (3), is

$$(14)\quad f(y, z, u) = \text{the expression (13)}+(z-E)^{\lambda_1}[(b_1+\ldots)u^{n-1} +(z-E)^\lambda(b_2+\ldots)u^{n-2}+\ldots+(z-E)^{(n-1)\lambda}(b_n+\ldots)],$$

in which λ_1 is determined, and that uniquely, so that at least one of $b_1, b_2, \ldots, b_n$, which are series proceeding according to ascending integral powers of $y^{1/\delta'}$, does not vanish identically. The exponent λ_1 exceeds λ, and $\rho\lambda_1$ is integral. Moreover, each coefficient of a power of $z-E$ following b_t is a formal series proceeding according to ascending integral

powers of $y^{1/\delta'}$. From the identity in θ

$$(15) \qquad (b_1\theta^{n-1}+b_2\theta^{n-2}+\ldots+b_n)\prod_{s=1}^{r}\left(\frac{1}{\theta-a_s}\right)^{p_s}=\sum_{s=1}^{r}\sum_{t=1}^{p_s}\frac{A_{st}}{(\theta-a_s)^t},$$

corresponding to (4), n formal series A_{st} may be determined, from which the form

$$(16) \quad \prod_{s=1}^{r}\left[\{u-a_s(z-E)^{\lambda}\}^{p_s}+A_{s1}(z-E)^{\lambda_1}\{u-a_s(z-E)^{\lambda}\}^{p_s-1}+\ldots+A_{sp_s}(z-E)^{\lambda_1+(p_s-1)\lambda}\right],$$

corresponding to (5), may be determined. Form (16) may differ from (14) only in coefficients of powers of $z-E$ following $b_1, b_2, \ldots, b_n$. If (14) and (16) differ, $f(y, z, u)$ may be written in the form

$$(17) \quad f(y, z, u) = \text{the expression (16)}+(z-E)^{\lambda_2}[(c_1+\ldots)u^{n-1}+(z-E)^{\lambda}(c_2+\ldots)u^{n-2}+\ldots+(z-E)^{(n-1)\lambda}(c_n+\ldots)],$$

in which λ_2 is determined, and that uniquely, so that at least one of $c_1, c_2, \ldots, c_n$, which are series proceeding according to ascending integral powers of $y^{1/\delta'}$, does not vanish identically. The exponent λ_2 exceels λ_1 and $\rho\lambda_2$ is integral. Evidently the correspondence with the procedure in § 1 continues, and results in affording a representation for $f(y, z, u)$ in the form

$$(18) \quad f(y, z, u) = \prod_{s=1}^{r}\left[\{u-a_s(z-E)^{\lambda}\}^{p_s}+f_{s1}\{u-a_s(z-E)^{\lambda}\}^{p_s-1}+\ldots+f_{sp_s}\right],$$

corresponding to (9), in which f_{st} denotes

$$A_{st}(z-E)^{\lambda_1+(t-1)\lambda}+B_{st}(z-E)^{\lambda_2+(t-1)\lambda}+\ldots.$$

All coefficients of powers of $z-E$ in each f_{st} are formal series proceeding according to ascending integral powers of $y^{1/\delta'}$. Moreover, there is a finite number less than any of the powers of y in those terms of each f_{st} where the powers of $z-E$ do not exceed any number assigned in advance.

The method of formal factorization thus outlined may be applied to such of the factors in (18) as are not linear in u. If the method of formal factorization, however frequently applied, fails to represent a non-linear factor as a product of factors, it is because the non-linear factor is a power of a linear factor, and in that case the double series in y, $z-E$ proceed according to multiples of the same exponents for both the non-linear factor and the resulting linear factor. Hence the method of formal factorization, repeated if necessary, affords a representation of

$f(y, z, u)$ in the form

$$f(y, z, u) = (u-F_1)(u-F_2)\ldots(u-F_n), \tag{19}$$

corresponding to (11), in which each F_t is a double series in y, $z-E$, the powers of y being multiples of $1/\delta''$ where δ'' is a multiple of δ', and the powers of $z-E$ being multiples of $1/\rho'$, where ρ' is a multiple of ρ. Moreover, $f(y, z, F_t)$ vanishes identically for $t = 1, 2, \ldots, n$, and the factorization in (19) is unique.

Under certain additional limitations the linear factorization rests on a basis of value as well as form. It will be supposed that a number h exists such that, on expanding each f_t as a double series in y, $z-E/y^h$, there is a least exponent of y, and that each f_t is convergent for sufficiently small values of $|y|$, $|z-E/y^h|$ with possible exception of the value zero. Suppose $u-F$ is a factor of $f(y, z, u)$ and to the power $p+1$ exactly. The standard method* of Cauchy, applied to $f(y, z, u)$ if $p = 0$, and to the p-th partial derivative of $f(y, z, u)$ with regard to u if $p > 0$, leads to the conclusion that a number $H \geqslant h$ exists such that F is convergent for sufficiently small values of $|y|$, $|z-E/y^H|$ with possible exception of the value zero.

Important special cases arise from replacing $z-E$ in the argument by $z-a$ or $1/z$. Also E may be an algebraic function, and each f rational, or, more generally, algebraic in the variables y, z in the vicinity of $y = 0$, $z-E = 0$.

The manner of extending the treatment step by step to any number of remaining variables is evident.

* Hensel, *Acta Mathematica*, 23 (1900), 390–395.

THE CONVERGENCE OF CERTAIN INTEGRALS

By E. C. TITCHMARSH.

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1. An application of the theory of Fourier series to the theory of integrable functions by Lusin* has given the following theorem :—

If $f(x)$ is any function of integrable square, then the integral

$$\int_0^1 \frac{f(x+t)-f(x-t)}{t}\,dt,$$

considered as $\lim_{\epsilon\to 0}\int_\epsilon^1$, *exists for almost all values of* x.

A proof is also given by Plessner,† and a direct proof, without using the theory of Fourier series, by Besikovitch.‡

The theorem gives rise to the following observations :—

(i) The condition that the square of $f(x)$ should be integrable is not essential. The result is extended by Plessner to all functions which have a Lebesgue integral, whether their squares are integrable or not.

(ii) The function $1/t$ cannot be replaced by any function which tends to infinity more rapidly as $t\to 0$; for then the integral may diverge in a set of positive measure, even if the function $f(x)$ is continuous. The proof of this is the first object of the present paper (Theorem I).

(iii) The convergence of the integral is due not to the smallness of $f(x+t)-f(x-t)$ for small values of t, but to the interference of positive

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* N. Lusin, *Comptes Rendus*, 156 (June 2nd, 1913), 1655–1658.

† A. Plessner, *Inauguraldissertation*, Giessen (1923).

‡ A. Besikovitch, *Fundamenta Mathematicae*, 4 (1923), 172–195.

and negative values; for the integral

$$\int_0^1 \frac{|f(x+t)-f(x-t)|}{t}\,dt$$

may diverge in a set of positive measure. This is stated without proof by Lusin and Besikovitch. The result is included in Theorem II below.

(iv) The apparently similar integral

$$\int_0^1 \frac{f(x+t)-f(x)}{t}\,dt$$

may diverge in a set of positive measure. This has been deduced from the theory of Fourier series by Hardy and Littlewood.* It is included in Theorem III below.

2. I prove the following three theorems :—

THEOREM I.—*If $\phi(t)$ is a positive continuous increasing function of t, and $\phi(t)/t \to 0$ when $t \to 0$, then there is a continuous function $f(x)$ such that the integral*

$$\int_0^1 \frac{f(x+t)-f(x-t)}{\phi(t)}\,dt \tag{2.1}$$

is divergent for almost all values of x.

THEOREM II.—*If $\phi(t)$ is a positive continuous increasing function of t, and the integral*

$$\int_0^1 \frac{dt}{\phi(t)}$$

is divergent, then there is a continuous function $f(x)$ such that the integral

$$\int_0^1 \frac{|f(x+t)-f(x-t)|}{\phi(t)}\,dt \tag{2.2}$$

is divergent for almost all values of x.

THEOREM III.—*If $\phi(t)$ satisfies the conditions of Theorem* II, *then there is a continuous function $f(x)$ such that the integral*

$$\int_0^1 \frac{f(x+t)-f(x)}{\phi(t)}\,dt \tag{2.3}$$

is divergent for almost all values of x.

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* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 211–246 (235).

For example, there is a continuous function for which

$$\int_0^1 \{f(x+t)-f(x-t)\} \log(1/t)\, \frac{dt}{t}$$

diverges almost everywhere; and a continuous function for which

$$\int_0^1 |f(x+t)-f(x-t)|\, \frac{dt}{t \log 1/t}$$

diverges almost everywhere, while

$$\int_0^1 \{f(x+t)-f(x-t)\}\, \frac{dt}{t \log 1/t}$$

converges almost everywhere; but

$$\int_0^1 |f(x+t)-f(x-t|\, \frac{dt}{t \log^2 1/t}$$

converges almost everywhere, since

$$\int_0^1 \frac{dt}{t \log^2 1/t}$$

is convergent.

The method may be extended to show that, in Theorem II, $|f(x+t)-f(x-t)|$ may be replaced by $|f(x+t)-f(x-t)|^p$ $(p>1)$, or by differences of higher order; and Theorem III is also true with differences of higher order, provided they contain a term independent of t, *e.g.* $f(x+2t)+f(x-2t)-2f(x)$.

3. *The method of proof.*—The beginning of each proof is the same. Let

$$\sum_{n=1}^{\infty} c_n \tag{3.1}$$

be a convergent series of positive terms; $r_1, r_2, \ldots$ an increasing sequence of positive integers; and $l_1, l_2, \ldots$ a decreasing sequence of positive numbers such that $2r_n l_n < 1$ for all values of n. We define a sequence of functions $f_1(x), f_2(x), \ldots$ as follows. Let

$$f_n(x) = \frac{c_n}{l_n}\left(x - \frac{q}{r_n} + l_n\right) \quad \left(\frac{q}{r_n} - l_n \leqslant x \leqslant \frac{q}{r_n}\right),$$

$$= \frac{c_n}{l_n}\left(\frac{q}{r_n} + l_n - x\right) \quad \left(\frac{q}{r_n} \leqslant x \leqslant \frac{q}{r_n} + l_n\right),$$

for $q = -r_n, -r_n+1, \ldots, 0, 1, \ldots, 2r_n-1$; and $f_n(x) = 0$ elsewhere.

Thus $f_n(x)$ is a continuous function, and

$$0 \leqslant f_n(x) \leqslant c_n.$$

Let
$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

The series is uniformly convergent, so that $f(x)$ is a continuous function. It will be proved that the sequences r_n, l_n and c_n, can be so chosen that $f(x)$ has the properties stated. In each case it is arranged that the integral is dominated by one particular term of the series for $f(x)$.

4. *Proof of Theorem* I.—If $0 < a_n < b_n < 1$, it follows from the uniform convergence of the series for $f(x)$ that

$$\int_{a_n}^{b_n} \frac{f(x+t)-f(x-t)}{\phi(t)}\,dt = \sum_{\nu=1}^{\infty} \int_{a_n}^{b_n} \frac{f_\nu(x+t)-f_\nu(x-t)}{\phi(t)}\,dt$$

$$= \int_{a_n}^{b_n} \frac{f_n(x+t)-f_n(x-t)}{\phi(t)}\,dt + \sum_{\nu=1}^{n-1} \int_{a_n}^{b_n} \frac{f_\nu(x+t)-f_\nu(x-t)}{\phi(t)}\,dt$$

$$+ \sum_{\nu=n+1}^{\infty} \int_{a_n}^{b_n} \frac{f_\nu(x+t)-f_\nu(x-t)}{\phi(t)}\,dt$$

$$= J_1+J_2+J_3,$$

say. We shall consider each of these terms separately.

4.1. J_1.—We shall in the first place choose a_n and b_n so that the interval (a_n, b_n) contains just one of the intervals where $f_n(x+t) \neq 0$, and no interval where $f_n(x-t) \neq 0$. Now $f_n(x+t) = 0$ for

$$\frac{q-1}{r_n}+l_n < x+t < \frac{q}{r_n}-l_n, \quad \frac{q}{r_n}+l_n < x+t < \frac{q+1}{r_n}-l_n,$$

and $f_n(x-t) = 0$ for

$$\frac{q-1}{r_n}+l_n < x-t < \frac{q}{r_n}-l_n.$$

Hence, writing $\xi = q/r_n - x$, we have $f_n(x+t) = 0$ for

$$\xi+l_n-\frac{1}{r_n} < t < \xi-l_n, \quad \xi+l_n < t < \xi+\frac{1}{r_n}-l_n$$

and $f_n(x-t) = 0$ for

$$-\xi+l_n < t < \frac{1}{r_n}-l_n-\xi.$$

Hence the required condition is fulfilled if

$$\xi+l_n-\frac{1}{r_n}<a_n<\xi-l_n,$$

$$\xi+l_n<b_n<\xi+\frac{1}{r_n}-l_n,$$

and
$$-\xi+l_n<a_n<b_n<\frac{1}{r_n}-l_n-\xi,$$

i.e. if

$$\text{Max}\left(a_n+l_n,\ b_n+l_n-\frac{1}{r_n},\ l_n-a_n\right)$$

$$<\xi<\text{Min}\left(a_n+\frac{1}{r_n}-l_n,\ b_n-l_n,\ \frac{1}{r_n}-l_n-b_n\right).$$

Suppose that

(4.11) $$0<l_n<a_n<b_n<\frac{1}{2r_n}.$$

Then the required condition is fulfilled if

$$a_n+l_n<\xi<b_n-l_n,$$

and we have then

$$\int_{a_n}^{b_n}\frac{f_n(x+t)-f_n(x-t)}{\phi(t)}\,dt=\int_{\xi-l_n}^{\xi+l_n}\frac{f_n(x+t)}{\phi(t)}\,dt$$

$$>\frac{1}{\phi(\xi+l_n)}\int_{\xi-l_n}^{\xi+l_n}f_n(x+t)\,dt>\frac{l_nc_n}{\phi(b_n)}.$$

Hence

(4.12) $$J_1>\frac{l_nc_n}{\phi(b_n)}$$

in a set E_n consisting of r_n intervals, each of length $b_n-a_n-2l_n$, the left-hand end-points being at equal distances $1/r_n$.

4.2. J_2.—A typical term of J_2 is

$$\int_{a_n}^{b_n}\frac{f_\nu(x+t)-f_\nu(x-t)}{\phi(t)}\,dt,$$

and this is zero if $f_\nu(u)$ is zero for $x-b_n<u<x+b_n$, *i.e.* if the interval $(x-b_n,\ x+b_n)$ does not include any part of any of the r_ν intervals between 0 and 1, each of length $2l_\nu$, where $f_\nu(u)\neq 0$. Hence this term is different from zero if x lies in one of the intervals where $f_\nu(u)\neq 0$, which are

included in the set F where $f(x) \neq 0$, or if x lies in one of $2r_\nu$ intervals each of length b_n, adjacent to the intervals where $f_\nu(u) \neq 0$. This set is of measure $2r_\nu b_n$. It follows that $J_2 = 0$, except possibly when x lies in the set F, or when x lies in a set G_n of measure

$$2b_n \sum_{\nu=1}^{n-1} r_\nu.$$

4.3. J_3.—We now require the following

LEMMA.—*For all values of a and b*

$$\left| \int_a^b \{f_n(x+t)-f_n(x-t)\}\, dt \right| \leqslant 4l_n c_n. \tag{4.31}$$

If an interval of length $b-a$ contains just p intervals where $f_n(x) \neq 0$, then

$$r_n(b-a-2l_n)-1 < p \leqslant r_n(b-a-2l_n)+1;$$

for the distance from the left of the first interval to the right of the p-th is $(p-1)/r_n+2l_n$ and this is not greater than $b-a$; but (adding a new interval at each end) we have $(p+1)/r_n+2l_n$ greater than $b-a$.

It follows that

$$\{r_n(b-a-2l_n)-1\}\, l_n c_n \leqslant \int_a^b f_n(x \pm t)\, dt \leqslant \{r_n(b-a-2l_n)+3\}\, l_n c_n,$$

which proves the lemma.

From this lemma and the Second Mean Value Theorem it follows that

$$\left| \int_{a_n}^{b_n} \frac{f_\nu(x+t)-f_\nu(x-t)}{\phi(t)}\, dt \right| = \frac{1}{\phi(a_n)} \left| \int_{a_n}^{b} \{f_\nu(x+t)-f_\nu(x-t)\}\, dt \right| \leqslant \frac{4l_\nu c_\nu}{\phi(a_n)}.$$

Hence

$$|J_3| \leqslant \frac{4}{\phi(a_n)} \sum_{\nu=n+1}^{\infty} l_\nu c_\nu. \tag{4.32}$$

4.4. *The set where the integral* (2.1) *diverges.*—Let

$$\frac{4}{\phi(a_n)} \sum_{\nu=n+1}^{\infty} l_\nu c_\nu < \tfrac{1}{2} \frac{l_n c_n}{\phi(b_n)}. \tag{4.41}$$

Then $|J_3| < \frac{1}{2}|J_1|$, so that

$$\left| \int_{a_n}^{b_n} \frac{f(x+t)-f(x-t)}{\phi(t)}\, dt \right| > \tfrac{1}{2} \frac{l_n c_n}{\phi(b_n)}$$

at all points of E_n not belonging to G_n or to F. Let E'_n be the set of points of E_n not belonging to G_n, and let

$$\frac{l_n c_n}{\phi(b_n)} \to \infty . \tag{4.42}$$

Then the integral (2.1) is divergent at any point which belongs to an infinity of the sets E'_n, and which is not a point of F. Let H be the set of points belonging to an infinity of sets E'_n. Then the integral is divergent in a set D, where

$$m(D) \geqslant m(H) - m(F) \geqslant m(H) - 2 \sum_{\nu=1}^{\infty} r_\nu l_\nu,$$

supposing that the last series is convergent. But

$$f(x) = \sum_{\nu=1}^{p} f_\nu(x) + \sum_{p+1}^{\infty} f_\nu(x) = f^{(1)}(x) + f^{(2)}(x),$$

say. Now the integral involving $f^{(1)}(x)$ can only diverge at a finite number of points, whereas, applying the above argument to $f^{(2)}(x)$, we see that the integral involving $f^{(2)}(x)$ diverges in a set of measure not less than

$$m(H) - 2 \sum_{\nu=p+1}^{\infty} r_\nu l_\nu,$$

which may be made as near to $m(H)$ as we please. Hence the integral is divergent almost everywhere in H.

4.5. *Proof that* $m(H) = 1$.—Let

$$b_n \sum_{\nu=1}^{n-1} r_\nu = o\{r_n(b_n - a_n - 2l_n)\}, \tag{4.51}$$

i.e.

$$m(G_n) = o\{m(E_n)\},$$

and suppose that the series

$$\sum_{n=1}^{\infty} r_n(b_n - a_n - 2l_n) \tag{4.52}$$

is divergent. Let H_n be the set of points belonging to at least one of E'_{n+1}, E'_{n+2}, ... Then it is sufficient to prove that $m(H_n) = 1$ for all values of n.

Let L_m be the set of points belonging to E'_{n+m}, but not to any of the

sets E'_{n+1}, E'_{n+2}, ..., E'_{n+m-1}. Then

$$m(H_n) = \sum_{m=1}^{\infty} m(L_m).$$

Now E_n consists of r_n intervals, and G_n of at most

$$2\sum_{\nu=1}^{n-1} r_\nu = o(r_n)$$

intervals. Hence (for $n > n_0$) E'_n consists of at most $2r_n$ intervals. Now, as in the lemma of § 4.3, an interval of length L contains the whole of or part of at most

$$r_{n+m}(L-b_{n+m}+a_{n+m}+2l_{n+m})+3 < r_{n+m}L+3$$

of the intervals which make up E_{n+m}. Hence the set

$$E'_{n+1}+\ldots+E'_{n+m-1} = L_1+\ldots+L_{m-1}$$

contains the whole or part of at most

$$r_{n+m}\{m(L_1)+\ldots+m(L_{m-1})\}+3\sum_{\nu=n+1}^{n+m-1} r_\nu$$

such intervals. Hence the number of intervals of E_{n+m} lying entirely outside $L_1+\ldots+L_{m-1}$ is at least

$$r_{n+m}\{1-m(L_1)-\ldots-m(L_{m-1})\}-o(r_{n+m}).$$

Hence

$$m(L_m) \geqslant r_{n+m}(b_{n+m}-a_{n+m}-2l_{n+m})\{1-m(L_1)-\ldots-m(L_{m-1})\},$$

$$-o\{r_{n+m}(b_{n+m}-a_{n+m}-2l_{n+m})\}-o\{m(G_{n+m})\}$$

$$> \tfrac{1}{2}r_{n+m}(b_{n+m}-a_{n+m}-2l_{n+m})\{1-m(L_1)-\ldots-m(L_{m-1})\}$$

for $m > m_0$. But if $\sum_1^{\infty} m(L_m) = A < 1$,

it follows that

$$m(L_m) > \tfrac{1}{2}(1-A)\,r_{n+m}(b_{n+m}-a_{n+m}-2l_{n+m})$$

which is impossible, since the series (4.52) is divergent. Hence

$$m(H_n) = \Sigma m(L_m) = 1.$$

4.6. *Consistency of the conditions assumed.*—We have now only to prove that the various conditions assumed, viz. (4.11), (4.41),

(4 . 42), (4 . 51), and the convergence of

$$\Sigma c_n, \quad \Sigma r_n l_n$$

and the divergence of $\Sigma r_n(b_n - a_n - 2l_n)$

are consistent.

Observe first that we may suppose that, for a sequence of values of t tending to zero,

$$\phi(t) \leqslant 4\phi(\tfrac{1}{2}t).$$

For suppose, on the contrary, that for $t \leqslant t_0$

$$\phi(t) > 4\phi(\tfrac{1}{2}t).$$

Then $$\phi(t_0) > 2^{2n}\phi(t_0/2^n).$$

Hence, if $$t_0/2^n > t > t_0/2^{n+1},$$

then $$\phi(t_0/2^n) > \phi(t), \quad 2^{2n} > (t_0/2t)^2.$$

Hence $$\phi(t_0) > t_0^2\,\phi(t)/4t^2,$$

$$\frac{1}{\phi(t)} > \frac{t_0^2}{4\phi(t_0)} \cdot \frac{1}{t^2},$$

and the integral (2 . 1) is divergent for all values of x if $f(x) = x$.

Take then a decreasing sequence b_n such that

$$\phi(b_n) \leqslant 4\phi(\tfrac{1}{2}b_n)$$

for all n, and let $a_n = \frac{1}{2}b_n$. Then (4 . 41) is satisfied if

$$\sum_{\nu=n+1}^{\infty} l_\nu c_\nu < \tfrac{1}{3\,2} l_n c_n.$$

Let $$c_n = \frac{1}{n^2}, \quad r_n = \frac{1}{4n}\left[\frac{1}{b_n}\right], \quad l_n = \frac{1}{4n^2 r_n}.$$

Then the convergence and divergence conditions are satisfied, (4 . 11) is satisfied if $b_1 < \frac{1}{2}$, and the remaining conditions become

$$\sum_{\nu=n+1}^{\infty} \frac{b_\nu}{\nu^3} = o\left(\frac{b_n}{n^3}\right),$$

$$\phi(b_n) = o(b_n/n^3),$$

$$\sum_{\nu=1}^{n-1} \frac{1}{\nu b_\nu} = o\left(\frac{1}{nb_n}\right).$$

All these can be satisfied by substituting for the sequence b_n a subsequence which tends to zero sufficiently rapidly. If, for example, $\phi(t) = -t/\log t$, we may take $b_n = e^{-n^4}$. This completes the proof. Note that the proof breaks down at the last stage if $\phi(t)/t$ does not tend to zero.

5. *Proof of Theorem* II.—We have

$$(5.1)\quad \int_{a_n}^{b_n} \frac{|f(x+t)-f(x-t)|}{\phi(t)}\,dt \geqslant \int_{a_n}^{b_n} \frac{|f_n(x+t)-f_n(x-t)|}{\phi(t)}\,dt$$

$$-\sum_{\nu=1}^{n-1}\int_{a_n}^{b_n} \frac{|f_\nu(x+t)-f_\nu(x-t)|}{\phi(t)}\,dt - \sum_{\nu=n+1}^{\infty}\int_{a_n}^{b_n} \frac{|f_\nu(x+t)-f_\nu(x-t)|}{\phi(t)}\,dt.$$

Now

$$(5.2)\quad \int_{a_n}^{b_n} \frac{|f_n(x+t)-f_n(x-t)|}{\phi(t)}\,dt = \int_{a_n}^{b_n} \frac{f_n(x+t)}{\phi(t)}\,dt + \int_{a_n}^{b_n} \frac{f_n(x-t)}{\phi(t)}\,dt$$

provided that $f_n(x+t)$ and $f_n(x-t)$ are never both different from zero in the interval of integration; and $f_n(x+t) = 0$ if

$$q/r_n + l_n < x+t < (q+1)/r_n - l_n,$$

and $f_n(x-t) \neq 0$ if

$$q'/r_n - l_n < x-t < q'/r_n + l_n.$$

Hence (5.2) is true if an interval $\{q/r_n + l_n - x,\ (q+1)/r_n - l_n - x\}$ where $f_n(x+t) = 0$ includes an interval $(x-q'/r_n - l_n,\ x-q'/r_n + l_n)$, where $f_n(x-t) \neq 0$, *i.e.* if

$$(q+q')/2r_n + l_n < x < (q+q'+1)/2r_n - l_n.$$

For different values of q and q' there are $2r_n$ intervals of this type in $(0, 1)$, each of length $1/2r_n - 2l_n$. Hence (5.2) is true if x lies in a set of measure $1 - 4r_n l_n$.

Suppose now that

$$(5.3)\qquad l_n = o(1/r_n), \quad 1/r_n = o(a_n), \quad a_n = o(b_n),$$

and, for a particular value of n, let k and p be integers such that

$$\frac{k-1}{r_n} - l_n < a_n \leqslant \frac{k}{r_n} - l_n \qquad \frac{k+p}{r_n} + l_n \leqslant b_n < \frac{k+p+1}{r_n} + l_n.$$

Then

$$\int_{a_n}^{b_n} \frac{f_n(x+t)}{\phi(t)}\,dt \geqslant \sum_{q=0}^{p} \int_{(k+q)/r_n-l_n}^{(k+q)/r_n+l_n} \frac{f_n(x+t)}{\phi(t)}\,dt$$

$$= \sum_{q=0}^{p} \frac{1}{\phi\{(k+q)/r_n+\theta_q l_n\}} \int_{(k+q)/r_n-l_n}^{(k+q)/r_n+l_n} f_n(x+t)\,dt \quad (|\theta_q| \leqslant 1)$$

$$= \sum_{q=0}^{p} l_n c_n \Big/ \phi\left(\frac{k+q}{r_n}+\theta_q l_n\right)$$

$$\geqslant l_n c_n \sum_{q=0}^{p} \int_{k+q}^{k+q+1} \frac{dt}{\phi(t/r_n+l_n)}$$

$$= r_n l_n c_n \int_{k/r_n+l_n}^{(k+p+1)/r_n+l_n} \frac{dt}{\phi(t)}$$

$$\geqslant r_n l_n c_n \int_{a_n+1/r_n+2l_n}^{b_n} \frac{dt}{\phi(t)}.$$

The same inequality may be obtained for the last term in (5 . 2). Hence

$$\int_{a_n}^{b_n} \frac{|f_n(x+t)-f_n(x-t)|}{\phi(t)}\,dt \geqslant 2r_n l_n c_n \int_{a_n+1/r_n+2l_n}^{b_n} \frac{dt}{\phi(t)} \tag{5 . 4}$$

except possibly in a set of measure $4r_n l_n$.

Also as in § 4 . 2

$$\sum_{\nu=1}^{n-1} \int_{a_n}^{b_n} \frac{|f_\nu(x+t)-f_\nu(x-t)|}{\phi(t)}\,dt = 0 \tag{5 . 5}$$

unless x lies in the set F, or in a set G_n of measure

$$2b_n \sum_{\nu=1}^{n-1} r_\nu.$$

Also

$$\sum_{\nu=n+1}^{\infty} \int_{a_n}^{b_n} \frac{|f_\nu(x+t)-f_\nu(x-t)|}{\phi(t)}\,dt < 2\int_{a_n}^{b_n} \frac{dt}{\phi(t)} \sum_{\nu=n+1}^{\infty} c_\nu. \tag{5 . 6}$$

From (5 . 1), (5 . 4), (5 . 5), (5 . 6) it follows at once that the integral (2 . 2) diverges almost everywhere, provided that (5 . 3) is satisfied, and

$$r_n l_n c_n \int_{a_n+1/r_n+2l_n}^{b_n} \frac{dt}{\phi(t)} \to \infty, \tag{5 . 71}$$

$$b_n \sum_{\nu=1}^{n-1} r_\nu \to 0, \tag{5 . 72}$$

$$\int_{a_n}^{b_n} \frac{dt}{\phi(t)} \sum_{\nu=n+1}^{\infty} c_\nu = o\left\{r_n l_n c_n \int_{a_n+1/r_n+2l_n}^{b_n} \frac{dt}{\phi(t)}\right\}. \tag{5 . 73}$$

Let $l_n = 1/(2^n r_n)$, $c_n = 2^{n-n^2}$. Then (5 . 71), (5 . 73) hold if

$$2^{-n^2}\int_{a_n+2/r_n}^{b_n}\frac{dt}{\phi(t)}\to\infty, \quad \int_{a_n}^{b_n}\frac{dt}{\phi(t)} = o\left\{2^n\int_{a_n+2/r_n}^{b_n}\frac{dt}{\phi(t)}\right\}.$$

Suppose r_ν chosen for $r = 1, 2, \ldots, n-1$. Let

$$1/b_n = 2^n \sum_1^{n-1} r_\nu.$$

Having fixed b_n, we can choose a_n so small that

$$\int_{a_n}^{b_n}\frac{dt}{\phi(t)} = 2^{n+n^2}.$$

We can then choose r_n so large that

$$\int_{a_n+2/r_n}^{b_n}\frac{dt}{\phi(t)} > 2^{n^2+n-1}, \quad r_n > 2^n/a_n.$$

The theorem then follows.

6. *Proof of Theorem* III.—We have

$$\int_{a_n}^{b_n}\frac{f(x+t)-f(x)}{\phi(t)}\,dt = \int_{a_n}^{b_n}\frac{f_n(x+t)-f_n(x)}{\phi(t)}\,dt$$
$$+\sum_{\nu=1}^{n-1}\int_{a_n}^{b_n}\frac{f_\nu(x+t)-f_\nu(x)}{\phi(t)}\,dt+\sum_{\nu=n+1}^{\infty}\int_{a_n}^{b_n}\frac{f_\nu(x+t)-f_\nu(x)}{\phi(t)}\,dt$$
$$= J_1+J_2+J_3.$$

Take x to be in one of the intervals where $f_n(x) = 0$, and suppose that (5 . 3) again holds. Then, as in § 5,

$$J_1 = \int_{a_n}^{b_n}\frac{f_n(x+t)}{\phi(t)}\,dt \geqslant r_n l_n c_n \int_{a_n+1/r_n+2l_n}^{b_n}\frac{dt}{\phi(t)},$$

and $J_2 = 0$ unless x lies in the set F, or in a set of measure

$$2b_n \sum_1^{n-1} r_\nu.$$

Also
$$|J_3| < 2\int_{a_n}^{b_n}\frac{dt}{\phi(t)}\sum_{\nu=n+1}^{\infty} c_\nu.$$

The theorem may now be proved in the same way as Theorem II.

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1925.

Liabilities.	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				350	4	0
Life Compositions Fund				2911	16	6
Reserve Fund in 1924	593	8	9			
Depreciation of Investments	133	5	6			
Reserve Fund				460	3	3
Sundry Creditors				33	8	6
Reserve for next year's printing				300	0	0
Revenue Account balance in hand				316	17	7
				£5071	17	3

Assets.	£	s.	d.
London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 79 ...	304	3	0
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 58	406	4	1
Funding Loan £2500 4 per Cent. at $87\frac{1}{4}$ ...	2181	5	0
Conversion Loan £1700 $3\frac{1}{2}$ per Cent. at $76\frac{1}{4}$	1296	5	0
The above-mentioned stocks are valued at the present market price.			
Deposit at Bank	600	0	0
Cash at Bank	284	0	2
	£5071	17	3

REVENUE ACCOUNT for the year 1st November, 1924, to 31st October, 1925.

	Year 1923-24.							Year 1923-24.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	137	0	4	*332*	*9*	*7*	Printing and Circulating *Proceedings*, &c.	874	8	3	*1140*	*1*	*6*
Income tax returned	23	0	3	*51*	*9*	*11*	Purchases and Binding for Library ...	44	19	0	*138*	*9*	*4*
Dividends on—							Rent	30	0	0	*30*	*0*	*0*
Lord Rayleigh's Fund ...	38	18	4	*35*	*14*	*3*	Postages and Sundries	4	18	3	*6*	*12*	*11*
The Society's Investments	200	18	5	*170*	*1*	*3*	Anniversary Celebration...	4	18	10		—	
Entrance Fees	27	6	0	*25*	*4*	*0*	Reserve for next year's printing ...	300	0	0		—	
Annual Subscriptions	429	13	8	*434*	*1*	*0*	Balance in hand on 31st October, 1925...	316	17	7	*137*	*0*	*4*
Sale of *Proceedings*, &c.	391	4	11	*296*	*4*	*1*							
Advertisements revenue	28	0	0	*7*	*0*	*0*							
Grant from Royal Society	300	0	0	*100*	*0*	*0*							
	£1576	1	11	*1452*	*4*	*1*		£1576	1	11	*1452*	*4*	*1*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance in hand 31st October, 1924 ...	3	1	6	Cash at Bank 31st October, 1925	6	1	10
Dividends received	2	6	10				
Income tax returned		13	6				
	£6	1	10		£6	1	10

STATEMENT OF TRUST FUNDS, 31st October, 1925.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.

£229. 13*s.* 4*d.* 4 per Cent. Funding Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

12th November, 1925.

Signed on behalf of the Council:—

A. L. DIXON, *President.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM Lt.-Col. R.E.,
Auditor.

27th November, 1925.

FORMULAE FOR THE INTERSECTION OF LINE SYSTEMS IN HIGHER SPACE

By C. G. F. JAMES.

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IN the first part of this paper we define a system of characteristics, in terms of which can be expressed the orders of the constructs generated by the lines of a system which satisfy simple incidence and other conditions. We proceed to obtain in terms of these characteristics the order of the construct common to two line systems of suitable dimension. From these results and some systematic preliminary formulae can be read off, when required, formulae for all the characteristics of the system of intersection of any two systems, though these actual formulae are not recorded owing to the great number of sub-cases necessary for their expression.

1. A system of lines in a space $[r]$ of r dimensions* can have any dimension from zero to $2r-2$, the last being that of the totality of lines in the space. A system ∞^1 we shall call a scroll,† an ∞^2 a congruence, and an ∞^3 a complex. Similarly a system ∞^{2r-3} will be called a hypercomplex, while the intermediate cases will be denoted by such symbols as $\{i\}$, where i is the dimension of the system in question. Systems of dimension i less than $r-1$ will be said to be *restricted*, in that the loci of their lines are definite constructs, in general V_{i+1}, though they may exceptionally be of lower dimension, in which case they will be called special.‡ Systems of dimension greater than $r-2$ are similarly described as *unrestricted*, since in general at least a finite number of lines pass

* Generally we denote by $[t]$ a linear space of dimension t immersed in the $[r]$.

† To be distinguished as a line system from the surface filled.

‡ A construct of points of dimension α and order β is denoted by a symbol as V_α^β. Examples of special systems are the complex of lines on a V_3^2 immersed in [5], or a plane system regarded as a system in [4].

through an arbitrary point of space. *Special unrestricted systems* are those whose lines lie on a construct.* The first of the unrestricted systems, the $\{r-1\}$, is very important, and is such that a finite number of its lines pass through an arbitrary point of space, while the lines in general are $(r-1)$-tangent to a certain V_{r-1}, the focal construct or *focal form*.†

1 *a. Definition of the system of characteristics.*—The lines of a system $\{i\}$ which belong to a space $[s]$ of $[r]$ form a system of dimension $i+2s-2r$. If this sub-system generate a form‡ of the $[s]$ we have $i+2s-2r=s-2$, or $s=2r-i-2$. The original system being unrestricted $(2r-3 \geqslant i \geqslant r-1)$, the sub-system in the $[s]$, for a general value of s, is unrestricted§ when $s>2r-i-2$, and restricted when $s \leqslant 2r-i-2$.

We define then as *characteristics of an unrestricted system* :—

The order equal to the order of the cone of its ∞^{i-r+1} lines through an arbitrary point. It will be denoted by m.

The classes n_1, n_2, etc., where

n_1 the first class is the order of the V_{2r-i-3} of its lines in a $[2r-i-2]$,

n_2 the second class is the order of the V_{2r-i-5} of its lines in a $[2r-i-3]$,

and generally

n the s-th class is the order of the $V_{2r-i-1-2s}$ of its lines in a $[2r-i-s-1]$.

The final class is that for which

$$s=[r-1-\tfrac{1}{2}i]=r-1-[\tfrac{1}{2}(i+1)],$$

with the usual notation for the greatest integer contained in a positive number. The corresponding space is of dimension

$$r-[\tfrac{1}{2}i].$$

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* For example, the complex on a V_3^2 in [4].

† C. Segre, "Un' osservazione sui sistemi . . .", *Rend. di Palermo*, 2 (1888), 148. We shall call these systems *generalized congruences*.

‡ V_{s-1}.

§ Namely, as a system of $[s]$.

1 *b*. In analogous manner we define as *characteristics of a restricted system* $(i < r-1)$:—

The order m, equal to the order of the construct generated, this construct being a V_{i+1}.

The first class n_1, equal to the order of the V_{i-1} of its lines in an $[r-1]$ of $[r]$;

the second class n_2, the order of the V_{i-3} of its lines in an $[r-2]$;

and generally

the s-th class n_s, equal to the order of the V_{i-2s+1} of its lines in an $[r-s]$.

The last case is that for which s equals $[\frac{1}{2}i]$, the space in question being of dimension $r-[\frac{1}{2}i]$.

In either case the order of a special system is to be taken as zero, and the same applies to any characteristic, when the system of lines implied in its definition lies on a construct of lower dimension than in the general case. Notice also that these definitions are for the general case when the construct of lines generated by $\{i\}$, or by one of the subsystems of the definitions, is such that only one line passes through each point of the construct. In exceptional cases more than one such line may pass through each point, and in these cases we shall regard the construct as counted multiply according to this number, or alternatively the characteristics could be defined in terms of the number of lines meeting suitable linear spaces.*

2. *Characteristics of some Simple Systems.*

(i) *Characteristics of the system of all lines of* $[r]$ *meeting a* $[j]$.— The system is of dimension $j+r-1$, and is therefore unrestricted. We have

$$m = 1, \quad n_s = 0 \quad \text{(for all values of } s\text{)}, \qquad (1)$$

* Notice in this connexion that a construct of lines of the system may contain other systems of any dimension, which may contain the system of these lines. Cases of this are very numerous in higher spaces though rare in the first few.

since n_s is the order of the construct of lines in a $[2r-(j+r-1)-1-s]$ or $[n-j-s]$; and this is zero since such a space does not meet the $[j]$.

(ii) *Characteristics of the system of all lines in a* $[t]$ *of* $[r]$ *as a system in* $[r]$:—

(*a*) $2t-2 \geqslant r-1$ (when the system is unrestricted) :—

$$m = n_s = 0 \quad (s < r-t), \quad n_{r-t} = 1, \tag{2}$$

(*b*) $2t-2 < r-1$:—

$$m = n_s = 0 \quad (s < t-1), \quad n_{t-1} = 1, \tag{3}$$

for in each case the construct of lines in a space of $[r]$ is of fewer dimensions than for the general case, unless it be a single line.

(iii) *Characteristics* $(m', n_1', n_2', \ldots)$ *of an* ∞^i *of a* $[t]$ *in* $[r]$ *(as a system in* $[r]$*)* in terms of its characteristics $(m, n_1, n_2, \ldots)$ as a system in $[t]$.

Case (a).—The system, (say) Γ, *is unrestricted in both spaces* $(i \geqslant r-1)$. Then $m' = 0$; and

> n_s equals the number of lines in a $[2r-i-s-1]$ of $[r]$ meeting a $[s]$ of the lower space, or, in our case, the number of lines in a $[r+t-i-s-1]$ of $[t]$ meeting a $[s-r-t]$ of the lower space.

Further, the sub-system in the $[r+t-i-s-1]$ is unrestricted in this space if $s \leqslant n-t$, and for the limiting value $s = S = r-t$ it is a generalized congruence therein. Finally, observing that

$$r+t-i-s-1 = 2t-i-1-(s+t-r),$$

we have

$$m' = 0; \quad n_1' = n_2' = \ldots = n_{S-1}' = 0, \quad n_S' = m, \quad n_{S+j}' = n_j \quad (j = 1, 2, \ldots),$$

the series ending simultaneously with the last n' and the last n.

Case (b).—Γ *is unrestricted in* $[t]$, *but not in* $[r]$, $(r-1 > i \geqslant t-1)$

In precisely the same way it is found that

$$m' = n_1' = n_2' = \ldots = n_{S-1}' = 0, \quad n_S' = m, \quad n_{S+j}' = n_j \quad (j = 1, 2, \ldots),$$

where, however, $S = i-t+1$.

Case (c).—Γ *is restricted in* $[t]$, $(t-1>i)$.

We have at once $\quad m'=m; \quad n'_j=n_j \quad (\text{all } j).$

(iv) *Characteristics of the system* Γ' *of all lines of an* $\{i\}$, Γ, *of* $[r]$ *in a space* $[t]$, *as a system in* $[t]$.

Γ' is of dimension $(2t-2r+i)$, so that, when Γ is unrestricted, we have two sub-cases according as Γ' is unrestricted in $[t]$, or otherwise.

Case (a).—Γ *unrestricted,* Γ' *unrestricted in* $[t]$,

$$(i \geqslant r-1, \quad t \geqslant 2r-i-1).$$

If the characteristics of Γ' as a system in $[t]$ are denoted by accented symbols, we have without difficulty

$$m'=m; \quad n'_s=n_s \quad (s=1, 2, \ldots [r-1-\tfrac{1}{2}i]). \quad (4)$$

Case (b).—Γ *unrestricted,* Γ' *restricted in* $[t]$,

$$(i \geqslant r-1, \quad t<2r-i-1).$$

$$m'=n_T; \quad n'_1=n_{T+1}, \quad n'_2=n_{T+2} \ldots n'_s=n_{T+s}, \quad (5)$$

where T equals $2r-i-t-1$, and s takes the values 1, ... $[t-r+\frac{1}{2}i]$.

Case (c).—Γ *restricted,* $\quad (i<r-1)$.

The values are as in Case (*b*), where, however, T equals $r-t$, and s takes the same set of values.

3. *Two preliminary results.*—On the two following results depends the determination of the characteristics of sub-systems of given systems meeting linear spaces, and thence of those of the systems of intersection of given systems.

To find the order of the construct of lines of an $\{i\}$ *meeting a linear space* $[t]$ *of* $[r]$, $(t<r-1)$.

Case I.—*The system* Γ *is restricted in* $[r]$, $(i<r-1)$.

The sub-system is of dimension $t+i+1-r$, and therefore describes in general a construct of dimension $t+i+2-r$, meeting the $[t]$ in a construct of dimension $t+i+1-r$ and order m, namely the section of

$[t]$ and the locus of lines of the $[i]$. An $[r-1]$ through $[t]$ cuts the original construct in this $V^m_{t+i+1-r}$ and in a $V'_{i+t+1-r}$. Considering the section of this construct by an arbitrary $[r-2]$ through the $[t]$ (when possible), and continuing in this way, we have finally

The order of the construct sought equals

$$m+n_1+n_2+\ldots n_\mu \quad (t+i+1-r>0),$$

where μ is the lesser of the numbers $i+t-r+1$, $r-t-1$.

II.—*The system* Γ *is unrestricted in* $[r]$.

The dimension of the sub-system is $t+i-r+1$ as before, and we must therefore suppose

$$t+i+1-r<r-1 \quad \text{or} \quad t<2r-i-2,$$

in order that the dimension of the sub-system shall not exceed $r-2$. The space $[t]$ is m-fold on the $V_{t+i+2-r}$, and we draw through the $[t]$ a space of such dimension that it contain a construct of lines of Γ meeting $[t]$ of dimension t, namely of ∞^{t-1} lines. If y is this dimension, we have $t+(i+2y-2r)+1-y=t-1$ or $y=2r-i-2$, provided $y>t$, and this is true by the previous inequality.

Now by I, using § 2 (iv), the order of this V_t is $n_1+n_2+\ldots$. Thus *the order of the construct sought equals* $(t>0)$

$$m+n_1+n_2+\ldots n_\nu,$$

where ν is the lesser of the two numbers t and $2r-i-t-2$.

All these results hold true for special systems, provided the construct is of the same dimension as in the general case. Otherwise the formulae give zero.

4. We now pass on to the final group of preliminary results, namely *the characteristics of the sub-systems of lines of an* $\{i\}$, Γ, *of* $[r]$ *which meet prescribed linear spaces.* The linear space in question we denote by $[t]$, the sub-system by Γ', and its characteristics by accented letters in the usual notation. The dimension of Γ' is always $t+i+1-r$, but there are a number of sub-cases for the expression of the results. We suppose $t<r-1$.

Case I.—Γ *and* Γ' *are both unrestricted in* $[r]$,

$$(i \geqslant r-1, \quad t \geqslant 2r-i-2).$$

Then m' equals m; and n'_s equals the order of the construct of lines of Γ' in a space $[\theta]$, where θ equals $3r-t-i-2-s$, or the order of the construct of lines of Γ in this space meeting a $[2r-i-2-s]$. The last characteristic is that for which s equals $[\frac{1}{2}(3r-i-t-3)]$, but in most cases several of the classes are actually zero. There are in fact two conditions for a non-zero n'_s, which come into operation under separate conditions.

(i) When the system of Γ in $[\theta]$ is unrestricted, the $[t]$ and the $[\theta]$ must meet. Since they meet in a $[2r-i-2-s]$, we must have

$$s \leqslant 2r-i-2 = s_1. \tag{6}$$

(ii) When the system of Γ in $[\theta]$ is restricted in $[\theta]$, the $[t]$ must meet its locus, and this requires $(2\theta-2r+i+1)+t-r \geqslant 0$, or, on reduction,

$$s \leqslant [\tfrac{1}{2}(3r-t-i-3)] = s_2, \tag{7}$$

and the maximum value of s actually is the final value of s theoretically occurring.

If the system of Γ in $[\theta]$ is unrestricted, we have $2\theta-2r+i \geqslant \theta-1$, or

$$s \leqslant r-t-1 = s_0, \tag{8}$$

while it is restricted if s exceed this value. We therefore subdivide into cases according as the limit in (i) is reached before (including *at*) or after this dividing value s_0 for s.

Case I *aa*.—

$$2r-i-2 \leqslant r-t-1 \quad \text{or} \quad t \leqslant i-r+1.$$

Using the results of § 2, (iv), and § 3, we have at once

$$m' = m; \qquad n'_s = m+n_1+n_2+\ldots+n_{\mu_s} \quad (1 \leqslant s \leqslant s_1),$$

where

$$\mu_s = \text{Min}\{s, s_1-s\}.$$

$$n'_{s_1+1} = n_{s_2+1} = \ldots = n_{s_2} = 0.$$

Case I *ba.*— $t = i-r+2,$

and $m' = m\,; \quad n'_s = m+n_1+n_2+\ldots+n_{\mu_s} \quad (1 \leqslant s \leqslant s_0),$

noticing that $s_0 = s_2 = s_1-1.$

Case I *bb.* $t > i-r+2.$

We are here to consider principally the characteristics for $s > s_0.$ We have by the same preliminary results

$$n'_s = n_{2r-i-1-\theta}+n_{2r-i-\theta}+\ldots \quad (s > r-t-1),$$

$$= n_{s+t+1-r}+n_{s+t+2-r}+\ldots,$$

though in the case when $s = s_2 = \frac{1}{2}(3r-i-t-3)$, the last characteristic n_s equals the first of these terms only. The sequence is thus

$$m' = m\,; \quad n'_s = m+n_1+n_2+\ldots+n_{\mu_s} \quad (1 \leqslant s \leqslant s_0),$$

$$n'_s \equiv n'_{s_0+i} = n_i+n_{i+1}+\ldots+n_{i+\nu_s} \quad (s_0 < s \leqslant s_2),$$

where $\nu_s = \text{Min}\{3r-i-t-3-2s,\ r-t-1\},$

the last term being $n'_{s_2} = n_\alpha+n_{\alpha+1}$ or $n_\alpha,$

in the last case according as $3r-i-t$ is even or odd. The value of α is

$$[\tfrac{1}{2}(r+t-i-1)].$$

Case II.—Γ *is still unrestricted, but* Γ' *is now restricted in* $[r]$.

$$i \geqslant r-1 : t < 2r-i-2.$$

The order m' equals $m+n_1+n_2+\ldots$, when it equals m (§ 3, Theorem II). The s-th class, n'_s, equals the order of the construct of lines of Γ in a $[r-s]$ meeting a $[t-s]$; and, as before, the sub-system of Γ in this space is unrestricted or restricted therein as $s \leqslant i+1-r$, or $s > i+1-r$, which we denote by S_0. In the former case we have the condition $s \leqslant t$ for the $[t]$ to meet the $[r-s]$, and in the latter case we

must have, as in Case I,

$$s \leqslant [\tfrac{1}{2}(t+i+1-r)] = S_2, \tag{9}$$

which is actually the index of the last class occurring. We have therefore the following sub-cases.

Case II *a*.—The restriction $s \leqslant t$ comes into operation before the division index S_0 is reached, namely $t \leqslant i+1-r$.

The sequence of characteristics is thus

$$m' = m+n_1+n_2+\ldots n_{\mu'_0},$$

where $$\mu'_0 = \text{Min}\{t,\ 2r-i-t-2\},$$

$$n'_s = m+n_1+n_2+\ldots n_{\mu'_s} \quad (1 \leqslant s \leqslant t),$$

$$n'_{t+1} = n'_{t+2} = \ldots n_{s_2} = 0,$$

μ's being $$\text{Min}\{t-s,\ 2r-i-t+s-2\}.$$

Case II *ba*.— $$t = i+2-r.$$

$$m' = m+n_1+n_2+\ldots+n_{\mu'_0},$$

$$n'_s = m+n_1+n_2+\ldots+n_{\mu'_s} \quad (1 \leqslant s \leqslant t-1),$$

noticing that $S_0 = S_2 = i+1-r$.

Case II *bb*.— $$t > i+2-r.$$

$$m' = m+n_1+n_2+\ldots+n_{\mu'_0},$$

$$n'_s = m+n_1+n_2+\ldots+n_{\mu'_s} \quad (1 \leqslant s \leqslant S_0),$$

$$n'_s \equiv n'_{S_0+i} = n_i+n_{i+1}+\ldots+n_{\nu'_s} \quad (S_0 < s \leqslant S_2),$$

where ν'_s equals $$\text{Min}\{i+t+1-r-2s,\ r-t-1\},$$

the sequence ending with

$$n'_{S_2} = n_a+n_{a+1} \quad \text{or} \quad n_a,$$

according* as S_2 equals $\frac{1}{2}(t+i-r)$, or $\frac{1}{2}(t+i+1-r)$. The value of α is $\frac{1}{2}(r-i+t-1)$.

Case III.—Γ *and therefore* Γ' *are restricted in* $[r]$: $(i<r-1)$.

Writing $m = n_0$, $m' = n'_0$, we have, by § 3, Theorem I, for all values of s.

$$n'_s = n_s+n_{s+1}+\ldots n_{s+\nu'_s} \quad (0 \leqslant s \leqslant S_2),$$

except for $s = S_2 = \frac{1}{2}(t+i+1-r)$, *when*

$$n'_{S_2} = n_{S_2},$$

which is the last characteristic. When $S_2 = \frac{1}{2}(t+i-r)$, *the last value is**

$$n'_{S_2} = n_{S_2}+n_{S_2+1}.$$

5. *To find the characteristics of the system common to two line systems in* $[r]$.

We shall be able to write down all the characteristics if we determine merely the order of the system of intersection of two given systems in any space, and owing to the multiplication of sub-cases we confine ourselves to recording the solution of this special problem, from which the complete solution can be read off in any special case.†

A direct solution for the present problem is only required when the two systems share a finite number of lines and their dimensions i, j therefore satisfy $i+j = 2r-2$. There are two possibilities :

(1) Both systems are unrestricted, when each is necessarily a generalized congruence, or $i = j = r-1$; or

(2) One system, say the $\{i\}$, is restricted, and the other, $\{j\}$, unrestricted.

The methods of proof differ slightly in the two cases, but are constructed on the same general lines, using the lemmas about to be given. We shall prove them first under the assumption that neither system is special (as defined in § 1).

* *I.e.* we have to distinguish between the cases $t+i-r$ odd and even.

† *I.e.* the results of § 4 give the minimum multiplicity of sub-cases consistent with ready practical application of the formulae of this paper.

LEMMA I. *A theorem of Schubert's.*—Given a system ∞^r of pairs of points (A, B) in the space $[r]$, such that an arbitrary point A determines α pairs, and an arbitrary point B β pairs, while γ is the order of the generalized congruence of the joins of its pairs in a space $[r-1]$: then the order of the generalized congruence of limiting joining lines for the pairs of coinciding points equals* $\alpha+\beta+\gamma$ (in general).†

Notice in connexion with this theorem that :—

(1) Points,‡ in which A and B coincide with arbitrary limiting joins, are called *complete coincidences* of the system, and count once in the formula. Further, not all coincidences enter into the result, for there exist systems with coincidences for which the system of limiting joins is restricted, or, if still of dimension $r-1$, is special.

(2) Either index α, β, or both, can be zero. In this case the corresponding points, A, etc., lie on constructs, and any point of the construct has an infinite number of correspondents. Such correspondences we shall call *singular*.§ Most of the correspondences we deal with are singular.

LEMMA II.—*Characteristics of the system derived by projecting an $\{i\}$ of $[r]$ from a point on to a $[r-1]$, considered as a system in this latter space.*‖ $(i \leqslant 2r-5)$.

Taking three cases, these are readily found to be :

Case I.— $i \leqslant r-3$;

* This theorem appears to have been first given in "Die n-dimensionale Verallgemeinerung der Anzahl für die vielpünktig . . . ", *Math. Annalen*, 26 (1885), 52–73. The general theory of these correspondences is given by M. Pieri in "Formule di coincidenza per le serie algebriche di coppie di punti", *Rend. di Palermo*, 5 (1895), 252 (§ 3).

† Or symbolically, in the usual notation :

$$(0)(r)' + (0)'(r) + (0, r-1) = \epsilon(0, r)(r).$$

‡ They are necessarily finite in number, if the theorem is applicable.

§ By extension from the singular correspondences on a line. For example, if we consider the system formed by every pair of a construct of dimension k, associated with every point of one of dimension k', we can construct a new proof of the generalized Bezout's Theorem for the number of their intersections, proceeding by induction from the case of a curve and a form.

‖ By repetition we have the characteristics of projection on to any space.

and

Case II.— $i = r-2$:

$$m' = m; \quad n'_s = n'_s \quad \text{(all values of } s\text{)}.$$

Case III.— $i \geqslant r-1$:

$$m' = n_1; \quad n'_s = n'_{s+1} \quad \text{(all values of } s\text{)}.$$

6. *To find the number of lines common to two generalized congruences* Γ, Γ' *in* $[r]$.

The characteristics of Γ are taken to be $(m; n_1, n_2, \ldots, n_k)$, where k is $[\frac{1}{2}(r-1)]$, in our standard notation, those of Γ' the same with accents.

Two arbitrary spaces $[r-1]$, Π and Π', are chosen in the space $[r]$ of the two systems, and cutting in a space ϖ of dimension $r-2$. Points A and B of Π are associated when through A passes a line of Γ meeting on Π' a line of Γ' through B; and to the $[r-1]$-correspondence so obtained is applied the theorem of Schubert quoted (§ 5, Lemma I). Since m lines of Γ, and m' of Γ' pass through assigned points, the indices α, β are each equal to* mm'; and the theorem gives

$$2mm' + V = F_r + U,$$

or

$$F_r = 2mm' + V - U,$$

where F_r denotes the number of lines common to the two generalized congruences, their intersections with Π being complete coincidences of the correspondence, and U denotes the order of the generalized congruence of Π of limiting joining lines for the coincidences in ϖ, every point of which is a limited coincidence. These lines are the intersection with Π of the planes defined by a line of Γ, and one of Γ' concurrent on ϖ; and the order sought is the number of planes through a point Q of Π.

Finally V denotes the number of pairs (A, B) in a $[r-2]$ of Π, whose join passes through a fixed point Q' in the $[r-2]$, namely the number of pairs (A, B) in a $[r-2]$, whose join passes through a fixed point, and through which pass rays of Γ, Γ' respectively concurrent on a space† Π'', of $r-1$ dimensions, arbitrarily situated.

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* But the correspondence is not involutory.

† In the first place identical with π'.

Now let Q lie in ϖ. Then

$$U = mm' + U',$$

where U' equals the number of pairs of rays, one of each system, concurrent on any $[r-2]$, lying in a (variable) $[r-1]$ through it, and such that their plane passes through a point of the lower space.

The numbers U', V are now defined independently of the original figure. Let us choose as a special position of Π'' one passing through the $[r-2]$. We have at once

$$V = U' + W,$$

where W equals the number of pairs of rays of Γ, Γ', one of each system, in a fixed $[r-1]$, and such that they intersect, while their planes pass through a fixed point Q'' of this space.

The conclusion of the foregoing part of the proof is :—

$$F_r = mm' + W.$$

Now the sub-systems of Γ, Γ' in this space are restricted and of characteristics* $(n_1;\ n_2,\ n_3,\ \ldots)$, $(n_1';\ n_2',\ n_3',\ \ldots)$ respectively. Projecting these systems on to a fixed $[r-2]$ from the point Q'', we obtain therein generalized congruences of the same characteristics.† The lines common to these systems arise exclusively from pairs of rays in the $[r-1]$ whose common plane passes through Q''. If F_{r-2} denotes this number, we have

$$F_r = m \,.\, m' + F_{r-2}.$$

Repeating the process,

$$F_r = m \,.\, m' + n_1 \,.\, n_1' + F_{r-4},$$

where F_{r-4} denotes the number of lines common to two generalized congruences in $[r-4]$, of characteristics :

$$(n_2;\ n_3,\ n_4,\ n_5.\ \ldots) : (n_2';\ n_3',\ n_4',\ n_5',\ \ldots).$$

This process continues until the dimension of the space into which we project equals three for n odd, or two for n even. In the former case the sequence of added terms closes with

$$n_{k-1} n_{k-1} + n_k' n_k',$$

* § 2 (iv).

† § 5, Lemma II.

by Halphen's Theorem.* In latter case we end with two envelopes in a plane of classes n_k, n_k' respectively. Hence in each case *the number of lines common to the two generalized congruences in* $[r]$ *equals*

$$m \cdot m' + \sum_{1}^{k} n_i \cdot n_1', \quad k = [\tfrac{1}{2}(r-1)]. \tag{10}$$

§6a. In any particular case the theorem can be proved without indeterminancy of the numbers entering. Thus in case of an ordinary space the result is known, but the present method gives a new and simplified proof. The number V equals the order of the residual curve of intersection of the scrolls of lines of the respective congruences meeting a fixed line in Π (now a plane), namely, $(m+n)(m'+n')-mm'$. The number U' is obtained by a simple correspondence argument, and is now the number of planes through a fixed line g, containing a ray of each system concurrent on g.† In fact, if we associate planes through g, which respectively contain lines of the two systems concurrent on g, the united planes are the planes sought and are $mn'+m'n$ in number.

In four dimensions we have two complexes, or $\{3\}$, of characteristics $(m, n)(m', n')$, where the single class is the order of the scroll of lines in an ordinary space.‡ Our correspondence is one between the points of a space,§ and the two orders of generalized congruences for the general theorem of Schubert are replaced by classes of envelopes in a plane. The number U' equals the class of the envelope of sections with a fixed plane ϖ of planes incident with ϖ, and containing a line of each complex concurrent on ϖ. Consider the following correspondence. We take a point S on ϖ, and the m rays of Γ through S: the plane ϖ and these lines define m spaces, each of which contains n' rays of Γ' meeting the associated line, and meeting ϖ in as many points T. These we associate to S, and conversely given T we derive $m'n$ points S. To this correspondence we apply the Schubert theorem for a plane. The only united points are the points of concurrence on ϖ of rays of the two complexes cospacial with ϖ, and the envelope of limiting joining lines is that whose class is sought. Hence

$$U' = mn' + m'n + \psi,$$

where ψ is the number of pairs on a fixed line g. It is equal to the number of pairs of intersection of the scrolls of orders $m+n$, $m'+n'$ of the two complexes meeting g (by § 3); and thus by a theorem‖ of Severi's to

$$(m+n)(m'+n')-mm'-X,$$

where X is the class of the envelope of common tangent spaces in points of g. It is subsequently eliminated, but for consistency we record its value. In fact, by a second theorem¶ from the same source we have

$$X = z_1 m' + z_1' m,$$

where z_1 is the *class of immersion* of g for the first scroll, or the number of planes, tangent to

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* See C. M. Jessop, *A Treatise on the Line Complex* (Cambridge), 1903, ch. 14, § 234, p. 259. The present theorem is the general extension of the theorem quoted.

† This number might be described as the *mutual rank* of the congruences.

‡ These are established definitions for space of four dimensions.

§ Used temporarily to denote an ordinary space.

‖ "Sulle intersezioni delle varietà . . .", *Mem. Torino* (2), 52 (1903), p. 61, § 4, n. 11.

¶ Or X can be found, in the present case, by an easy correspondence method.

the scroll in points of g, and meeting a line. Now these tangent planes are the planes of g and the rays of Γ meeting it. Hence $z_1 = n$, and so also $z_1' = n'$. Thus $X = mn' + m'n$, $\psi = nn'$, and finally

$$U' = mn' + m'n + nn'.$$

The number V is obtained as follows :—Let 0 be a fixed point in ϖ, and h a line through it in ϖ. It is met by a scroll of rays of Γ of order $m+n$ cutting Π in a curve of this order on which the point $h\Pi'$ is m-fold. This curve is met by a scroll of lines of Γ' of order*

$$(m+n)(m'+n'),$$

having this point as mm'-fold, and cutting ϖ in $mn' + m'n + nn'$ points other than $h\Pi'$. These we join to 0, and associate to h. We have in the pencil $(0, \varpi)$ a correspondence of this index in each direction, whose united lines arise from :—

(i) the lines of the envelope sought, passing through 0;

(ii) the points on the line $\varpi\Pi'$, at which two rays of separate complexes lie in a plane passing through 0, in number precisely X. Thus $V = 2(mn' + m'n + nn') - X$, or $2nn' + mn' + m'n$. Substituting these values we have for the number of lines common to the two complexes $mm' + nn'$, in accordance with the general theorem.

7. We turn now to the case of *the intersection of a restricted system* $\{i\}$ *and an unrestricted system* $\{j\}$ where $i+j = 2r-2$.

It is convenient to take separately the case $i = 1$, $j = 2r-3$, namely, we seek first *the number of lines common to a scroll of order* m, *and a hypercomplex* k.† Since the condition that a line may belong to a hypercomplex is represented by a single extra relation of degree k between the Plücker coordinates‡ p_{ij} of the line, it is seen that a permissible degeneration of the hypercomplex is into k linear systems, in particular into k systems defined by the condition of meeting k arbitrary $[r-2]$. From this it follows that the number of lines common to the systems is $m \,.\, k$. The following synthetic proof is almost as simple. Taking two spaces Π, Π' as before, let A be a point on the section by Π of the scroll; let the generator through this point cut Π' in P, and let the k-tic hypercomplex cone of P cut the curve in points B which we associate to A. The correspondence (A, B) is of indices $k \,.\, m$, and zero valency. The $2km$ coincidences are at the intersections of the common

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* This is seen at once by a degeneration argument.

† For a hypercomplex we have a single characteristic, its degree k equal to the order of the cone of its lines through a point. Its sub-systems in all spaces are hypercomplexes of the same degree.

‡ Defined as in ordinary space by the quantities $(x_i y_j - x_j y_i)$, where $(x_1, \ldots, x_{r+1})$, $(y_1, \ldots, y_{r+1})$ are the coordinates of two points on the line. For space of four dimensions, see G. Castelnuovo, *Atti Ist. Veneto* (7), 2, Part I (1891), 855.

rays, and at the m points where the scroll meets the space of intersection of Π and Π', each counted k times. Hence again *the number of lines of an m-tic scroll contained in an arbitrary hypercomplex of degree k equals m . k.*

8. Suppose now $i > 1$, and that κ lines of the system $\{i\}$ pass through an arbitrary point* of the locus filled, and that its characteristics are $(m; n_1, \ldots)$. The locus in question is thus of order $m' = m/\kappa$. Let the characteristics of $\{j\}$ be $(M; N_1, N_2, \ldots)$. The systems are again supposed to be non-special.

The correspondence we now construct is defined exactly as in § 6, but the correspondence so obtained is singular. The points A having correspondents lie on the $V_i^{m'}$ of section of Π and the locus $V_{i+1}^{m'}$, and to each correspond the points of κ, V_{j+1-r}^{M}, sections by Π and the M-tic cones of the points where the rays of $\{i\}$ through A cut Π'. The points A corresponding to a given point B of Π are κ in number for each point of intersection of the M-tic cone of B for $\{j\}$ and the $*V_i^{m'}$, section of $V_{i+1}^{m'}$ and Π'. The indices are thus zero and mM respectively.

The united points are where the common rays meet Π, and the points of the $V_{i-1}^{m'}$, section of $V_{i+1}^{m'}$ and ϖ (the section of Π and Π'). In finding the order of the generalized congruence of limiting joins for these coincidences it is now immaterial whether we seek the number of lines through an arbitrary point of Π or through a point of the $[r-2]$. Exactly as before we show that the difference between the order of the generalized congruence of joins of pairs in an arbitrary $[r-2]$, and the order of the previous generalized congruence equals the numbers of pairs of rays, one of each system, lying in a plane of an arbitrary $[r-1]$, and passing through a fixed point.

Since the sub-systems of $\{i\}$ and $\{j\}$ in the $[r-1]$ are $\{i-2\}$, $\{j-2\}$, respectively of characteristics

$$(n_1; n_2, n_3, \ldots) : (M; N_1, N_2, \ldots),$$

we seek on projection the number of lines common to an $\{i-2\}$ of characteristics $(n_1; n_2, n_3, \ldots)$, and a $\{j-2\}$ of characteristics $(N; N_2, N_3, \ldots)$ in an $[r-2]$. If, as before, F_r denotes the number of lines common to the original systems and F_{r-2} the number common to the new systems, we have

$$F_r = mM + F_{r-2},$$

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* This is technically a special case. In general, $\kappa = 1$ (§ 1*b*).

since, as explained above, U is now equal to U' merely. By repetition

$$F_r = mM + n_1N_1 + F_{r-4},$$

where F_{r-4} denotes the number of lines common to an $\{i-4\}$ of characteristics $(n_2; n_3, \ldots)$ and a $[j-4]$ of characteristics $(N_2; N_3, \ldots)$ in a space $[r-4]$. The solution is finally reduced to finding :—

For i even the number of times a ray of the $\{j-i\}$ of lines derived by this succession of operations from the $\{j\}$, and one of the finite number of rays derived from the $\{i\}$ lie in a plane through a fixed point in a space $[r-i+1]$, the succession being arrested before an *operation of projection.* There are n_s such lines derived from the $\{i\}$, and the $\{j-i\}$ being a $\{2(r-i+1)-4\}$ has two characteristics, namely N_{s-1}; N_s, the second being the number of its lines in a plane. The number sought is therefore n_sN_s, where s is the suffix of the last characteristic of either system, namely $[\frac{1}{2}i]$.

For i odd we seek finally the number of common rays for a scroll derived from $\{i\}$ and a $\{j-i+1\}$ derived from $\{j\}$ in a space $[r-i+1]$, the succession of operations being arrested *after a projection.* The scroll is of order n_s and, since $j-i+1 = 2(r-i+1)-3$, the $\{j-i+1\}$ is a hypercomplex of degree N_s. By the result of § 7 the number sought is again n_sN_s, where s has the same value as in the preceding case.

We have therefore the theorem :

The number of lines common to a restricted system $\{i\}$ *and an unrestricted system* $\{j\}$, *where* $i+j = 2r-2$ *equals* (in all non-special cases)

$$mM + \sum_1^s n_1N_1 \quad \{s = [\tfrac{1}{2}i]\}, \tag{11}$$

where the (m, n), (M, N) *are the respective characteristics.*

8*a*. *The case of space of four dimensions.*—The problem just solved does no arise in ordinary space. In the case of space of four dimensions, it can be solved without indeterminancy in the numbers involved in the proof. The only case is that of the number of lines common to a congruence $(i = 2)$ and a hypercongruence* $(j = 4)$. The characteristics of a congruence are its order m, or the number of its lines meeting a fixed line,† and its class n,

* The nomenclature we have adopted for line systems in space of four dimensions is by analogy with those of circle-systems in ordinary space. J. L. Coolidge, *A Treatise on the Circle and the Sphere* (Oxford, 1916). No nomenclature has yet been standardized.

† This diverges from the notation of Segre, "Varietà cubiche dello spazio a quattro dimensione . . ", *Mem. Torino* (2), 39 (1883), 3.

or the number of lines in a space. Those of the hypercongruence are its order M, or the order of the cone of its lines through a point, and its class (first class), N, or the number of its lines in a plane. We have to determine :—

(i) The class, V, of the envelope cut on a plane ϖ by the planes of a pair of rays, one of each system, concurrent on a space Π. Let Q be a fixed point in the plane; we shall construct a correspondence between the lines of the pencil (Q, ϖ), whose united lines include the lines of the envelope through Q. Take any line g of this pencil, and let it cut the section of the form F, described by the congruence in m' points. Through each pass κ rays of the system, cutting Π in $\kappa m'$ or m points in all. The M-tic cones* of these points cut ϖ in mM points, and these points joined to Q give the correspondents g' of g. Inversely a line g' is met by a form of lines of the hypercongruence of order $(M+N)$, which cuts the section of Π' and the $(m+n)$-tic scroll of lines of F meeting ϖ in $(m+n)(M+N)$ points, each of which gives a single point where the conjoint generator of the scroll meets ϖ, and thence a single line g. The united lines of the correspondence (g, g') are the lines sought, and the lines joining Q to the points of intersection of the line $\varpi\Pi'$ and F, each counting M times. The class in question is thus

$$V = mM + (m+n)(M+N) - mM = (m+n)(M+N).$$

(ii) The class, U, of the congruence of limiting joining lines for the coincidences ϖF of the main correspondence, or, by considering the number through a point Q in ϖ, the class of the envelope cut on a plane ϖ by the planes of pairs of rays concurrent on and cospacial with ϖ. We again establish a correspondence in ϖ, whose coincidences include the same points. Let A be a point on the curve of section of ϖ and F; each one of the κ congruence lines through it is met by a form of lines of the hypercongruence of order $(M+N)$ cutting ϖ in a curve on which A is M-fold. We associate to A all points of all these curves. If B is any points in ϖ, its correspondents A are given by taking the intersections of the M-tic cone of B, and the $(m+n)$-tic scroll of lines of F meeting ϖ, each giving a single point A. The indices are thus 0 and $(m+n)M$. The number of pairs on a fixed line equals the order of the locus of A multiplied by the number of correspondents of a point A on a line through it or $m' . \kappa . (M+N-M)$, namely mN. Applying the Schubert Theorem to this singular correspondence the class sought is found to be $mM + nM + mN$. Applying the same theorem to the main correspondence of the general case the number of common lines is found to be $mM + nN$, which is the form the general result takes for this special space.

9. *A note on the cases when one or other system is special.*

The proof in § 8 is inapplicable as it stands in the cases when the locus of the $\{i\}$ is of fewer dimensions than in general, and when the system of the $\{j\}$ has a locus not identical with the whole of space. The correspondence is now singular in both senses in either case, so that the second index is also zero, and the term mM is missing from the final result. The subsequent discussion is unaltered, or undergoes modification of the same nature. Since, however, $m = 0$, $M = 0$, in the respective cases the theorem as stated is still true. In the same way it is found that the result of § 6 remains true if either or both generalized congruences are special systems having definite loci.

* Two dimensional.

In all cases the order and a succession of the classes may vanish, but it is readily seen that, if any one class (including the order as the 0-th class) does not vanish, none of the subsequent classes can do so in virtue of the sub-systems to which they correspond having loci of lower dimension than in general.*

10. *The order of the system* Γ_3 *of dimension* $i+j-2r+2$ *common to two systems* Γ_1, Γ_2 *of dimensions* i, j *respectively* $(i \leqslant j)$.

Let the respective characteristics be $(m\,;\ n_1,\ n_2,\ \ldots)$ and $(M\,;\ N_1,\ N_2,\ \ldots)$. Then if Γ_3 is unrestricted so also must be Γ_1 and Γ_2; and the order of Γ_3 is mM. If, however, Γ_3 is restricted, we are to determine the number of its lines meeting a $[3r-j-i-3]$. Let Γ_1' be the sub-system of lines of Γ_1 meeting this space, and sharing a finite number of lines with Γ_2, namely the lines whose number is sought. Now § 4 enables us to write down the characteristics of Γ_1' in all cases, and thence by § 8 we pass at once to the solution sought. The problem may be considered completely solved, since, as we have already remarked, no advantage is gained by formal expression of the results, owing to the large number of sub-cases which would have to be considered.

Application of the same process to the sub-systems of Γ_1 and Γ_2 in linear spaces gives the complete system of characteristics of their common system Γ_3.

These results may also be found *ab initio* by a correspondence constructed in the same way as in §§ 6, 8, replacing the first space II by an arbitrary space of dimension t, where $t = 3r-i-j-3$. We select one example for illustration.

To find the order of the system common to an $\{r-1\}$ *and a* $\{2r-4\}$, *in space of dimension* r.

The common system is of dimension $r-3$, and generates in general a construct of dimension $r-2$, whose order is sought, namely the number of its lines meeting a plane η. Let $(m\,;\ n_1,\ n_2,\ \ldots)$ be the characteristics of the generalized congruence, $(M\,;\ N)$ those of the $\{2r-4\}$. Taking an arbitrary space Π' of dimension $r-1$, let us establish in η the usual correspondence of pairs (A, B). The correspondents of a point A are mM in number, and those of a point B are obtained by drawing the

* Though, of course, such a class can vanish in virtue of the sub-system being non-existent, in which case all subsequent classes vanish.

M-tic cone of B cutting Π' in a V^M_{r-3} which is met by a V_{r-2}, of order $(m+n_1+n_2)M$, of lines of the generalized congruence. $M(m+n_1+n_2)$ of these lines meet the plane η, giving the points A sought. The number of pairs on a fixed line g is given by the number of points of intersection on Π', of a scroll of order $m+n_1$, and a form of order $M+N$, having g as m- M-fold respectively, excluding those on g. There are therefore $n_1(M+N)+mN$ such pairs.

The united points of the correspondence are the complete coincidences arising from the intersections with η of the V_{r-2}, together with the points of the line h common to η and Π'. We are therefore to find the class of the envelope of limiting joining lines for these coincidences. By considering the lines of the envelope through a point of h it is seen that the class equals mM, together with the number of planes through h containing a ray of each system concurrent thereon. This latter number can be found by a correspondence among the ∞^{r-2} planes through h, associating two such planes ξ, ξ' when they contain rays of the two systems concurrent on h. The locus of ξ is a n_1-tic [1]-cone,* C, and to each of its planes correspond the planes of a M-tic cone. Conversely to each of the latter planes (namely to an arbitrary plane through the line h) correspond Nm planes ξ'. If more especially† we associate to ξ the planes ξ'_* of intersection of the cone C and the cone of planes corresponding to the ξ, Mn_1 in number, we have among the planes of C a correspondence (Nm, Mn_1) of zero valency. The n_1M+mN *united* planes are the planes sought. The class of the envelope dealt with is thus $mM+n_1M+mN$.

Substituting, we obtain finally $M(m+n_1+n_2)+n_1N$ as the order of the V_{r-2}, which we seek. In the particular case of space of four dimensions the term in n_2 is absent (since there is no such characteristic), but the proof remains otherwise unaltered.

11. *Some important non-additive characteristics of Line Systems.*

The orders and classes previously defined have the additive property in that, if such a system is degenerate, these characteristics for the total system equal the sums of the corresponding characteristics of its components. This property is not shared by the following additional characteristic numbers.

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* *I.e.* a cone of planes with a line as centre, ∞^1 in the present cases.

† The correspondence may also be taken as a singular correspondence in the [1]-star, centre h.

I. Consider first a generalized congruence $\{r-1\}$ of $[r]$. We define as *its rank, or first rank, the number of times a pair of its rays concur on a fixed line, and lie in a (variable) $[r-1]$ through a fixed $[r-2]$ containing the line.* We denote it by $R_{1,0}$, and for a congruence in [3] it is the rank of the ordinary definition. The possibility of such a characteristic appears later. For the present consider the focal form, locus of points (foci) where a ray meets a consecutive ray, and touched $r-1$ times by every ray of the system.* Its order θ can be expressed in terms of $R_{1,0}$ by the following extension of a well known theorem.† Taking the figure of the definition, let us associate spaces $[r-1]$ through the $[r-2]$ when they contain rays concurrent on the line. We have at once

$$\theta = 2n_1(m-1)-2R_{1,0}. \tag{12}$$

We take further a system‡ $[r-2+i]$ in $[r]$, and consider its lines meeting a space $[t]$. They form a restricted system when $t<r-i$, and their locus possesses a double construct, locus of intersections of pairs of generators. For the higher values of t, this construct passes through $[t]$, the condition for this being that the cone V_i of rays of the system passing through an arbitrary point of the $[t]$ possess at least a pair of rays lying in a $[t+1]$ through the $[t]$, and this is the case if $t>r-2i$.

Consider therefore the system of lines meeting a $[r-2i-h]$, where h is zero or positive. They generate a V_{r-i-h}, which possesses a $V_{r-2i-2h}$ of improper double points,§ cutting the space in a $V_{r-2i-2h-1}$. This gives then the following definition.

II. *The i-th rank of species h, $R_{i,h}$, of an $\{r-2+i\}$ in $[r]$ equals the number of the pairs of its rays concurrent on a $[h+1]$, and lying in a $[r-2i-h+1]$ through a given $[r-2i-h]$ containing the $[h+1]$.* $R_{i,h}$ is, in fact, the order of the above $V_{r-2i-2h-1}$. For the definition to be significant we must have $i \leqslant [\frac{1}{2}(r-1]$, and $h \leqslant [\frac{1}{2}(r-2i-1)]$. This definition suggests a series of correspondence arguments (especially for the generalized congruence), similar to the one constructed above, but

* C. Segre, "Un' osservazione . . .", *loc. cit.*

† C. M. Jessop, *op. cit.*, 262. The method of proof of the present extension is immediately suggested by the proof for the theorem in ordinary space.

‡ Where i ranges from unity to a certain value. For restricted systems the ranks are replaced by the orders of the loci of intersection of two or more rays, either for the whole system or for sub-systems.

§ F. Severi, "Sulle intersezioni delle varietà algebriche . . .", *Mem. Torino* (2), 52 (1903), 61.

their discussion would lead us outside the limits we have assigned to this paper.

III. Consider finally any unrestricted system $\{j\}$ in r. *Its sub-system in a space* $[N]$, *of dimension* $N = 2r-j+i-2$, *is of dimension* $N+i-2$, *and we define as the ranks* $R_{i,h}$ *(for various values of* i*) of the original system the ranks* $R_{i,h}$ *of this sub-system.* The notation has been so arranged that the ranks of the sub-system of the $\{j\}$ in any space are included in those of the $\{j\}$ and the equal ranks are denoted by the same symbol. The values of i are restricted by two conditions :— (i) As above, we must have $2i+1 \leqslant N = 2r-j+i-2$, or $i \leqslant 2r-j-3$. (ii) Since $N \leqslant r$, we have $2r-j+i-2 \leqslant r$, or $i \leqslant j+2-r$. There are therefore two sub-cases according as to which condition operates first.

(1) $2j \leqslant 3r-5$. The second condition operates before the first, and, writing $j = r+k-2$, the values of i range from 1 to k.

(2) $2j > 3r-5$. The reverse is the case, and i ranges from unity to $r-k-1$.

In either case for a given value of i, j ranges from zero to

$$[\tfrac{1}{2}(N-2i-1)] = [\tfrac{1}{2}(r-k-i-1)].$$

Other systems of numbers arise from concurrencies of more than two rays, and can be defined in the same way. These numbers, however, are not all independent, but are connected with each other and with the ranks. The further development of the theory of line systems will require the determination of the complete system of relations of this nature.

THE INVERSION FORMULAE OF HARDY AND TITCHMARSH

By RICHARD G. COOKE.

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1. The inversion formulae of Hardy* are

$$\phi(x) = \int_0^\infty F_\nu(xu)\, u f(u)\, du, \tag{1.1}$$

$$f(x) = \int_0^\infty C_\nu(tx)\, t\, \phi(t)\, dt,$$

where $$C_\nu(z) = \cos a\pi\, J_\nu(z) + \sin a\pi\, Y_\nu(z),$$

and $$F_\nu(z) = \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{1}{2}z)^{\nu+2a+2p}}{\Gamma(a+1+p)\,\Gamma(a+1+\nu+p)}$$

$$= \frac{2^{1-\mu}\, s_{\mu,\nu}(z)}{\Gamma\{(\mu+\nu+1)/2\}\,\Gamma\{(\mu-\nu+1)/2\}},$$

$s_{\mu,\nu}(z)$ being Lommel's function,† and

$$\mu = \nu + 2a - 1.$$

The inversion formulae of Titchmarsh‡ are

$$\phi(x) = \pi \int_0^\infty \Xi_\nu(xu)\, u f(u)\, du, \tag{1.2}$$

$$f(x) = \int_0^\infty \frac{d}{dx}\, \{x\, G_\nu(tx)\}\, t\, \phi(t)\, dt,$$

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* G. H. Hardy, "Some formulae in the theory of Bessel functions", *Proc. London Math. Soc.* (2), 23 (1925), lxi (*Records* for June 12th, 1924).

† G. N. Watson, *Theory of Bessel Functions* (1922), 345.

‡ E. C. Titchmarsh, "An Inversion Formula involving Bessel Functions", *Proc. London Math. Soc.* (*Records* for November 13th, 1924), iii.

where $$\Xi_\nu(z) = \sin a\pi \{J_\nu^2(z) - Y_\nu^2(z)\} - 2\cos a\pi\, J_\nu(z)\, Y_\nu(z),$$

and* $$G_\nu(z) = \sum_{p=0}^{\infty} \frac{(-1)^p\, \Gamma(\nu+p+a+\frac{1}{2})\, z^{2\nu+2a+2p}}{\Gamma(\frac{1}{2})\,\Gamma(a+p+1)\,\Gamma(\nu+a+p+1)\,\Gamma(2\nu+a+p+1)}.$$

Hardy has pointed out† that $F_\nu(z)$ reduces to $J_\nu(z)$ when $a = 0$, and to $\mathbf{H}_\nu(z)$ when $a = \frac{1}{2}$; it may be added that it reduces to $J_{-\nu}(z)$ when $a = -\nu$. The case $a = -\frac{1}{2}$ is also of interest. In this case we derive the formula

$$(1.3) \qquad f(x) = -\int_0^\infty \left\{ \frac{(\frac{1}{2}xu)^{\nu-1}}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} - \mathbf{H}_\nu(xu) \right\} u\, du \int_0^\infty Y_\nu(tu)\, t f(t)\, dt,$$

which reduces to Fourier's integral formula for the function $x^{\frac{1}{2}} f(x)$ when $\nu = \frac{1}{2}$; $\mathbf{H}_\nu(z)$ is Struve's function. If $f(t)$ satisfy conditions which ensure the truth *both* of the formula (1.3) *and* of the formula‡

$$f(x) = \int_0^\infty \mathbf{H}_\nu(xu)\, u\, du \int_0^\infty Y_\nu(tu)\, t f(t)\, dt,$$

then we have the result

$$(1.4) \qquad \int_0^\infty u^\nu\, du \int_0^\infty Y_\nu(tu)\, t f(t)\, dt = 0.$$

This may be verified by inverting the order of integration, and then writing $\mu = 2\nu+1$ in the formula§

$$\int_0^\infty \frac{Y_\nu(t)}{t^{\nu-\mu+1}}\, dt = -\frac{\Gamma(\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}\mu-\nu)\cos(\frac{1}{2}\mu-\nu)\pi}{2^{\nu-\mu+1}\pi}.$$

Titchmarsh has pointed out‖ that $G_\nu(z)$ reduces to $J_\nu^2(z)$ when $a = 0$; it may be added that it reduces to $J_\nu(z) J_{-\nu}(z)$ when $a = -\nu$, and to $J_{-\nu}^2(z)$ when $a = -2\nu$.

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* For some special cases of (1.1) and (1.2), see G. H. Hardy, *loc. cit.*, and E. C. Titchmarsh, "Extensions of Fourier's Integral Formula to formulae involving Bessel Functions", *Proc. London Math. Soc.* (2), 23 (1925), xxii (*Records* for January 17th, 1924).

† G. H. Hardy, *loc. cit.*

‡ By inspection of the conditions given below for the truth of the formula (A), we see that if $t^{-\nu} f(t)$ is integrable over $(0, \delta)$ $(\delta > 0)$, and $t^{\frac{1}{2}} f(t)$ is integrable over (δ, ∞), where $|\nu| \leqslant \frac{1}{2}$, then both formulae under consideration will be true. See also § 3 (*i*).

§ G. N. Watson, *loc. cit.*, 393 (5).

‖ E. C. Titchmarsh, "An Inversion Formula involving Bessel Functions", *Proc. London Math. Soc.* (*Records* for November 13th, 1924), iii.

In the former case, $a = -\nu$, we derive the formula

(1.5)

$$f(x) = \frac{\pi}{\sin \nu\pi} \int_0^\infty \{J^2_{-\nu}(xu) - J^2_\nu(xu)\}\, u\, du \int_0^\infty \frac{d}{du} \{u\, J_\nu(tu)\, J_{-\nu}(tu)\}\, t f(t)\, dt,$$

on expressing Y_ν in terms of J_ν and $J_{-\nu}$.

As in the case of Hardy's formula, we can deduce a "null" formula from Titchmarsh's formula in the cases $a = \frac{1}{2}$, $a = -\frac{1}{2}$.

If the conditions for the truth of the formula in both these cases are satisfied, then

$$\int_0^\infty u^{2\nu}\, du \int_0^\infty \{J^2_\nu(tu) - Y^2_\nu(tu)\}\, t f(t)\, dt = 0.$$

This may be verified by inverting the order of integration, and then applying the formula*

$$\int_0^\infty \frac{J_\mu(at)\, J_\nu(at)}{t^\lambda}\, dt = \frac{(\frac{1}{2}a)^{\lambda-1}\, \Gamma(\lambda)\, \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\lambda + \frac{1}{2})}{2\Gamma(\frac{1}{2}\lambda + \frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2})\, \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2})\, \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})},$$

after expressing Y_ν in terms of J_ν and $J_{-\nu}$.

The two formulae

(A) $$f(x) = \int_0^\infty F_\nu(xu)\, u\, du \int_0^\infty C_\nu(tu)\, t f(t)\, dt,$$

(A′) $$f(x) = \int_0^\infty C_\nu(xu)\, u\, du \int_0^\infty F_\nu(tu)\, t f(t)\, dt$$

are evidently obtained from Hardy's formulae; whilst

(B) $$f(x) = \pi \int_0^\infty u\, \Xi_\nu(xu)\, du \int_0^\infty \frac{d}{du} \{u\, G_\nu(tu)\}\, t f(t)\, dt,$$

(B′) $$f(x) = \pi \int_0^\infty \frac{d}{du} \{u\, G_\nu(xu)\}\, u\, du \int_0^\infty \Xi_\nu(tu)\, t f(t)\, dt$$

are obtained from Titchmarsh's formulae.

It is the object of this paper to establish conditions for the truth of

* G. N. Watson, *loc. cit.*, 403 (2).

these formulae We shall always suppose that $f(t)$ is of bounded variation in the neighbourhood of the point $t = x$.

If, in any of the four formulae, $f(t)$ is not continuous at $t = x$, we must then replace $f(x)$ by $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

The following theorems will be proved :—

(1) *Let* (i) $a > -1, \quad a+\nu > -1$;

(ii) $|2a+\nu| < \frac{3}{2}$;

(iii) $t^\sigma f(t)$ *be integrable over* $(0, \delta)$,

where $\delta > 0$ *and* $\sigma = \text{Min}\,(1-|\nu|,\ 1-2a-\nu,\ \frac{1}{2})$;

(iv) $t^{\frac{1}{2}} f(t)$ *be integrable over* (δ, ∞);

then formula (A) *is true.*

(2) *Let* (i) $a > -1, \quad a+\nu > -1$;

(ii) $\nu+2a < \frac{3}{2}$;

(iii) $|\nu| \leqslant \frac{3}{2}$;

(iv) $t^\sigma f(t)$ *be integrable over* $(0, \delta)$,

where $\delta > 0$ *and* $\sigma = \text{Min}\,(1+\nu+2a,\ \frac{1}{2})$;

(v) $t^{\frac{1}{2}} f(t)$ *be integrable over* (δ, ∞);

then formula (A′) *is true.*

(3) *Let* (i) $a > -1, \quad a+2\nu > -1$;

(ii) $1 > a+\nu \geqslant -\frac{1}{2}$;

(iii) $|\nu| \leqslant 1$;

(iv) $t^\sigma f(t)$ *be integrable over* $(0, \delta)$,

where $\delta > 0$ *and* $\sigma = \text{Min}\,(1+2\nu+2a,\ 1)$;

(v) $tf(t)$ *be integrable over* (δ, ∞);

then formula (B) *is true.*

(4) *Let* (i) $a > -1, \quad a+2\nu > -1, \quad \nu > -\frac{1}{2}$;

(ii) $1 > a+\nu \geqslant -\frac{1}{2}$;

(iii) $t^\sigma f(t)$ *be integrable over* $(0, \delta)$,

where $\delta > 0$ *and* $\sigma = \mathrm{Min}\,(0, 1-|2\nu|)$;

(iv) *$f(t)$ be integrable over* (δ, ∞);

then formula (B′) *is true.*

The methods to be adopted are those given by Titchmarsh in his proof of Weber's Integral Theorem.*

2. In this section we will deal with the asymptotic forms required for the various functions which arise. In the first place, it is easily deduced from the integral†

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{iz-\frac{1}{2}\nu\pi-\frac{1}{4}\pi}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty e^{-u} u^{\nu-\frac{1}{2}} \left(1-\frac{iu}{2z}\right)^{\nu-\frac{1}{2}} du$$

that each of the functions $z^{\frac{1}{2}}J_\nu(z)$, $z^{\frac{1}{2}}Y_\nu(z)$ is expressible in the form

$$\cos z \,.\, f_1(z) + \sin z \,.\, f_2(z), \tag{2.1}$$

where f_1, f_2 are functions of bounded variation over $(1, \infty)$ for all values of ν.‡

$F_\nu(z)$, a Lommel Function, has for its asymptotic expansion

$$\begin{aligned} &\cos a\pi\, J_\nu(z) + \sin a\pi\, Y_\nu(z) \\ &\qquad + \frac{(\frac{1}{2}z)^{\nu+2a-2}}{\Gamma(a)\,\Gamma(a+\nu)} \sum_{m=0}^{p-1} \frac{(-)^m \Gamma(m-a+1)\,\Gamma(m-a+1-\nu)}{(\frac{1}{2}z)^{2m}\,\Gamma(1-a)\,\Gamma(1-a-\nu)} \\ &\qquad + O(z^{\nu+2a-1-2p})\S \qquad (2.2) \\ &= C_\nu(z) + \frac{(\frac{1}{2}z)^{\nu+2a-2}}{\Gamma(a)\,\Gamma(a+\nu)} \\ &\qquad + Az^{\nu+2a-2} \int_{-\infty i-\frac{1}{2}}^{\infty i-\frac{1}{2}} \frac{\Gamma(1-a-s)\,\Gamma(1-a-\nu-s)}{\Gamma(1-a)\,\Gamma(1-a-\nu)} \cdot \frac{\pi(\frac{1}{2}z)^{2s}\,ds}{\sin s\pi},\| \end{aligned}$$

where, in the integral, p is given its least value, viz. unity.

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* E. C. Titchmarsh, "Weber's Integral Theorem", *Proc. London Math. Soc.* (2), 22 (1924), 15-28.

† G. N. Watson, *loc. cit.*, 168 (3).

‡ This was pointed out to me by Mr. Titchmarsh.

§ This may be gathered from G. N. Watson, *loc. cit.*, 347 and 351–2.

‖ *Ibid.*, 351.

The integral may be differentiated under the integral sign with respect to z, and the differential coefficient is

$$O(z^{\nu+2a-4}).$$

Now $O(z^{\nu+2a-\frac{5}{2}})$ is integrable up to infinity if $\nu+2a<\frac{3}{2}$, and then we may write

$$z^{\frac{3}{2}}F_\nu(z) = z\cos z\,.f_1(z)+z\sin z\,.f_2(z)+f_3(z), \tag{2.3}$$

f_1, f_2 being the functions of (2.1), and f_3 being of bounded variation over (δ, ∞), $(\delta>0)$, if

$$\nu+2a<\tfrac{3}{2}. \tag{2.4}$$

We now require an asymptotic expansion for the function $G_\nu(z)$. Consider

$$-\frac{z^{2a+2\nu}}{2\pi i}\int_{-\infty i-p+\frac{1}{2}}^{\infty i-p+\frac{1}{2}} \frac{\Gamma(-a-s)\,\Gamma(-a-\nu-s)\,\Gamma(-a-2\nu-s)}{\Gamma(-a-\nu-s+\frac{1}{2})}\cdot\frac{\pi z^{2s}\,ds}{\sin s\pi}, \tag{2.5}$$

where the contour is drawn by taking p to be an integer so large that the only poles of the integrand to the left of the contour are poles of cosec $s\pi$, the poles of the Gamma Functions being on the right of the contour.

By the usual methods of dealing with integrals of the Barnes type, it may be shown that the integral (2.5) is convergent when $|\arg z|<\pi$, and is $O(z^{2a+2\nu+1-2p})$; and that the integrand when integrated round a semicircle of centre $-p+\frac{1}{2}$, and radius R, on the right of the contour, tends to zero as $R\to\infty$, provided that $R\to\infty$ in such a way that the semicircle never passes through any of the poles of the integrand.

It follows that the integral (2.5) is equal to the sum of the residues of

$$-z^{2a+2\nu+2s}\,\frac{\Gamma(-a-s)\,\Gamma(-a-\nu-s)\,\Gamma(-a-2\nu-s)}{\Gamma(-a-\nu-s+\frac{1}{2})}\cdot\frac{\pi}{\sin s\pi}$$

at the points

$$-1,\quad -2,\quad -3,\quad \ldots\quad -(p-1),$$

$$0,\quad 1,\quad 2,\quad 3,\quad \ldots,$$

$$-a,\quad -a+1,\quad -a+2,\quad \ldots,$$

$$-a-\nu,\quad -a-\nu+1,\quad -a-\nu+2,\quad \ldots,$$

$$-a-2\nu,\quad -a-2\nu+1,\quad -a-2\nu+2,\quad \ldots$$

and this is equal to

$$-z^{2a+2\nu}\sum_{m=1}^{p-1}\frac{(-)^m\,\Gamma(m-a)\,\Gamma(m-a-\nu)\,\Gamma(m-a-2\nu)}{z^{2m}\,\Gamma(m-\nu-a+\frac{1}{2})}$$

$$+\frac{\pi^2\cot(a+\nu)\pi\,.\,z^{2a+2\nu}}{\sin a\pi\,\sin(a+2\nu)\,\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma(\nu+m+a+\frac{1}{2})\,z^{2m}}{\Gamma(a+m+1)\,\Gamma(\nu+a+m+1)\,\Gamma(2\nu+a+m+1)}$$

$$-\frac{\pi^2 z^{2\nu}}{2\sin a\pi\,\sin^2\nu\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma(\frac{1}{2}+m+\nu)\,z^{2m}}{m!\,\Gamma(m+\nu+1)\,\Gamma(m+2\nu+1)}$$

$$+\frac{\pi^2}{\sin(a+\nu)\,\pi\,\sin^2\nu\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma(m+\frac{1}{2})\,z^{2m}}{m!\,\Gamma(m+\nu+1)\,\Gamma(m-\nu+1)}$$

$$-\frac{\pi^2 z^{-2\nu}}{2\sin(a+2\nu)\,\pi\,.\,\sin^2\nu\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma(\frac{1}{2}+m-\nu)\,z^{2m}}{m!\,\Gamma(m-\nu+1)\,\Gamma(m-2\nu+1)}$$

$$=-z^{2a+2\nu}\sum_{m=1}^{p-1}\frac{(-1)^m\,\Gamma(m-a)\,\Gamma(m-a-\nu)\,\Gamma(m-a-2\nu)}{z^{2m}\,\Gamma(m-a-\nu+\frac{1}{2})}$$

$$+\frac{\pi^2\cot(a+\nu)\,\pi}{\sin a\pi\,\sin(a+2\nu)\,\pi}\,\Gamma(\tfrac{1}{2})\,G_\nu(z)-\frac{\pi^2\,\Gamma(\frac{1}{2})}{2\sin a\pi\,\sin^2\nu\pi}\,J_\nu^2(z)$$

$$+\frac{\pi^2\,\Gamma(\frac{1}{2})}{\sin(a+\nu)\,\pi\,\sin^2\nu\pi}\,J_\nu(z)\,J_{-\nu}(z)-\frac{\pi^2\,\Gamma(\frac{1}{2})}{2\sin(a+2\nu)\,\pi\,\sin^2\nu\pi}\,J_{-\nu}^2(z).*$$

We consequently have the asymptotic expansion

$$(2.6)\quad G_\nu(z)=\operatorname{cosec}^2\nu\pi\,[\tfrac{1}{2}\sin(a+2\nu)\,\pi\tan(a+\nu)\,\pi\,J_\nu^2(z)$$

$$-\sin a\pi\,\sin(a+2\nu)\,\pi\sec(a+\nu)\,\pi\,J_\nu(z)\,J_{-\nu}(z)$$

$$+\tfrac{1}{2}\sin a\pi\tan(a+\nu)\,\pi\,J_{-\nu}^2(z)]$$

$$+\frac{\sin a\pi\,\sin(a+2\nu)\,\pi\tan(a+\nu)\,\pi}{\pi^2}\,z^{2\nu+2a}$$

$$\times\sum_{m=1}^{p-1}\frac{(-1)^m\,\Gamma(m-a)\,\Gamma(m-a-\nu)\,\Gamma(m-a-2\nu)}{z^{2m}\,\Gamma(m-\nu-a+\frac{1}{2})}$$

$$+O(z^{2a+2\nu+1-2p}).$$

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* G. N. Watson, *loc. cit.*, 147.

The last two terms of (2.6), when multiplied by z,

$$= -z^{2\nu+2a-1}$$

$$\times \frac{\sin a\pi \sin (a+2\nu)\pi \tan (a+\nu)\pi\, \Gamma(1-a)\,\Gamma(1-a-\nu)\,\Gamma(1-a-2\nu)}{\pi^{\frac{3}{2}}\,\Gamma(\frac{3}{2}-\nu-a)}$$

$$+Az^{2\nu+2a+1}\int_{-\infty i-\frac{3}{2}}^{\infty i-\frac{3}{2}} \frac{\Gamma(-a-s)\,\Gamma(-a-\nu-s)\,\Gamma(-a-2\nu-s)}{\Gamma(-a-\nu-s+\frac{1}{2})}\cdot\frac{\pi z^{2s}\,ds}{\sin s\pi},$$

taking $p=2$ (its least value).

The differential coefficient of this with respect to z is

$$O(z^{2\nu+2a-3}),$$

and this is integrable up to infinity if $2\nu+2a-2<0$; thus we must have

$$a+\nu<1.$$

Hence we may write, using (2.1),

$$zG_\nu(z)=\cos 2z\,\phi_1(z)+\phi_2(z), \tag{2.7}$$

where ϕ_1, ϕ_2 are functions of bounded variation over $(1,\infty)$ for all values of ν [since they involve products of the functions f_1, f_2 of the formula (2.1)], provided

$$a+\nu<1.$$

We next require an asymptotic expansion for

$$\frac{d}{dz}\{zG_\nu(z)\}.$$

We have

$$\frac{d}{dz}\{zG_\nu(z)\}=G_\nu(z)+zG'_\nu(z) \tag{2.8}$$

$$= G_\nu(z)+Az\,J_\nu(z)\{J_{\nu-1}(z)-J_{\nu+1}(z)\}$$

$$+Bz[J_{-\nu}(z)\{J_{\nu-1}(z)-J_{\nu+1}(z)\}+J_\nu(z)\{J_{-\nu-1}(z)-J_{-\nu+1}(z)\}]$$

$$+Cz\{J_{-\nu-1}(z)-J_{-\nu+1}(z)\}J_{-\nu}(z)$$

$$+Dz\frac{d}{dz}\left[z^{2\nu+2a}\int_{-\infty i-p+\frac{1}{2}}^{\infty i-p+\frac{1}{2}}\frac{\Gamma(-a-s)\,\Gamma(-a-\nu-s)\,\Gamma(-a-2\nu-s)}{\Gamma(-a-\nu-s+\frac{1}{2})}\cdot\frac{\pi z^{2s}ds}{\sin s\pi}\right]$$

since $$J'_\nu(z)=\tfrac{1}{2}\{J_{\nu-1}(z)-J_{\nu+1}(z)\};$$

A, B, C, D being independent of z.

Hence, by (2 . 7),

(2 . 9)

$$\frac{d}{dz}\{zG_\nu(z)\} = \frac{\cos 2z}{z}\,.\,\phi_1(z)+\frac{\phi_2(z)}{z}+\cos 2z\,.\,\phi_3(z)+\phi_4(z)+Kz^{2\nu+2a}$$

$$\times\int_{-\infty i-p+\frac{1}{2}}^{\infty i-p+\frac{1}{2}} \frac{\Gamma(-a-s)\,\Gamma(1-a-\nu-s)\,\Gamma(-a-2\nu-s)}{\Gamma(-a-\nu-s+\frac{1}{2})}\cdot\frac{\pi z^{2s}\,ds}{\sin s\pi}.$$

The differential coefficient of the last term on the right-hand side of (2 . 9) is

$$O(z^{2a+2\nu-2p}),$$

and giving p its least value, 2, we see that this is integrable up to infinity if $a+\nu<\frac{3}{2}$.

Hence we may write

(2 . 10) $$\frac{d}{dz}\{z\,G_\nu(z)\} = \cos 2z\,.\,\psi_1(z)+\psi_2(z),$$

where ψ_1, ψ_2 are of bounded variation over $(1, \infty)$ for all values of ν such that

(2 . 11) $$a+\nu<1;$$

for $\psi_1(z)\equiv\{\phi_1(z)/z\}+\phi_3(z)$, which is clearly of bounded variation over $(1, \infty)$, if $a+\nu<1$; similarly for $\psi_2(z)$.

Finally, an asymptotic expansion for the function

$$T_\nu(z)\equiv\frac{1}{z^2}\int_0^z t\,G_\nu(t)\,dt$$

will be required.

This is obtained by integrating the asymptotic expansion* of $zG_\nu(z)$ term by term; this gives the asymptotic expansion of $z^2T_\nu(z)$. We find that

(2 . 12) $$T_\nu(z) = \psi_3(z)+\sin 2z\,.\,\psi_4(z),$$

where ψ_3, ψ_4 are of bounded variation over $(1, \infty)$, if $a+\nu<1$.

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* The more useful form to employ is (2 . 7).

3. *Proof of the Theorem (A).*

(α) We have first to evaluate a definite integral. Consider

$$\int_\Gamma \{F_\nu(cz)-e^{a\pi i}J_\nu(cz)\}\left\{H^{(1)}_{\nu+1}(bz)+i\frac{\Gamma(\nu+1)}{\pi}\left(\frac{2}{bz}\right)^{\nu+1}\right\}dz,$$

where Γ is the contour formed by the real axis, a semicircular indentation (γ) above the origin, and a large semicircle above the real axis, with centre at the origin. The integral round the large semicircle certainly vanishes if $a<\frac{3}{4}-\frac{1}{2}\nu$;* whilst the integral round γ

$$=\pi i\,.\operatorname*{Lim}_{z\to 0}\left[-ze^{a\pi i}J_\nu(cz)\left\{iY_{\nu+1}(bz)+i\frac{\Gamma(\nu+1)}{\pi}\left(\frac{2}{bz}\right)^{\nu+1}\right\}\right]=0,$$

if $a+\nu>-1$ and $\nu>-1$, since

$$\operatorname*{Lim}_{z\to 0} z^{\nu+1}Y_{\nu+1}(bz)=-\frac{2^{\nu+1}}{\pi b^{\nu+1}}\Gamma(\nu+1). \tag{3.1}$$

Hence

$$\int_{-\infty}^{\infty}\{F_\nu(cx)-e^{a\pi i}J_\nu(cx)\}\left\{H^{(1)}_{\nu+1}(bx)+i\frac{\Gamma(\nu+1)}{\pi}\left(\frac{2}{bx}\right)^{\nu+1}\right\}dx=0,$$

so that

$$\int_0^\infty\Big[\{F_\nu(cx)-e^{a\pi i}J_\nu(cx)\}\left\{J_{\nu+1}(bx)+iY_{\nu+1}(bx)+i\frac{\Gamma(\nu+1)}{\pi}\left(\frac{2}{bx}\right)^{\nu+1}\right\}$$

$$-\{e^{(\nu+2a)\pi i}F_\nu(cx)-e^{(\nu+a)\pi i}J_\nu(cx)\}\,e^{-(\nu+1)\pi i}\left\{J_{\nu+1}(bx)-iY_{\nu+1}(bx)\right.$$

$$\left.+i\frac{\Gamma(\nu+1)}{\pi}\left(\frac{2}{bx}\right)^{\nu+1}\right\}\Big]dx=0.$$

By equating imaginary parts, we get, on division by $2\sin a\pi$,

$$\int_0^\infty F_\nu(cx)\left\{C_{\nu+1}(bx)+\frac{\sin a\pi}{\pi}\Gamma(\nu+1)\left(\frac{2}{bx}\right)^{\nu+1}\right\}dx \tag{3.2}$$

$$=\int_0^\infty J_\nu(cx)\,J_{\nu+1}(bx)\,dx$$

$$=\frac{c^\nu}{b^{\nu+1}}\ (c<b),\quad \frac{1}{2c}\ (c=b),\quad 0\ \ (c>b).\dagger$$

---

* This, as will be seen, is one of the conditions for the truth of the theorem.

† See G. N. Watson, *loc. cit.*, 406 (8). By the theory of analytic continuation, the result (3.2) is valid for (complex) values of a and ν, such that $R(a+\nu)>-1$, $R(2a+\nu)<\frac{3}{2}$, and hence it is valid when $a+\nu>-1$, $2a+\nu<\frac{3}{2}$.

(β) We will now verify the theorem in the particular case

$$f(t) = t^\nu \quad (0 \leqslant t < \kappa), \quad \tfrac{1}{2}t^\nu \quad (t = \kappa), \quad 0 \quad (t > \kappa).$$

This will serve to indicate the part played by the definite integral (3 . 2).

In this particular case the repeated integral

$$\int_0^\infty F_\nu(xu)\, u\, du \int_0^\infty C_\nu(tu)\, t f(t)\, dt$$

$$= \int_0^\infty F_\nu(xu) \left\{ \cos a\pi\, J_{\nu+1}(\kappa u) \,.\, \kappa^{\nu+1} + \sin a\pi \,.\, Y_{\nu+1}(\kappa u) \,.\, \kappa^{\nu+1} \right.$$

$$\left. + \frac{\sin a\pi}{\pi}\, \Gamma(\nu+1) \left(\frac{2}{u}\right)^{\nu+1} \right\} du.$$

Hence, by (3 . 2), the repeated integral

$$\int_0^\infty F_\nu(xu)\, u\, du \int_0^\kappa C_\nu(tu)\, t^{\nu+1}\, dt = 0 \quad (x > \kappa); \quad \tfrac{1}{2}x^\nu \quad (x = \kappa), \quad x^\nu \quad (x < \kappa),$$

which verifies the theorem in the particular case.

Preliminary Lemmas.

(γ) *The function of t and λ*

$$t^{\sigma'} \int_0^\lambda F_\nu(xu)\, C_\nu(tu)\, u\, du$$

where x is constant, and $|x-t|$ remains greater than a constant, is bounded for all values of t such that $t \geqslant \delta > 0$, if

$$a > -1, \quad a+\nu > -1, \quad |2a+\nu| < \tfrac{3}{2},$$

and $$\sigma' = \tfrac{1}{2};$$

and is bounded for all values of t such that $0 \leqslant t < \delta$ if

$$a > -1, \quad a+\nu > -1, \quad |2a+\nu| < \tfrac{3}{2},$$

and $$\sigma' = \text{Max}\,(|\nu|,\ 2a+\nu,\ \tfrac{1}{2}).$$

(i) We first consider the finite range $(0, \lambda_0)$ of u.

(a) Let $$t < t_0.$$

Then $$F_\nu(xu) = O(u^{\nu+2a}), \quad C_\nu(tu) = O(tu)^\nu + O(tu)^{-\nu}.$$

Thus

$$t^{\sigma'}\int_0^{\lambda_0} F_\nu(xu)\,C_\nu(tu)\,u\,du = O\left[t^{\sigma'}\int_0^{\lambda_0} u^{2\alpha+\nu}\{A(tu)^\nu + B(tu)^{-\nu}\}\,u\,du\right],$$

where A and B are independent of t and u;

$$= O\left[t^{\sigma'+\nu}\int_0^{\lambda_0} Au^{2\nu+2\alpha+1}\,du + t^{\sigma'-\nu}\int_0^{\lambda_0} Bu^{2\alpha+1}\,du\right].$$

For convergence at the lower limit we must have

$$\nu+\alpha > -1, \quad \alpha > -1.$$

Hence the result follows if $\sigma' \geqslant |\nu|$.

(*b*) Let $t > t_0$.

We here divide the range of integration into the two parts

$$(0,\ 1/t), \quad (1/t,\ \lambda_0).$$

In the first part we employ forms near the origin for both functions, as in (*a*), and we have

$$t^{\frac{1}{2}}\int_0^{1/t} F_\nu(xu)\,C_\nu(tu)\,u\,du = O\left[t^{\frac{1}{2}+\nu}\int_0^{1/t} u^{2\nu+2\alpha+1}\,du + t^{\frac{1}{2}-\nu}\int_0^{1/t} u^{2\alpha+1}\,du\right].$$

As before, we must have

$$\nu+\alpha > -1, \quad \alpha > -1,$$

and the expression is then seen to be

$$O(t^{-\nu-2\alpha-\frac{3}{2}}),$$

so that we must have $\nu+2\alpha > -\frac{3}{2}$.

In the second part we employ the form near the origin for $F_\nu(xu)$, and write

$$C_\nu(tu) = O(tu)^{-\frac{1}{2}}.$$

Then

$$t^{\frac{1}{2}}\int_{1/t}^{\lambda_0} F_\nu(xu)\,C_\nu(tu)\,u\,du = O\left[t^{\frac{1}{2}}\int_{1/t}^{\lambda_0} u^{2\alpha+\nu}(tu)^{-\frac{1}{2}}\,u\,du\right]$$

$$= O(t^{-\frac{3}{2}-\nu-2\alpha}).$$

Hence, as before, we must have

$$\nu+2a > -\tfrac{3}{2},$$

and the result then follows.

(ii) Let $\lambda > \lambda_0, \quad t > t_0.$

Then, by the formulae (2 . 1) and (2 . 3),

$$t^{\frac{1}{2}}\int_{\lambda_0}^{\lambda} F_\nu(xu)\, C_\nu(tu)\, u\, du$$

$$= \frac{1}{\sqrt{(x)}}\int_{\lambda_0}^{\lambda} [\cos xu\,.f_1(xu)+\sin xu\,.f_2(xu)+(xu)^{-1}f_3(xu)]$$

$$\times[\cos tu\,.f_1(tu)+\sin tu\,.f_2(tu)]\, du,$$

if (2 . 4) is satisfied.

All the integrals involved are bounded; for products such as $f_1(tu)\, f_2(xu)$ are of bounded variation over (λ_0, λ).

(iii) Let $t < t_0, \quad t\lambda > A.$

We here divide the range of integration into the two parts

$$(\lambda_0,\ 1/t), \quad (1/t,\ \lambda).$$

In the first part we see that

$$\left| t^{\sigma'}\int_{\lambda_0}^{1/t} F_\nu(xu)\, C_\nu(tu)\, u\, du \right| < Kt^{\sigma'-|\nu|}\left| \int_{\lambda_0}^{1/t} F_\nu(xu)\, u^{1-|\nu|}\, du \right|,$$

where K is a constant, since $(tu)^{|\nu|}\, C_\nu(tu)$ is of bounded variation over $(\lambda_0, 1/t)$. Also

$$t^{\sigma'-|\nu|}\int_{\lambda_0}^{1/t} F_\nu(xu)\, u^{1-|\nu|}\, du$$

$$= \frac{t^{\sigma'-|\nu|}}{\sqrt{(x)}}\int_{\lambda_0}^{1/t} [\cos xu f_1(xu)+\sin xu f_2(xu)+Au^{2a+\nu-\frac{3}{2}}+O(u^{2a+\nu-\frac{5}{2}})]\, u^{\frac{1}{2}-|\nu|}\, du,$$

by (2 . 1) and (2 . 2), taking $p = 1$.

The integrals involved are seen to be

$$O(t^{\sigma'-\frac{1}{2}}), \quad O(t^{\sigma'-2a-\nu}), \quad O(t^{\sigma'+1-2a-\nu}), \quad O(t^{\sigma'-|\nu|}).$$

Hence $$t^{\sigma'}\int_{\lambda_0}^{1/t} F_\nu(xu)\, C_\nu(tu)\, u\, du$$

is bounded, provided

$$\sigma' \geqslant \text{Max}\,(2a+\nu,\ \tfrac{1}{2},\ |\nu|).$$

In the second part, $(1/t,\ \lambda)$, we have

$$t^{\sigma'}\int_{1/t}^{\lambda} F_\nu(xu)\,C_\nu(tu)\,u\,du$$

$$= \frac{t^{\sigma'-\frac{1}{2}}}{\sqrt{(x)}}\int_{1/t}^{\lambda}[\cos xu\,f_1(xu)+\sin xu\,f_2(xu)+(xu)^{-1}f_3(xu)]$$

$$\times[\cos tu\,.f_1(tu)+\sin tu\,.f_2(tu)]\,du,$$

and this is bounded if $\sigma' \geqslant \frac{1}{2}$.

(iv) Let $\qquad t < t_0, \quad t\lambda < A.$

This can be dealt with as in the first part of Case (iii).

It will be observed from the above argument that we must have $t^{\frac{1}{2}}f(t)$ integrable over (δ, ∞), where $\delta > 0$, and $t^{\sigma}f(t)$ integrable over $(0, \delta)$, where $\sigma = \text{Min}\,(1-|\nu|,\ 1-2a-\nu,\ \frac{1}{2})$ and $\sigma+\sigma' = 1$.

This completes the proof.

In the following lemmas, the conditions of Lemma 3(γ) will be assumed to hold.

(δ) *If* $0 \leqslant A \leqslant \alpha < \beta \leqslant B$, *and* x *is not a point of the closed interval* (A, B), *then, when* $\lambda \to \infty$,

$$\int_\alpha^\beta t^{\nu+1}\,dt\int_0^\lambda F_\nu(xu)\,C_\nu(tu)\,u\,du \to 0$$

uniformly in α *and* β.

We have

$$\int_\alpha^\beta t^{\nu+1}\,dt\int_0^\lambda F_\nu(xu)\,C_\nu(tu)\,u\,du = \int_0^\lambda F_\nu(xu)\{\beta^{\nu+1}\,C_{\nu+1}(\beta u)-\alpha^{\nu+1}\,C_{\nu+1}(\alpha u)\}\,du.$$

When $\lambda \to \infty$, this becomes, by (3 . 2), equal to zero, whether $x < \beta$ or $x > \beta$. The asymptotic expansions show that the convergence is uniform.

(ϵ) *If* x *is not a point of the closed interval* $(A.\ B)$, *then*

$$\lim_{\lambda\to\infty}\int_A^B t\,f(t)\,dt\int_0^\lambda F_\nu(xu)\,C_\nu(tu)\,u\,du = 0,$$

We here employ Hobson's Convergence Theorem* :—

If $\psi(t)$ is integrable over (A, B), $\phi(\lambda, t)$ is integrable over (A, B) for all values of λ, and bounded uniformly with respect to λ in (A, B); and

$$\int_\alpha^\beta \phi(\lambda, t)\, dt \to 0$$

when $\lambda \to \infty$, uniformly in α and β, for $A \leqslant \alpha < \beta \leqslant B$; then

$$\lim_{\lambda \to \infty} \int_A^B \phi(\lambda, t)\, \psi(t)\, dt = 0.$$

Suppose first that $A > 0$, and B is finite. Then the conclusion follows from Hobson's theorem, and § § 3 (γ) and 3 (δ), putting

$$\psi(t) = t^{-\nu} f(t),$$

$$\phi(\lambda, t) = t^{\nu+1} \int_0^\lambda F_\nu(xu)\, C_\nu(tu)\, u\, du.$$

If $A > 0$, and B is infinite, we can, by § 3 (γ), choose Δ so that

$$\left| \int_\Delta^\infty t f(t)\, dt \int_0^\lambda F_\nu(xu)\, C_\nu(tu)\, u\, du \right| < \epsilon,$$

for all values of λ.

The result then follows by applying Hobson's theorem to the range (A, Δ).

We may, again by § 3 (γ), deal similarly with the case $A = 0$.

(η) *We have*

$$\int_0^\infty F_\nu(xu)\, u\, du \int_0^\infty C_\nu(tu)\, t f(t)\, dt = \lim_{\lambda \to \infty} \int_0^\infty t f(t)\, dt \int_0^\lambda F_\nu(xu)\, C_\nu(tu)\, u\, du,$$

provided the limit on the right exists.

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* E. W. Hobson, *Proc. London Math. Soc.* (2), 12 (1913), 166.

We have to prove that, for a finite λ,

$$\int_0^\lambda F_\nu(xu)\,u\,du\int_0^\infty C_\nu(tu)\,t f(t)\,dt = \int_0^\infty t f(t)\,dt\int_0^\lambda F_\nu(xu)\,C_\nu(tu)\,u\,du.$$

We divide the range of integration for t into the two parts

$$(0,\ 1/u),\quad (1/u,\ \infty).$$

In the first part, we have

$$|\,(tu)^{|\nu|}\,C_\nu(tu)\,| < A.$$

Thus
$$u^{|\nu|}\int_0^{1/u} C_\nu(tu)\,t f(t)\,dt$$

is uniformly convergent, and we may multiply by $u^{1-|\nu|}F_\nu(xu)$ and integrate from 0 to λ under the integral sign.

In the second part, we have

$$|\,(tu)^{\frac{1}{2}}\,C_\nu(tu)\,| < B.$$

Thus
$$u^{\frac{1}{2}}\int_{1/u}^\infty C_\nu(tu)\,t f(t)\,dt$$

is uniformly convergent,* and we may multiply $u^{\frac{1}{2}}F_\nu(xu)$ and integrate from 0 to λ under the integral sign.

Hence the lemma is established, on adding the above results.

(ζ) *The repeated integral*

$$\int_A^B t^{\frac{1}{2}}\,dt\int_0^\lambda F_\nu(xu)\,C_\nu(tu)\,u\,du$$

tends to a limit when $\lambda\to\infty$.

On inserting the asymptotic formulae for $F_\nu(xu)$ and $C_\nu(tu)$, we see that all the integrals with respect to u, when multiplied by $t^{\frac{1}{2}}$, are convergent as $\lambda\to\infty$ for all values of t, except integrals of the type

$$\int_0^\lambda \cos xu\,\cos tu\,du.$$

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* The uniform convergence of this integral follows since $t^{\frac{1}{2}}f(t)$ is integrable over $(1/u,\ \infty)$, and that of $u^{|\nu|}\int_0^{1/u} C_\nu(tu)\,t f(t)\,dt$ follows since $t^{1-|\nu|}f(t)$ is integrable over $(0,\ 1/u)$.

This contributes to the repeated integral a constant multiple of

$$\int_A^B dt \int_0^\lambda \cos xu \cos tu \, du$$

$$= \tfrac{1}{2} \int_A^B \left[\frac{\sin \lambda(x+t)}{x+t} + \frac{\sin \lambda(x-t)}{x-t} \right] dt$$

$$= M \int_{\lambda(x+A)}^{\lambda(x+B)} \frac{\sin \theta}{\theta} \, d\theta + N \int_{\lambda(x+A)}^{\lambda(x+B)} \frac{\cos \theta}{\theta} \, d\theta + \tfrac{1}{2} \int_{\lambda(x-B)}^{\lambda(x-A)} \frac{\sin \theta}{\theta} \, d\theta,$$

M and N being finite and independent of λ.

These all tend to limits as $\lambda \to \infty$, and the lemma therefore follows.

(θ) *Proof of the Integral Formula.*

From §§ 3 (ϵ) and 3 (η), it follows that, in evaluating the repeated integral, we need only consider the value of $f(t)$ in an arbitrarily small neighbourhood of $t = x$.

Since $f(t)$ is of bounded variation in an interval of which x is an internal point, so also is $t^{\frac{1}{2}} f(t)$; so we may write, for the interval,

$$t^{\frac{1}{2}} f(t) = \chi_1(t) - \chi_2(t),$$

where $\chi_1(t)$ and $\chi_2(t)$ are positive increasing functions of t.

Having chosen a positive number ϵ, take δ so small that $f(t)$ is of bounded variation in $(x-\delta, x+\delta)$, and also

$$\chi_1(x+\delta) - \chi_1(x+0) < \epsilon, \quad \chi_1(x-0) - \chi_1(x-\delta) < \epsilon,$$

$$\chi_2(x+\delta) - \chi_2(x+0) < \epsilon, \quad \chi_2(x-0) - \chi_2(x-\delta) < \epsilon.$$

By the Second Mean-Value theorem there exists a number ξ, between 0 and δ, such that

$$\int_x^{x+\delta} \chi_1(t) \, t^{\frac{1}{2}} \, dt \int_0^\lambda F_\nu(xu) \, C_\nu(tu) \, u \, du$$

$$= \chi_1(x+0) \int_x^{x+\delta} t^{\frac{1}{2}} \, dt \int_0^\lambda F_\nu(xu) \, C_\nu(tu) \, u \, du$$

$$+ \{\chi_1(x+\delta) - \chi_1(x+0)\} \int_{x+\xi}^{x+\delta} t^{\frac{1}{2}} \, dt \int_0^\lambda F_\nu(xu) \, C_\nu(tu) \, u \, du.$$

It follows from § 3 (ξ) that the first term on the right tends to a limit when $\lambda \to \infty$ (δ remaining fixed); and at the same time the modulus of the second term remains less than a constant multiple of ϵ.

Hence

$$\operatorname*{Lim}_{\lambda\to\infty}\int_x^{x+\delta}\chi_1(t)\,t^{\frac{1}{2}}\,dt\int_0^{\lambda}F_\nu(xu)\,C_\nu(tu)\,u\,du$$
$$=\chi_1(x+0)\,.\operatorname*{Lim}_{\lambda\to\infty}\int_x^{x+\delta}t^{\frac{1}{2}}\,dt\int_0^{\lambda}F_\nu(xu)\,C_\nu(tu)\,u\,du$$
$$=\chi_1(x+0)\,.\,Px^{-\frac{1}{2}},$$

say, where P is a function of x to be determined.

We can treat $\chi_2(t)$ in a similar way, and also apply similar reasoning to the interval $(x-\delta,\ x)$.

Adding up the results, we have

$$\operatorname*{Lim}_{\lambda\to\infty}\int_{x-\delta}^{x+\delta}t\,f(t)\,dt\int_0^{\lambda}F_\nu(xu)\,C_\nu(tu)\,u\,du=Pf(x+0)+Qf(x-0),$$

where
$$Qx^{-\frac{1}{2}}=\operatorname*{Lim}_{\lambda\to\infty}\int_{x-\delta}^{x}t^{\frac{1}{2}}\,dt\int_0^{\lambda}F_\nu(xu)\,C_\nu(tu)\,u\,du.$$

To calculate P, put $f(t)=t^\nu$ in $(x,\ x+\delta)$, and zero elsewhere, in the above result. We get

$$\operatorname*{Lim}_{\lambda\to\infty}\int_x^{x+\delta}t^{\nu+1}\,dt\int_0^{\lambda}F_\nu(xu)\,C_\nu(tu)\,u\,du=Px^\nu;$$

or, writing y for $x+\delta$, and inverting the order of integration,

$$Px^\nu=\int_0^\infty F_\nu(xu)\,u\,du\int_x^y C_\nu(tu)\,t^{\nu+1}\,dt$$
$$=\int_0^\infty F_\nu(xu)\left\{y^{\nu+1}\,C_{\nu+1}(yu)-x^{\nu+1}\,C_{\nu+1}(xu)\right\}du$$
$$=\tfrac{1}{2}x^\nu,$$

by (3 . 2), since $y>x$. Hence $P=\frac{1}{2}$.

Similarly, writing z for $x-\delta$, we have

$$Qx^\nu=\int_0^\infty F_\nu(xu)\,u\,du\int_z^x C_\nu(tu)\,t^{\nu+1}\,dt$$
$$=\int_0^\infty F_\nu(xu)\left\{x^{\nu+1}\,C_{\nu+1}(xu)-z^{\nu+1}\,C_{\nu+1}(zu)\right\}du$$
$$=\tfrac{1}{2}x^\nu,$$

by (3'. 2), since $z<x$. Hence $Q=\frac{1}{2}$ also.

The theorem (A) is therefore established.

(*i*) With regard to the condition (iii) of the theorem (*A*), viz. that $t^\sigma f(t)$ should be integrable over $(0, \delta)$, where $\delta > 0$, and

$$\sigma = \text{Min}\,(1-|\nu|,\ 1-2a-\nu,\ \tfrac{1}{2}),$$

it may be pointed out that in the case $a = \frac{1}{2}$, when we have Titchmarsh's formula

$$f(x) = \int_0^\infty \mathbf{H}_\nu(xu)\,u\,du \int_0^\infty Y_\nu(tu)\,t\,f(t)\,dt,$$

the least of the quantities $1-|\nu|$, $1-2a-\nu$, $\frac{1}{2}$ is $1-2a-\nu = -\nu$ if $\nu > -\frac{1}{2}$.

Hence, when $\nu \geqslant -\frac{1}{2}$, we must have

$$t^{-\nu} f(t) \text{ integrable over } (0, \delta).$$

When $-\frac{3}{2} < \nu \leqslant -\frac{1}{2}$, the least of the above three quantities is $1-|\nu|$, and so we must then have $t^{\nu+1} f(t)$ integrable over $(0, \delta)$.

4. *Proof of the theorem (A').*

(*a*) We first have to evaluate a definite integral. Consider

$$\int_\Gamma \{F_{\nu+1}(cz) - e^{a\pi i} J_{\nu+1}(cz)\}\, H_\nu^{(1)}(bz)\,dz,$$

where Γ is the contour formed by the real axis, a semi-circular indentation (γ) above the origin, and a large semi-circle above the real axis with centre at the origin.

The integral round the large semi-circle vanishes.*

We find, then, that

$$(4.1)\quad \int_{-\infty}^{\infty} \{F_{\nu+1}(cx) - e^{a\pi i} J_{\nu+1}(cx)\}\, H_\nu^{(1)}(bx)\,dx + \int_\gamma \{F_{\nu+1}(cz) - e^{a\pi i} J_{\nu+1}(cz)\}\, H_\nu^{(1)}(bz)\,dz = 0.$$

* This requires that $a < \frac{3}{4} - \frac{1}{2}\nu$, which, as we shall see, is one of the conditions for the truth of the theorem.

The integral round γ certainly vanishes if $a > -1$, $\nu > -1$, $a+\nu > -1$. Thus (4 . 1) gives

$$\int_0^\infty [\{F_{\nu+1}(cx) - e^{a\pi i} J_{\nu+1}(cx)\}\{J_\nu(bx) + iY_\nu(bx)\}$$
$$- \{e^{(\nu+2a+1)\pi i} F_{\nu+1}(cx) - e^{(\nu+a+1)\pi i} J_{\nu+1}(cx)\} e^{-\nu\pi i}\{J_\nu(bx) - iY_\nu(bx)\}]\,dx = 0.$$

On equating real parts, we get

$$\int_0^\infty [(1+\cos 2a\pi) F_{\nu+1}(cx) J_\nu(bx) + \sin 2a\pi\, F_{\nu+1}(cx) Y_\nu(bx)]\,dx$$
$$= 2\cos a\pi \int_0^\infty J_{\nu+1}(cx) J_\nu(bx)\,dx.$$

Hence we have the formula*

$$(4.2) \quad \int_0^\infty F_{\nu+1}(cx) C_\nu(bx)\,dx = 0 \ (c < b), \quad \frac{1}{2c} \ (c = b), \quad \frac{b^\nu}{c^{\nu+1}} \ (c > b).$$

(β) We will now verify the theorem in the particular case

$$f(t) = t^\nu \ (0 \leqslant t < \kappa), \quad \tfrac{1}{2}t^\nu \ (t = \kappa), \quad 0 \ (t > \kappa).$$

In this case the repeated integral

$$\int_0^\infty C_\nu(xu)\, u\,du \int_0^\infty F_\nu(tu)\, t f(t)\,dt = \kappa^{\nu+1} \int_0^\infty C_\nu(xu) F_{\nu+1}(\kappa u)\,du\dagger$$
$$= 0 \ (x > \kappa), \quad \tfrac{1}{2}x^\nu \ (x = \kappa), \quad x^\nu \ (x < \kappa),$$

by (4 . 2).

This verifies the theorem in the particular case.

Preliminary Lemmas.

(γ) *The function of t and λ,*

$$t^\sigma \int_0^\lambda C_\nu(xu) F_\nu(tu)\, u\,du,$$

where x is a constant, and $|x-t|$ remains greater than a constant, is bounded for all values of t such that $t \geqslant \delta > 0$ if $a > -1$, $a+\nu > -1$,

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* See G. N. Watson, *loc. cit.*, 406 (8). By the theory of analytic continuation, as in the case of the formula 3.2, the result (4 . 2) is valid if $a > -1$, $a+\nu > -1$, $\nu + 2a < \frac{3}{2}$.

† This is seen at once by integrating the series for $t^{\nu+1} F_\nu(tu)$ term by term between limits 0 and κ.

$|\nu| \leqslant \frac{3}{2}$, $\nu+2\alpha < \frac{3}{2}$, *and* $\sigma' = \frac{1}{2}$; *and is bounded for all values of* t *such that* $0 \leqslant t < \delta$, *if* $\alpha > -1$, $\alpha+\nu > -1$, *and* $\sigma' = \mathrm{Max}\,(-2\alpha-\nu, \frac{1}{2})$.

(i) We first consider the finite range $(0, \lambda_0)$ of u.

(a) Let $$t < t_0.$$

Then $$C_\nu(xu) = O(x^\nu)+O(x^{-\nu}), \quad F_\nu(tu) = O(tu)^{\nu+2\alpha}.$$

Thus

$$t^{\sigma'} \int_0^{\lambda_0} C_\nu(xu)\, F_\nu(tu)\, u\, du = O\Big[t^{\sigma'+\nu+2\alpha} \int_0^{\lambda_0} \{A u^{2\nu+2\alpha+1}+B u^{2\alpha+1}\}\, du\Big],$$

where A and B are independent of t and u.

For convergence at the lower limit, we must have

$$\nu+\alpha > -1, \quad \alpha > -1.$$

Hence the result follows if

$$\sigma' \geqslant -\nu-2\alpha.$$

(b) Let $$t > t_0.$$

We here divide the range of integration into the two parts

$$(0,\ 1/t), \quad (1/t,\ \lambda_0).$$

In the first part, we employ the forms near the origin for both functions, as in (a), and we have

$$t^{\frac{1}{2}} \int_0^{1/t} C_\nu(xu)\, F_\nu(tu)\, u\, du = O\Big[t^{\frac{1}{2}+\nu+2\alpha} \int_0^{1/t} \{A u^{2\nu+2\alpha+1}+B u^{2\alpha+1}\}\, du\Big].$$

As before, we must have $\nu+\alpha > -1$, $\alpha > -1$, and the expression is then seen to be

$$O(t^{+\nu-\frac{3}{2}}),$$

so that we must have $$|\nu| \leqslant \tfrac{3}{2}.$$

In the second part, we employ the form near the origin for $C_\nu(xu)$, and the form (2 . 3) for $F_\nu(tu)$, and the integral

$$t^{\frac{1}{2}} \int_{1/t}^{\lambda_0} C_\nu(xu)\, F_\nu(tu)\, u\, du$$

$$= O\Big[\int_{1/t}^{\lambda_0} \{A u^{\frac{1}{2}+\nu}+B u^{\frac{1}{2}-\nu}\}\, \{\cos tu\, f_1(tu)+\sin tu\, f_2(tu)+(tu)^{-1} f_3(tu)\}\, du\Big]$$

$$= O(t^{\pm\nu-\frac{3}{2}}),$$

so that we must have $|\nu| \leqslant \frac{3}{2}$ as before.

(ii) Let $\lambda > \lambda_0, \quad t > t_0.$

Then, by the formulae (2 . 1) and (2 . 3),

$$t^{\frac{1}{2}} \int_{\lambda_0}^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du$$
$$= \frac{1}{\sqrt{(x)}} \int_{\lambda_0}^{\lambda} [\cos xu\, f_1(xu) + \sin xu\, f_2(u)]$$
$$\times [\cos tu\, f_1(tu) + \sin tu\, f_2(tu) + (tu)^{-1} f_3(tu)]\, du,$$

if (2 . 4) is satisfied, *i.e.* if $\nu + 2a < \frac{3}{2}$.

All the integrals involved are bounded; the result therefore follows in this case.

(iii) Let $t < t_0, \quad t\lambda > A.$

We here divide the range of integration into the two parts

$$(\lambda_0,\ 1/t), \quad (1/t,\ \lambda).$$

In the first part, we see that

$$\left| t^{\sigma'} \int_{\lambda_0}^{1/t} C_\nu(xu)\, F_\nu(tu)\, u\, du \right| < K t^{\sigma' + \nu + 2a} \left| \int_{\lambda_0}^{1/t} C_\nu(xu)\, u^{2a+\nu+1}\, du \right|$$

since $(tu)^{\nu-2a} F_\nu(tu)$ is of bounded variation over $(\lambda_0,\ 1/t)$. Also, by (2 . 1),

$$t^{\sigma'+\nu+2a} \int_{\lambda_0}^{1/t} C_\nu(xu)\, u^{\nu+2a+1}\, du$$
$$= \frac{t^{\sigma'+\nu+2a}}{\sqrt{(x)}} \int_{\lambda_0}^{1/t} [\cos xu\, f_1(xu) + \sin xu\, f_2(xu)]\, u^{\nu+2a+\frac{1}{2}}\, du$$
$$= O(t^{\sigma'+\nu+2a}) + O(t^{\sigma'-\frac{1}{2}}).$$

Hence we must have

$$\sigma' \geqslant \text{Max}\,(-\nu - 2a,\ \tfrac{1}{2}).$$

In the second part, $(1/t,\ \lambda)$, we have

$$t^{\sigma'} \int_{1/t}^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du$$
$$= \frac{t^{\sigma'-\frac{1}{2}}}{\sqrt{(x)}} \int_{1/t}^{\lambda} [\cos xu\, f_1(xu) + \sin xu\, f_2(xu)]$$
$$\times [\cos tu\, f_1(tu) + \sin tu\, f_2(tu) + (tu)^{-1} f_3(tu)]\, du,$$

which is bounded if $\sigma' \geqslant \frac{1}{2}$.

(iv) Let $$t < t_0, \quad t\lambda < A.$$

This can be dealt with as in the first part of case (iii).

It will be observed from the above argument that we must have $t^{\frac{1}{2}} f(t)$ integrable over (δ, ∞), where $\delta > 0$, and $t^{\sigma} f(t)$ integrable over $(0, \delta)$, where $\sigma = \text{Min}\,(1+\nu+2a, \frac{1}{2})$, $\sigma+\sigma' = 1$.

This completes the proof.

The remainder of the lemmas and the proof of the integral formula now follow as in the proof of Theorem (A), and it will be sufficient merely to state the lemmas :—

(δ) *If* $0 \leqslant A \leqslant \alpha < \beta \leqslant B$, *and* x *is not a point of the closed interval* (A, B), *then, when* $\lambda \to \infty$,

$$\int_{\alpha}^{\beta} t^{\nu+1}\, dt \int_0^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du \to 0$$

uniformly in α *and* β.

(ϵ) *If* x *is not a point of the closed interval* (A, B), *then*

$$\operatorname*{Lim}_{\lambda\to\infty} \int_A^B t f(t)\, dt \int_0^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du = 0.$$

(η) *We have*

$$\int_0^{\infty} C_\nu(xu)\, u\, du \int_0^{\infty} F_\nu(tu)\, t f(t)\, dt = \operatorname*{Lim}_{\lambda\to\infty} \int_0^{\infty} t f(t)\, dt \int_0^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du,$$

provided the limit on the right exists.

(ζ) *The repeated integral*

$$\int_A^B t^{\frac{1}{2}}\, dt \int_0^{\lambda} C_\nu(xu)\, F_\nu(tu)\, u\, du$$

tends to a limit when $\lambda \to \infty$.

In the above lemmas, it is assumed that the conditions of Lemma 4 (γ) hold.

We have now accumulated precisely the same information which we possessed at a similar stage in the proof of Theorem (A).

The proof of the integral formula will then at once follow, and hence the theorem (A') is proved.

5. *Proof of the theorem (B).*

(α) It is proposed to make rigorous the procedure of verification of the integral formula given by Titchmarsh* in the particular case

$$f(t) = 1 \quad (0 < t < \kappa), \quad \tfrac{1}{2} \quad (t = \kappa), \quad 0 \quad (t > \kappa).$$

Let
$$\phi(x) = \pi \int_0^\infty \Xi_\nu(xu)\, u\, du \int_0^\infty \frac{d}{du} \{u\, G_\nu(tu)\}\, t f(t)\, dt$$

$$= \pi \int_0^\infty \Xi_\nu(xu)\, u\, du \int_0^\kappa \frac{d}{dt} \{t\, G_\nu(tu)\}\, t\, dt.$$

Now

$$\int_0^\kappa \frac{d}{dt} \{t\, G_\nu(tu)\}\, t\, dt$$

$$= \kappa^2 G_\nu(\kappa u) - \tfrac{1}{2} \left[t^2 \sum_{n=0}^{\infty} \frac{(-1)^n (tu)^{2\nu+2a+2n}\, \Gamma(\nu+a+n+\frac{1}{2})}{\Gamma(\frac{1}{2})\, \Gamma(a+n+1)\, \Gamma(\nu+a+n+2)\, \Gamma(2\nu+a+n+1)} \right]_0^\kappa,$$

on integrating by parts; the value of the integrated function at the lower limit being zero if $a+\nu < 1$.

We write this
$$\kappa^2 [G_\nu(\kappa u) - \tfrac{1}{2} T_\nu(\kappa u)].$$

Hence
$$\phi(x) = \pi\kappa^2 \int_0^\infty \Xi_\nu(xu) \{G_\nu(\kappa u) - \tfrac{1}{2} T_\nu(\kappa u)\}\, u\, du.$$

Multiply both sides by $x F_{2\nu}(2\xi x)$, and integrate from 0 to ∞; then, if we can invert the order of integration, we have

$$(5.1) \quad \int_0^\infty F_{2\nu}(2\xi x)\, x\, \phi(x)\, dx$$

$$= \pi\kappa^2 \int_0^\infty \{G_\nu(\kappa u) - \tfrac{1}{2} T_\nu(\kappa u)\}\, u\, du \int_0^\infty F_{2\nu}(2\xi x)\, \Xi_\nu(xu)\, x\, dx.$$

To justify the inversion (5.1) of the order of integration, we employ the Convergence Theorem of Lebesgue. Let

$$f_X(u) = u \{G_\nu(\kappa u) - \tfrac{1}{2} T_\nu(\kappa u)\} \int_X^\infty F_{2\nu}(2\xi x)\, \Xi_\nu(xu)\, x\, dx.$$

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* E. C. Titchmarsh, "An Inversion formula involving Bessel Functions", *Proc. London Math. Soc.* (*Records* for November 13th, 1924), iii.

Writing in the asymptotic formulae for $F_{2\nu}(2\xi x)$ and $\Xi_\nu(xu)$, we see that the integral exists if condition (2 . 4) is satisfied; *i.e.* if

$$a+\nu<\tfrac{3}{4}.$$

Hence $f_X(u)\to 0$ as $X\to\infty$.

(i) Let $$u>u_0.$$

The integral is then $O(u^{-2})$, (uniformly in X), and

$$|\,u\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,|<A \quad \text{if} \quad a+\nu<1,$$

by (2 . 7), (2 . 8), and (2 . 12).

Hence $$f_X(u)=O(u^{-2}).$$

(ii) Let $$0\leqslant u\leqslant u_0.$$

We here divide the range of integration into the two parts

$$(X,\ 1/u),\quad (1/u,\ \infty).$$

The second part is dealt with as in (i); the integral is then $O(u^{-\frac{1}{2}})$, (uniformly in X), and using the form near the origin for

$$u\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\},$$

we see that $$f_X(u)=O(u^{\frac{1}{2}+2\alpha+2\nu}).$$

In the first part, we employ forms near the origin for

$$u\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}$$

and $\Xi_\nu(xu)$, and the form (2 . 3) for $F_{2\nu}(2\xi x)$; we then see that the integral is equal to

$$O\left\{\int_X^{1/u}[\cos 2\xi x\, f_1(2\xi x)+\sin 2\xi x\, f_2(2\xi x)+(xu)^{-1}f_3(2\xi x)]\right.$$
$$\left.\times[A(xu)^{2|\nu|}+B(xu)^{-2|\nu|}+C]\,x^{\frac{1}{2}}\,dx\right\}$$
$$=O(u^{-\frac{1}{2}}),$$

A, B, C being independent of x and u.

Hence $$f_X(u)=O(u^{\frac{1}{2}+2\alpha+2\nu}).$$

Take $$\chi(u)=Ku^{-2}\ \ (u>u_0),\quad Ku^{\frac{1}{2}+2\alpha+2\nu}\ \ (u\leqslant u_0).$$

Then $\int_0^U \chi(u)du$ exists, if $a+\nu > -\frac{3}{4}$, and $|f_X(u)| < \chi(u)$ for all X.

Thus

$$\operatorname*{Lim}_{X\to\infty} \int_0^U u\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,du \int_X^\infty F_{2\nu}(2\xi x)\,\Xi_\nu(xu)\,x\,dx = 0.$$

Hence

$$\int_0^U u\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,du \int_0^\infty F_{2\nu}(2\xi x)\,\Xi_\nu(xu)\,x\,dx$$

$$= \int_0^\infty F_{2\nu}(2\xi x)\,x\,dx \int_0^U \Xi_\nu(xu)\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,u\,du.$$

We next have to prove that

$$\operatorname*{Lim}_{U\to\infty} \int_0^\infty F_{2\nu}(2\xi x)\,x\,dx \int_U^\infty \Xi_\nu(xu)\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,u\,du = 0$$

Let

$$f_U(x) = F_{2\nu}(2\xi x)\,x \int_U^\infty \Xi_\nu(xu)\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,u\,du.$$

Writing in the asymptotic formulae for $\Xi_\nu(xu)$, $G_\nu(\kappa u)$, and $T_\nu(\kappa u)$, we see that the integral exists if $a+\nu<1$ by (2.7) and (2.12), and so $f_U(x)\to 0$ as $U\to\infty$.

(i) Let $$x > x_0.$$

The integral is then $O(x^{-2})$, and if $a+\nu<\frac{3}{4}$, we have

$$|(2x\xi)^{\frac{1}{2}}F_{2\nu}(2x\xi)| < A;$$

so that $$f_U(x) = O(x^{-\frac{3}{2}}).$$

(ii) Let $$0 \leqslant x \leqslant x_0.$$

We here divide the range of integration into the two parts

$$(U,\ 1/x),\quad (1/x,\ \infty).$$

The second part is dealt with as in (i); here the integral is $O(x^{-1})$, and taking the form near the origin for $xF_{2\nu}(2\xi x)$ we see that

$$f_U(x) = O(x^{2\nu+2a}).$$

In the first part, we employ forms near the origin for $xF_{2\nu}(2\xi x)$ and $\Xi_\nu(xu)$, and forms (2 . 7), (2 . 12) respectively for $G_\nu(\kappa u)$ and $T_\nu(\kappa u)$; the integral then becomes

$$\int_U^{1/x} [A(xu)^{2|\nu|}+B(xu)^{-2|\nu|}+C][\cos 2\kappa u\,\phi_1(u)+\sin 2\kappa u\,\phi_2(u)+\phi_3(u)]\,du$$
$$= O(x^{-1}).$$

Hence $$f_U(x) = O(x^{2\nu+2\alpha}).$$

Take $$\chi(x) = Kx^{-\frac{3}{2}} \quad (x > x_0), \quad Kx^{2\nu+2\alpha} \quad (x \leqslant x_0).$$

Then $\int_0^\infty \chi(x)\,dx$ exists if $\nu+\alpha > -\frac{1}{2}$, and $|f_U(x)| < \chi(x)$ for all U.

The inversion of the order of integration (5 . 1) is therefore justified. Now we have the formula*

$$(5\,.\,2) \qquad \int_0^\infty F_{2\nu}(2\mu t)\,\Xi_\nu(\lambda t)\,dt = \frac{1}{\pi\mu\sqrt{(\mu^2-\lambda^2)}} \quad (\lambda < \mu),$$
$$= 0 \quad (\lambda > \mu).$$

Hence (5 . 1) gives

$$\int_0^\infty F_{2\nu}(2\xi x)\,x\phi(x)\,dx = \kappa^2 \int_0^\xi \frac{u}{\sqrt{(\xi^2-u^2)}} \{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,du$$
$$= \int_0^1 \frac{s\,ds}{\sqrt{(1-s^2)}} \int_0^\kappa \frac{d}{dt} \{tG_\nu(ts\xi)\}\,t\,dt$$
$$(5\,.\,3) \qquad = \int_0^\kappa t\,dt \int_0^1 \frac{s\,ds}{\sqrt{(1-s^2)}} \cdot \frac{d}{dt} \{tG_\nu(ts\xi)\}$$
$$= \int_0^\kappa F_{2\nu}(2\xi t)\,t\,dt,$$

integrating with respect to s term by term.

The inversion of the order of integration (5 . 3) presents no difficulties, as the integrals are finite.

Applying Hardy's formula, we have

$$\varphi(x) = 1 \ (x < \kappa), \quad \tfrac{1}{2} \ (x = \kappa), \quad 0 \ (x > \kappa).$$

* E. C. Titchmarsh, *loc. cit.*

(β) The procedure of 5(α) establishes the result

$$(5.4)\quad \int_0^\infty \Xi_\nu(xu)\{G_\nu(\kappa u)-\tfrac{1}{2}T_\nu(\kappa u)\}\,u\,du$$
$$= \frac{1}{\pi\kappa^2}\ (0<x<\kappa),\quad \frac{1}{2\pi\kappa^2}\ (x=\kappa),\quad 0\ (x>\kappa),$$

provided $-\frac{1}{2}<a+\nu<\frac{3}{4}$.*

Preliminary Lemmas.

(γ) *The function of t and λ*

$$t^{\sigma'}\int_0^\lambda \Xi_\nu(xu)\frac{d}{du}\{x\,G_\nu(tu)\}\,u\,du$$

where x is constant, and $|x-t|$ remains greater than a constant, is bounded for all values of t such that $0\leqslant t\leqslant\delta$, if $\sigma'=\operatorname{Max}(0,\ -2a-2\nu)$, $a>-1$, $a+2\nu\geqslant-1$, and $a+\nu\geqslant-\frac{1}{2}$; and is bounded for all values of t such that $t>\delta>0$, provided $\sigma'=0$, $|\nu|\leqslant 1$, $a>-1$, $a+2\nu>-1$, and $a+\nu<1$.

(i) We first consider the finite range $(0,\ \lambda_0)$ of u.

(a) Let $t<t_0$.

Then
$$t^{\sigma'}\int_0^{\lambda_0}\Xi_\nu(xu)\frac{d}{du}\{u\,G_\nu(tu)\}\,u\,du$$
$$= O\Big[t^{\sigma'}\int_0^{\lambda_0}(Au^{2\nu}+Bu^{-2\nu}+C)(tu)^{2\nu+2\alpha}\,u\,du\Big],$$

A, B, C being independent of t and u, since near the origin we have

$$\Xi_\nu(xu)=O(u^{2\nu})+O(u^{-2\nu})+O(1),$$

$$\frac{d}{du}\{u\,G_\nu(tu)\}=O(tu)^{2\nu+2\alpha};$$

hence the integral is

$$O\Big[t^{\sigma'+2\nu+2\alpha}\int_0^{\lambda_0}(Au^{4\nu+2\alpha+1}+Bu^{2\alpha+1}+Cu^{2\nu+2\alpha+1})\,du\Big]=O(t^{\sigma'+2\nu+2\alpha}),$$

* By the theory of analytic continuation, as in the case of the formula 3.2, the result 5.4 is valid if $a>-1$, $a+2\nu>-1$, $a+\nu<1$.

the integral being convergent at the lower limit if

$$a+2\nu > -1, \quad a > -1, \quad a+\nu > -1.$$

Hence we must have $\sigma' \geqslant -2\nu-2a$.

(*b*) Let $t > t_0$.

We here divide the range of integration into the two parts

$$(0,\ 1/t), \quad (1/t,\ \lambda_0).$$

Then $$\int_0^{1/t} \Xi_\nu(xu) \frac{d}{du} \{u G_\nu(tu)\} u\, du$$

$$= O\left[t^{2\nu+2a} \int_0^{1/t} (Au^{4\nu+2a+1} + Bu^{2a+1} + Cu^{2\nu+2a+1})\, du\right],$$

using the forms near the origin for both functions, as in (*a*). As before, for convergence at the lower limit, we must have

$$a+2\nu > -1, \quad a > -1, \quad a+\nu > -1.$$

and the intregral is then seen to be

$$O(t^{-2\nu-2}) + O(t^{2\nu-2}) + O(t^{-2}).$$

Hence we must have $|\nu| \leqslant 1$.

In the second part $(1/t, \lambda_0)$ we employ the form near the origin for $\Xi_\nu(xu)$, and the asymptotic form (2 . 10) for $d/du\{u G_\nu(tu)\}$. We then have

$$\int_{1/t}^{\lambda_0} \Xi_\nu(xu) \frac{d}{du} \{u G_\nu(tu)\} u\, du$$

$$= O\left[\int_{1/t}^{\lambda_0} (Au^{2\nu} + Bu^{-2\nu} + C)\{\cos 2tu \,.\, \phi_1(tu) + \phi_2(tu)\} u\, du\right],$$

and this is bounded if $|\nu| \leqslant 1$.

(ii) Let $\lambda > \lambda_0, \quad t > t_0$.

The application of the asymptotic forms (2 . 1) and (2 . 10) will not serve to prove the boundedness of the integral in this case; instead, using

the ordinary asymptotic formulae for $J_\nu(xu)$ and $Y_\nu(xu)$, we have

$$\Xi_\nu(xu) = A \cdot \frac{\cos(2xu - \nu\pi - a\pi)}{u} + O(u^{-3});$$

whilst from an examination of (2 . 8) we see that, if $a+\nu < 1$,

$$(5\,.\,5)\quad \frac{d}{du}\{u\dot{G}_\nu(tu)\} = \cos 2tu \cdot \frac{\psi_1(tu)}{tu} + \frac{\psi_2(tu)}{tu} + B\cos 2tu + C\sin 2tu + O(tu)^{-2} + O(tu)^{2a+2\nu-3},$$

the first two terms on the right being the expression given by (2 . 7) for $G_\nu(tu)$, the next three terms being those terms of (2 . 8) involving the function J, and the last term being the term of (2 . 8) containing the Barnes integral.

On employing these expansions, we see that

$$\int_{\lambda_0}^{\lambda} \Xi_\nu(xu)\frac{d}{du}\{u\dot{G}_\nu(tu)\}\,u\,du$$

is bounded, provided $a+\nu<1$.

(iii) Let $$t < t_0, \quad t\lambda > A.$$

We here divide the range of integration into the two parts

$$(\lambda_0,\ 1/t), \quad (1/t,\ \lambda).$$

In the first part, we see that

$$\left| t^{\sigma'}\int_{\lambda_0}^{1/t} \Xi_\nu(xu)\frac{d}{du}\{u\dot{G}_\nu(tu)\}\,u\,du \right| < Kt^{\sigma'-1}\left|\int_{\lambda_0}^{1/t}\Xi_\nu(xu)\,du\right|,$$

since $$tu\frac{d}{du}\{u\dot{G}_\nu(tu)\}$$

is of bounded variation over $(\lambda_0,\ 1/t)$, if $a+\nu \geqslant -\frac{1}{2}$. Also by (2 . 1),

$$t^{\sigma'-1}\int_{\lambda_0}^{1/t}\Xi_\nu(xu)\,du = t^{\sigma'-1}\int_{\lambda_0}^{1/t}[\cos 2xu\,\phi_1(xu)+\phi_2(xu)]\,du = O(t^{\sigma'-1}).$$

This is bounded if $\sigma' \geqslant 1$.

In the second part $(1/t,\ \lambda)$ the result follows as in (ii).

Hence the result is proved in this case.

(iv) Let $$t < t_0, \quad t\lambda < A.$$

This can be dealt with as in the first part of case (iii).

It will be observed from the above argument that we must have $tf(t)$ integrable over (δ, ∞), where $\delta > 0$, and $t^\sigma f(t)$, where

$$\sigma = \text{Min}\,(1+2a+2\nu,\ 1),$$

and $\sigma+\sigma' = 1$, integrable over $(0, \delta)$.

This completes the proof.

It will now, as in the proof of (A'), be sufficient to state the remaining lemmas.

(δ) *If* $0 \leqslant A \leqslant \alpha < \beta \leqslant B$, *and* x *is not a point of the closed interval* (A, B), *then when* $\lambda \to \infty$,

$$\int_\alpha^\beta t\,dt \int_0^\lambda \Xi_\nu(xu)\,\frac{d}{du}\{u\,G_\nu(tu)\}\,u\,du \to 0$$

uniformly in α *and* β.

(ϵ) *If* x *is not a point of the closed interval* (A, B), *then*

$$\operatorname*{Lim}_{\lambda\to\infty} \int_A^B t f(t)\,dt \int_0^\lambda \Xi_\nu(xu)\,\frac{d}{du}\{u\,G_\nu(tu)\}\,u\,du = 0.$$

(η) *We have*

$$\int_0^\infty \Xi_\nu(xu)\,u\,du \int_0^\infty \frac{d}{du}\{u\,G_\nu(tu)\}\,t f(t)\,dt$$
$$= \operatorname*{Lim}_{\lambda\to\infty} \int_0^\infty t f(t)\,dt \int_0^\lambda \Xi_\nu(xu)\,\frac{d}{du}\{u\,G_\nu(tu)\}\,u\,du,$$

provided the limit on the right exists.

(ζ) *The repeated integral*

$$\int_A^B t\,dt \int_0^\lambda \Xi_\nu(xu)\,\frac{d}{du}\{u\,G_\nu(tu)\}\,u\,du$$

tends to a limit as $\lambda \to \infty$.

In the above lemmas, the conditions of Lemma 5 (γ) are assumed to hold.

The proof of the integral formula now follows, as in (A), and hence Theorem (B) is proved.

6. *Finally, a proof will be given of the theorem* (B').

(α) We will first verify the formula (D)

$$f(x) = \pi \int_0^\infty \frac{d}{du}\{u\,G_\nu(xu)\}\,u\,du \int_0^\infty \Xi_\nu(tu)\,t\,f(t)\,dt.$$

Put

$$f(t) = \int_0^\infty F_{2\nu}(2st)\,s\,\psi(s)\,ds. \tag{6.1}$$

Then

$$\int_0^\infty \Xi_\nu(tu)\,t\,dt \int_0^\infty F_{2\nu}(2st)\,s\,\psi(s)\,ds$$

$$= \int_0^\infty s\,\psi(s)\,ds \int_0^\infty F_{2\nu}(2st)\,\Xi_\nu(tu)\,t\,dt \tag{6.2}$$

$$= \frac{1}{\pi}\int_u^\infty \frac{\psi(s)}{\sqrt{(s^2-u^2)}}\,ds, \quad \text{by (5.2)}.$$

Hence we have to verify that

$$f(x) = \int_0^\infty \frac{d}{du}\{u\,G_\nu(xu)\}\,u\,du \int_u^\infty \frac{\psi(s)}{\sqrt{(s^2-u^2)}}\,ds. \tag{6.3}$$

The right-hand side of (6.3)

$$= \int_0^\infty \psi(s)\,ds \int_0^s \frac{d}{du}\{u\,G_\nu(xu)\}\,\frac{u\,du}{\sqrt{(s^2-u^2)}} \tag{6.4}$$

$$= \int_0^\infty s\,\psi(s)\,ds \int_0^1 \frac{d}{dv}\{v\,G_\nu(xsv)\}\,\frac{v\,dv}{\sqrt{(1-v^2)}}$$

$$= \int_0^\infty F_{2\nu}(2sx)\,s\,\psi(s)\,ds.*$$

$$= f(x), \quad \text{by (6.1)}.$$

This verifies the formula.

(β) It is now proposed to make the procedure of § 6 (α) rigorous in the particular case

$$f(t) = t^{2\nu} \ (0 < t < \kappa), \quad \tfrac{1}{2}t^{2\nu} \ (t = \kappa), \quad 0 \ (t > \kappa).$$

---

* By term-by-term integration.

Let $$\phi(x) = \pi \int_0^\infty \frac{d}{du}\{u\, G_\nu(xu)\}\, u\, du \int_0^\kappa \Xi_\nu(tu)\, t^{2\nu+1}\, dt.$$

We have to justify the inversions of the order of integration (6 . 2) and (6 . 4), and, to do this, we again employ the convergence theorem of Lebesgue. Dealing first with (6 . 2), take

$$f_S(t) = \Xi_\nu(tu)\, t \int_S^\infty F_{2\nu}(2st)\, s\, \psi(s)\, ds.$$

Now, by Hardy's integral formula,

(6 . 5) $$\begin{aligned}\psi(s) &= \int_0^\kappa C_{2\nu}(st)\, t^{2\nu+1}\, dt \\ &= \frac{\kappa^{2\nu+1}}{s}\, C_{2\nu+1}(s\kappa) \\ &= s^{-\frac{3}{2}}[\cos s\kappa\,.\, f_1(s) + \sin s\kappa\,.\, f_2(s)], \quad \text{by (2 . 1).}\end{aligned}$$

Hence, by (2 . 1), (2 . 3), and (6 . 5), we have

$$\begin{aligned}f_S(t) = u^{-1}\, t^{-\frac{1}{2}}&[\cos 2tu\,.\,\phi_1(tu) + \phi_2(tu)] \\ &\times \int_S^\infty [\cos 2st\,.\, f_1(st) + \sin 2st\,.\, f_2(st) + (2st)^{-1} f_3(st)] \\ &\times [\cos s\kappa\,.\, f_1(s) + \sin s\kappa\,.\, f_2(s)]\, s^{-1}\, ds.\end{aligned}$$

Thus $$\int_S^\infty F_{2\nu}(2st)\, s\, \psi(s)\, ds$$

exists, and so $$f_S(t) \to 0 \quad \text{as} \quad S \to \infty.$$

(i) Let $$t > t_0.$$

Then $$f_S(t) = O(t^{-\frac{3}{2}}).$$

(ii) Let $$0 \leqslant t \leqslant t_0.$$

We here divide the range of integration into the two parts

$$(S,\ 1/t), \quad (1/t,\ \infty).$$

The second part is dealt with as in (i); the integral is then $O(t^{-\frac{1}{2}})$, and so, from the form near the origin for $t\Xi_\nu(tu)$,

$$f_S(t) = O(t^{\frac{1}{2}-2|\nu|}).$$

The first part shows, on using the form near the origin for $\Xi_\nu(tu)$ and $F_{2\nu}(2st)$, and the form (6 . 5) for $\psi(s)$, that

$$f_S(t) = O(t^{1-2|\nu|}) \int_S^{1/t} O(st)^{2\nu+2a} [\cos s\kappa \,.\, f_1(s) + \sin s\kappa \,.\, f_2(s)] s^{-\frac{1}{2}}\, ds$$

$$= O(t^{\frac{1}{2}-2|\nu|}).$$

Take $$\chi(t) = K \quad (t > t_0), \quad Kt^{\frac{1}{2}-2|\nu|} \quad (t \leqslant t_0).$$

Then $\int_0^T \chi(t)\, dt$ exists, T being finite, if $|\nu| < \frac{3}{4}$, and $|f_S(t)| < \chi(t)$ for all S.

Thus $$\operatorname*{Lim}_{S\to\infty} \int_0^T \Xi_\nu(tu)\, t\, dt \int_S^\infty F_{2\nu}(2st)\, s\, \psi(s)\, ds = 0.$$

Hence

$$\int_0^T \Xi_\nu(tu)\, t\, dt \int_0^\infty F_{2\nu}(2st)\, s\, \psi(s)\, ds = \int_0^\infty s\, \psi(s)\, ds \int_0^T F_{2\nu}(2st)\, \Xi_\nu(tu)\, t\, dt.$$

We next have to prove that

$$\operatorname*{Lim}_{T\to\infty} \int_0^\infty s\, \psi(s)\, ds \int_T^\infty F_{2\nu}(2st)\, \Xi_\nu(tu)\, t\, dt = 0.$$

To effect this, we have to show that

$$f_T(s) = s\, \psi(s) \int_T^\infty F_{2\nu}(2st)\, \Xi_\nu(tu)\, t\, dt$$

exists.

On substituting the expressions (2 . 1) and (2 . 3) in the integral (the latter requires that $a+\nu < \frac{3}{4}$), this is seen to be $O(s^{-\frac{3}{2}})$, all the integrals involved being convergent.

Hence $$f_T(s) \to 0 \quad \text{as} \quad T \to \infty .$$

(i) Let $$s > s_0.$$

Substituting (6 . 5) for $\psi(s)$, we have

$$f_T(s) = [\cos s\kappa \,.\, f_1(s) + \sin s\kappa \,.\, f_2(s)] s^{-1}$$

$$\times \int_T^\infty [\cos 2st \,.\, f_1(st) + \sin 2st \,.\, f_2(st) + (st)^{-1} f_3(st)]$$

$$\times [\cos 2tu \,.\, \phi_1(tu) + \phi_2(tu)] t^{-\frac{1}{2}}\, dt$$

$$= O(s^{-2}).$$

(ii) Let $0 \leqslant s \leqslant s_0$.

We here divide the range of integration into the two parts

$$(T,\ 1/s),\quad (1/s,\ \infty).$$

The second part is dealt with as in (i); the integral is then $O(s^{-1})$, and so, from the form near the origin for $\psi(s)$, we see that

$$f_T(s) = O(s^{-2\nu}),$$

The first part shows, on using the form near the origin for $F_{2\nu}(2st)$ and $\psi(s)$, and the asymptotic form for $\Xi_\nu(tu)$, that

$$f_T(s) = O(s^{1-2|\nu|})\left[O\left\{\int_T^{1/s} (st)^{2\nu+2\alpha}\, dt\right\}\right]$$

$$= O(s^{-2|\nu|}).$$

Take $\chi(s) = Ks^{-2}\ \ (s > s_0),\quad Ks^{-2|\nu|}\ \ (s \leqslant s_0),$

Then $\int_0^\infty \chi(s)\, ds$ exists, if $|\nu| < \frac{1}{2}$, and $|f_T(s)| < \chi(s)$ for all T.

This justifies the inversion of the order of integration (6 . 2).

With regard to (6 . 4), we have

$$\int_0^S \psi(s)\, ds \int_0^s \frac{d}{du}\{u G_\nu(xu)\} \frac{u\, du}{\sqrt{(s^2-u^2)}}$$

$$= \int_0^S \frac{d}{du}\{u G_\nu(xu)\}\, u\, du \int_u^S \frac{\psi(s)}{\sqrt{(s^2-u^2)}}\, ds.$$

We have to prove that

$$\operatorname*{Lim}_{S\to\infty} \int_0^S \frac{d}{du}\{u G_\nu(xu)\}\, u\, du \int_S^\infty \frac{\psi(s)}{\sqrt{(s^2-u^2)}}\, ds = 0.$$

Now, writing $u = Sv$, we have

$$\int_0^S \frac{d}{du}\{u G_\nu(xu)\}\, u\, du \int_S^\infty \frac{\psi(s)}{\sqrt{(s^2-u^2)}}\, ds$$

$$= S\int_0^1 \frac{d}{dv}\{v G_\nu(Sxv)\}\, v\, dv \int_S^\infty \frac{\psi(s)}{\sqrt{(s^2-S^2v^2)}}\, ds.$$

Take $f_S(v) = S\dfrac{d}{dv}\{v G_\nu(xSv)\}\, v \displaystyle\int_S^\infty \frac{\psi(s)}{\sqrt{(s^2-S^2v^2)}}\, ds.$

(i) If $Sv > A$, employ the form (2.10) for $d/dv\{v\,G_\nu(xSv)\}$, and then we have

$$f_S(v) = Sv\{\cos 2xSv\,\phi_1(xSv)+\phi_2(xSv)\}\int_{1/v}^{\infty}\frac{\psi(Svw)}{\sqrt{(w^2-1)}}\,dw, \quad \text{where} \quad w=\frac{s}{Sv},$$

$$= O(Sv)\cdot\frac{v}{\sqrt{(1-v^2)}}\int_{\xi}^{\infty}\psi(Svw)\,dw, \quad \text{where} \quad \xi > \frac{1}{v},$$

$$= O\left(\frac{v^{-\frac{1}{2}}}{\sqrt{(1-v^2)}}\right)S^{-\frac{3}{2}}, \quad \text{by (6.5).}$$

$O\{v^{-\frac{1}{2}}/\sqrt{(1-v^2)}\}$ is integrable over (0, 1), and thus $f_S(v)\to 0$ as $S\to\infty$.

(ii) If $Sv < A$, we have

$$\frac{d}{dv}\{v\,G_\nu(xSv)\} = O(Sv)^{2\nu+2\alpha},$$

and, as in (i), we see that

$$f_S(v) = O\left(\frac{v^{2\nu+2\alpha-\frac{1}{2}}}{\sqrt{(1-v^2)}}\right)S^{2\alpha+2\nu-\frac{3}{2}}.$$

This tends to zero as $S\to\infty$, if $\alpha+\nu<\frac{3}{4}$; and $O\{v^{2\nu+2\alpha-\frac{1}{2}}/\sqrt{(1-v^2)}\}$ is integrable over (0, 1) if $\alpha+\nu>-\frac{1}{4}$. We have therefore justified the inversion of the order of integration (6.4).

We have thus proved that if $-\frac{1}{4}<\alpha+\nu<\frac{3}{4}$, and $|\nu|\leqslant\frac{1}{2}$, then

$$\frac{1}{\pi} = \tfrac{1}{2}\kappa^2\int_0^{\infty}\frac{d}{du}\{u\,G_\nu(xu)\}\,\Omega_\nu(\kappa u)\,u\,du,$$

where

$$\tfrac{1}{2}\kappa^2\Omega_\nu(\kappa u) \equiv \int_0^{\kappa}\Xi_\nu(tu)\,t^{2\nu+1}\,dt.$$

Hence we have established the formula

$$\int_0^{\infty}\frac{d}{du}\{u\,G_\nu(xu)\}\,\Omega_\nu(\kappa u)\,u\,du$$

$$= \frac{2x^{2\nu}}{\pi\kappa^2}\ (0<x<\kappa), \quad \frac{1}{\pi\kappa^{2-2\nu}}\ (x=\kappa), \quad 0\ (x>\kappa),$$

provided $|\nu|\leqslant\frac{1}{2}$, $-\frac{1}{4}<\alpha+\nu<\frac{3}{4}$.*

* These restrictions may be removed by the theory of analytic continuation, as in the case of the formula 3.2; the result is seen to be valid if $\alpha>-1$, $\alpha+2\nu>-1$, $\alpha+\nu<1$.

Preliminary Lemmas.

(γ) *The function of* t *and* λ

$$t^{\sigma'} \int_0^{\lambda} \frac{d}{du} \{u\, G_\nu(xu)\}\, \Xi_\nu(tu)\, u\, du$$

is bounded for all values of λ *and for* $t \geqslant \delta > 0$, *if* x *is a constant, and* $|x-t|$ *remains greater than a constant, provided that*

$$\sigma' = 1, \quad a > -1, \quad 1 > a+\nu \geqslant -\tfrac{1}{2}, \quad a+2\nu > -1,$$

and is bounded for all values of t *such that*

$$0 \leqslant t < \delta, \quad \text{if} \quad a > -1, \quad 1 > a+\nu > -1, \quad a+2\nu > -1,$$

and

$$\sigma' = \text{Max}\,(1,\ |2\nu|).$$

(i) We first consider the finite range $(0,\ \lambda_0)$ of u.

(a) Let

$$t < t_0.$$

Then

$$t^{\sigma'} \int_0^{\lambda_0} \frac{d}{du} \{u\, G_\nu(xu)\}\, \Xi_\nu(tu)\, u\, du$$

$$= O\left[t^{\sigma'} \int_0^{\lambda_0} u^{2\nu+2a} \{A(tu)^{2\nu} + B(tu)^{-2\nu} + C\}\, u\, du \right],$$

where A, B, C are independent of t and u, on taking the forms near the origin for both functions, as given in § 5 (γ), (i).

Hence the integral

$$= O\left[A t^{\sigma'+2\nu} \int_0^{\lambda_0} u^{4\nu+2a+1}\, du + B t^{\sigma'-2\nu} \int_0^{\lambda_0} u^{2a+1}\, du + C t^{\sigma'} \int_0^{\lambda_0} u^{2a+2\nu+1}\, du \right]$$

$$= O(t^{\sigma'+2\nu}) + O(t^{\sigma'-2\nu}) + O(t^{\sigma'}),$$

the integrals converging at the lower limit if

$$a+2\nu > -1, \quad a > -1, \quad a+\nu > -1.$$

Thus we must have

$$\sigma' \geqslant |2\nu|.$$

(b) Let

$$t > t_0.$$

We here divide the range of integration into the two parts

$$(0,\ 1/t), \quad (1/t,\ \lambda_0).$$

Then

$$t\int_0^{1/t}\frac{d}{du}\{u\,G_\nu(xu)\}\,\Xi_\nu(tu)\,u\,du$$

$$=O\Big[At^{1+2\nu}\int_0^{1/t}u^{4\nu+2a+1}\,du+Bt^{-2\nu}\int_0^{1/t}u^{2a+1}\,du+Ct\int_0^{1/t}u^{2\nu+2a+1}\,du\Big],$$

using the forms near the origin for both functions as in (a); here, again, we must have

$$a+2\nu>-1,\quad a>-1,\quad a+\nu>-1.$$

Hence the integral is $O(t^{-2\nu-2a-1})$,

and so we must have $\nu+a\geqslant-\frac{1}{2}$.

In the second part, $(1/t,\ \lambda_0)$, we employ the form near the origin for

$$\frac{d}{du}\{u\,G_\nu(xu)\},$$

and the asymptotic form for $\Xi_\nu(tu)$.

We then have

$$t\int_{1/t}^{\lambda_0}\frac{d}{du}\{u\,G_\nu(xu)\}\,\Xi_\nu(tu)\,u\,du$$

$$=t\int_{1/t}^{\lambda_0}O(u^{2a+2\nu})\,[\cos 2tu\,\phi_1(tu)+\phi_2(tu)]\,\frac{1}{t}\,du,$$

and this is bounded if $a+\nu\geqslant-\frac{1}{2}$.

(ii) Let $\lambda>\lambda_0,\quad t>t_0.$

Then, by (2 . 1) and (2 . 10),

$$t\int_{\lambda_0}^{\lambda}\frac{d}{du}\{u\,G_\nu(xu)\}\,\Xi_\nu(tu)\,u\,du$$

$$=\int_{\lambda_0}^{\lambda}[\cos 2xu\,.\,\psi_1(xu)+\psi_2(xu)]\,[\cos 2tu\,.\,\phi_1(tu)+\phi_2(tu)]\,du,$$

the condition (2 . 11) being satisfied, *i.e.* $a+\nu<1$. The integrals involved are all bounded; the result therefore follows in this case.

(iii) Let $\lambda>\lambda_0,\quad t<t_0,\quad t\lambda>A.$

We here divide the range of integration into the two parts

$$(\lambda_0,\ 1/t),\quad (1/t,\ \lambda).$$

In the first part, we see that

$$\left| t^{\sigma'} \int_{\lambda_0}^{1/t} \frac{d}{du} \{u\, G_\nu(xu)\}\, \Xi_\nu(tu)\, u\, du \right| < K t^{\sigma' - |2\nu|} \left| \int_{\lambda_0}^{1/t} \frac{d}{du} \{u\, G_\nu(xu)\}\, u^{1-|2\nu|}\, du \right|,$$

since $(tu)^{|2\nu|}\, \Xi_\nu(tu)$ is of bounded variation over $(\lambda_0,\ 1/t)$.

Also, by (5 . 5),

$$t^{\sigma'-|2\nu|} \int_{\lambda_0}^{1/t} u^{1-|2\nu|} \left[\cos 2xu \,.\, \frac{\psi_1(xu)}{xu} + \frac{\psi_2(xu)}{xu} \right.$$

$$\left. + B \cos 2xu + C \sin 2xu + O(xu)^{-2} + O(xu)^{2\alpha+2\nu-3} \right] du,$$

provided $\alpha + \nu < 1.$

This is bounded if $\sigma' \geqslant \text{Max}\,(1,\ |\,2\nu\,|).$

In the second part, $(1/t,\ \lambda)$, the result follows as in (ii).

Hence the result is proved in this case.

(iv) Let $t < t_0, \quad t\lambda < A.$

This can be dealt with as in the first part of case (iii).

It will be observed from the above argument that we must have $f(t)$ integrable over $(\delta,\ \infty)$, where $\delta > 0$, and $t^\sigma f(t)$ integrable over $(0,\ \delta)$, where

$$\sigma = \text{Min}\,(0,\ 1 - |\,2\nu\,|),$$

and $$\sigma + \sigma' = 1.$$

This completes the proof.

It will now, as in the proofs of (A') and (B), be sufficient to state the remaining lemmas :—

(δ) *If* $0 \leqslant A \leqslant \alpha < \beta \leqslant B$, *and* x *is not a point of the closed interval* $(A,\ B)$, *then, when* $\lambda \to \infty$,

$$\int_\alpha^\beta t^{2\nu+1}\, dt \int_0^\lambda \frac{d}{du} \{u\, G_\nu(xu)\}\, \Xi_\nu(tu)\, u\, du \to 0$$

uniformly in α *and* β.

(ϵ) *If* x *is not a point of the closed interval* $(A,\ B)$, *then*

$$\lim_{\lambda \to \infty} \int_A^B t f(t)\, dt \int_0^\lambda \frac{d}{du} \{u\, G_\nu(xu)\}\, \Xi_\nu(tu)\, u\, du = 0.$$

(η) *We have*

$$\int_0^\infty \frac{d}{du}\{u\,G_\nu(xu)\}\,u\,du\int_0^\infty \Xi_\nu(tu)\,t\,f(t)\,dt$$
$$= \lim_{\lambda\to\infty}\int_0^\infty t\,f(t)\,dt\int_0^\lambda \frac{d}{du}\{u\,G_\nu(xu)\}\,\Xi_\nu(tu)\,u\,du,$$

provided the limit on the right exists.

(ξ) *The repeated integral*

$$\int_A^B t\,dt\int_0^\lambda \frac{d}{du}\{u\,G_\nu(xu)\}\,\Xi_\nu(tu)\,u\,du$$

tends to a limit as $\lambda\to\infty$.

In the above lemmas, the conditions of Lemma 6(γ) are assumed to hold.

The proof of the integral formula now follows as in (A), and hence the theorem (B') is proved.

THE PROGRESS OF MATHEMATICAL ANALYSIS IN THE TWENTIETH CENTURY

(*Presidential Address.*)

By W. H. YOUNG.

[Read November 13th, 1924.]

MR. PRESIDENT, LADIES AND GENTLEMEN,

In all times perhaps, but more especially in recent times, the opposition to progress in abstract science has been made by those who were so possessed by an end to be aimed at, that they had no inclination and had often lost the capacity to study the means by which this end could be secured. Their contentions are indeed not infrequently based on insufficient knowledge of the history and literature of their own special branch of research, and may be summed up in the proposition that all scaffolding is futile, because no scaffolding is to appear on the finished edifice. But their objections are also due—and this is where they are most forcibly met—to ignorance of the use to which the newer concepts and methods are being put.

It was my desire to give, in my Presidential Address, some insight into the nature and extent of the development to which such new concepts have led in mathematics, and to show how, consciously or unconsciously, explicitly or implicitly, such novel ideas have come to permeate the Analysis of the Twentieth Century. But I am only able to-day to give what I might call the preface and first chapter of the work I had in view.

That in the domain of physical and technical science there has been advance of the most momentous and revolutionary character, no one at the present day thinks of contesting, however familiar he may already be becoming with the mysteries of radiographs, flying machines, or wireless telegraphy and telephony. If it be asked

whether there has been progress in pure mathematical science comparable to this, the answer is: "Yes! In the mighty effort which has made us at last giants in the domain of thought, whose prowess the Greeks would have admired." Perhaps we may at last claim with some certainty to have gone beyond the Greeks. Till the present century it would seem, from a close study of their thought, in the greater freedom of modern modes of expression, that we could claim little more than that we had transformed ideas that were in the greatest Greek minds—ideas less embryonic than inarticulate in their original presentation—so as to render them available for calculation and formal development.

Apart from the isolated and more tangible progress in technique, in mathematics as in physical science, the gradual getting down to the bed-rock of thought is characteristic of modern advance. Thus, in physics, we have arising the molecular theories, about which all the most fruitful results may be expected to centre; and, in mathematics, the Theory of Sets, the acknowledgment and singling out of which for study may be said to form the characteristic great and notable progress in mathematics in this first quarter of a century.

Indeed, I could not perhaps easily find a more striking illustration of the plea which I have taken as the prompting theme of my present address than in tracing the steps by which these two points have been gradually and completely gained in this short period of years.

At the turn of the century *Mengenlehre* was a term of which the meaning was all but unknown. The ideas were already budding out here and there, and by Georg Cantor it was perhaps already conceived as a perfect whole. But the mathematical world at large seems to have been unprepared for it. Indeed, in planning as one of the first articles of the *Encyclopädie* the one so entitled, which stands effectually in the first part of the first volume, the editor, Felix Klein, showed a foresight quite exceptional in character, and all the more praiseworthy as he was himself unversed in the subject.

Georg Cantor complained bitterly, and with justice, of the more wilful ignorance of his work in Germany itself. I am tempted to translate a whole passage from a letter written by him in 1908. After saying that he meant to present his next paper on the *Mengenlehre* for publication in our *Proceedings*, he continues:—"I shall always remain grateful to the London Mathematical Society for having made me one of its members, as well as to the Royal Society for the award of the Sylvester Medal, three-and-a-half years ago. You are quite right when you suggest in your letter that the recent great development of mathematics in Great

Britain is heartily welcome to me. My greatest wish is to be able myself to see that country, with whose high-minded inhabitants I feel myself at one; quite otherwise is it with the Germans, who do not know me, although I have lived among them fifty-two years."

And yet Hurwitz had achieved notable success by using the practical distinction, first pointed out by Cantor, between *countability* and *non-countability*; in other words, utilizing the fact that it is impossible to catalogue the points of the continuum by means of the integers 1, 2, 3, and so on for ever.

In France also the concepts of the Theory of Sets had begun already to bring forth fruit. In the same letter from which I have just quoted, Cantor says, *apropos* of some rather tactless remarks of Poincaré's at the Congress at Rome :—"It will interest you to hear that what Poincaré says about Hermite rests on complete ignorance. During the last ten years of his life, I was on terms of intimate friendship with Hermite, for whom I have an unbounded respect. Hermite and Weierstrass also had entirely overcome their earlier prejudices against me, prejudices systematically stimulated by Kronecker. The praises Hermite pours out to me in his letters, which I still possess, on the subject of the *Mengenlehre*, are so great and, in my eyes, so undeserved, that I should not like to publish them for fear of incurring the reproach of being myself dazzled by them."

Hermite's is the one name mentioned in the preface to Borel's little book of 1898, *Leçons sur la Théorie des Fonctions*, one which, written with an elegance all its own, was destined to produce an effect quite beyond what might now be supposed from a cursory perusal of its contents. It has been, in fact, the starting point of most of the pioneers of our century in the Theory of Sets of Points, and offers a good example of the manner in which some knowledge of this theory was gradually diffused, without either itself or its principal originator being cited by name.

With exceptions indeed, but very few exceptions, the work of Georg Cantor, in so far as it had individually attracted attention at all, was regarded as an attempt of chiefly philosophical interest, and this attitude of mind can scarcely surprise us, since many of the more prominent mathematical professors of the day were as unenlightened for practical purposes with regard to the fundamental concepts of analysis as Williamson, the author of well-worn works on Differential and Integral Calculus, then still commonly used in our ancient Universities and not yet superseded in those parts of the globe in which the English influence is paramount. Williamson adopted the standpoint that a change of

order in the procedure to the limit could not affect the result, and its study was for him of the nature of a purely metaphysical question.

Even in France, one of the most prominent members of the Institut, known for his work on cognate matters, refused, as I have it from the brilliant mathematician Réné Baire at first hand, to believe that a function $f(x, y)$ of two variables which is continuous with respect to each variable separately, could have any point of discontinuity; I need hardly say that $f(x, y)$ may be continuous with respect to *every* straight line in the plane and yet present discontinuities.

A complete change has come over the world of mathematics since, but it dates from quite recently. It would be hard now to find a writer on Analysis who does not directly or indirectly utilize the concepts and even the theorems of the Theory of Sets of Points, and the tendency is felt more and more to relegate the discussion of that theory and the proof of those theorems to an independent volume. There is even a periodical publication, the *Fundamenta Mathematicae*, devoted exclusively to the domain of the *Mengenlehre*. In the *Jahrbuch über die Fortschritte der Mathematik*, the *Mengenlehre* now comes, with a whole section to itself, between the section entitled "Arithmetic and Algebra" and that entitled "Analysis." But it has only just disappeared from the place it occupied between Philosophy and Pedagogy—a place it has occupied ever since it was first promoted out of the section of Philosophy in the year 1905. Some of the earliest papers by Borel, Lindelöf, and myself on the subject, published in 1904 and reviewed in 1906, are still under the section "Philosophy," cheek by jowl with writings by Hardy and Jourdain on transfinite numbers; although, for instance, my Leipsig paper of 1903 already gave a tangible contribution to pure mathematical science, as distinct from philosophy, in that I there introduced as a definitely classified entity *the inner limiting set*, or *i-set* as I called it later, and showed that it is either countable or has the potency of the continuum: in other words, the points which are common to an infinite succession of sets of intervals can be catalogued either by means of the integers or by means of the points of the continuum. I showed, moreover, that the latter mode must be used, if the *i-set* has any component dense in itself.

It is remarkable, too, that, though the *Théorie des Ensembles* had been recognized comparatively early in France, and some of her most distinguished mathematicians owe their advance to its exploitation, even Lebesgue himself, in a published criticism, when "The Theory of Sets of Points" by my wife and myself appeared in 1906, more than suggested that a work on this subject was not called for, and that its treatment

should merely constitute a chapter, one of the introductory chapters, of a course of Analysis.

The reluctance with which the Theory of Sets was allowed to take its individual place in the scheme of mathematics was but a symptom of the same spirit which, in the domain of scientific studies generally, is dictating the demand to relegate mathematics to the rank of the handmaid of the other sciences (*Hilfswissenschaft*). This policy, if not reversed, can only lead to such a weakening of the centre of the intellectual army as will ultimately secure its defeat : the tendency, to which it properly belongs, to restrict the period of study, is one which cannot now be too actively opposed. We should rather be citing the length of studies in the Middle Ages, as an incentive to our contemporaries to show a like devotion to the far more interesting and extensive science of to-day. A lifetime was too short to run through the gamut of knowledge in those days, and at an age when our young men expect to be speaking *ex cathedra* to students not ten years their juniors, the corresponding devotee of those far-off times was eagerly working as the servant of some venerable sage, so as to gather some of his jealously guarded stores of knowledge.

At a time when the fight for efficiency in the domain of technique is leading to the application there of the highest and the most fruitful principles, it is not only with shame, but with real apprehension, that the more enlightened in our fast thinning ranks must come to realize the retrogression of the course which is being forced upon them. The necessity of using and of teaching the use of the finest and most perfect tools, renders each inadequate apprenticeship, each premature appointment to office in our scientific guild, a menace to its future and to the future of all learning.

To the plea I am thus led more explicitly to make for increasing our power and technical grasp by studying beyond the merely apparent exigencies of the moment, the history of the utilization of the Theory of Sets again affords most vivid illustrations; none more so perhaps than is to be found in the history of Osgood's theorem in the Theory of Functions of two complex variables. This theorem states that *a function which is a regular analytic function of x when y is constant, and of y when x is constant, is also a regular analytic function of the pair of independent variables (x, y).*

Osgood, who was one of the very earliest to utilize the Theory of Sets of Points, succeeded in 1899 in proving this remarkable theorem, postulating only the condition, known *a posteriori* to be verified, that $f(x, y)$ is a bounded function. He foresaw, however, that it must be unnecessary to suppose this known *a priori*, and in the ensuing year he

produced an unfinished embryo of a proof. At this stage he left the problem. The fact that he was able to get as far as he did was owing to his use of a theorem in the Theory of Sets of Points, generally now known by the name of Baire, which he had discovered independently and either before or simultaneously with the French mathematician. Baire embodied it in the theory of what are known, following his nomenclature, as sets of the first category. The theorem states that *the continuum is not a set of the first category.*

A few years back, when I made the acquaintance of this work of Osgood's, I was immersed in the theory of sequences of functions and Lebesgue integrals; and I set myself the problem, and, I think, solved it, of examining how much further one could go with this last machinery in extending the result of Osgood's first paper. The authorities I consulted, and these the highest, were apparently ignorant that Osgood's embryonic attempt had been followed up fifteen years before. Osgood's problem had been solved, and solved in the way he had himself suggested, in a masterly manner by Hartogs in his *Habilitationsschrift* at Munich in 1906. How it occurred that this paper, which appeared in Volume 62 of the *Math. Annalen*, so completely escaped notice, I am at a loss to divine. No copy reached me, although I am referred to in a foot-note; and, though my wife was at Göttingen at the time, no one there spoke of it to her. Two other papers besides my own have appeared in the interim, dealing with the subject and containing no reference to Hartogs.

Had I known in time, I should certainly not have published my own paper. I was all along extremely suspicious, and held back my proof-sheets even to the last moment. It was Plancherel who subsequently came across Hartogs's paper. It was not entirely ill that my work was published. Methods are as instructive, and sometimes even more instructive, than results, and it is well to gauge their limitations and the place they hold in the scale of methods. I hope, and I am inclined to believe, that my paper throws nothing but credit on the *Proceedings* of the London Mathematical Society, though the actual result obtained had been already surpassed.

This is one of the cases that indicate the *Mengenlehre* as an apparatus surpassing in power even the subtle processes of Analysis in its most modern form. The theorem by which Hartogs was able to solve the problem was a direct generalization of one of Arzelà's, which I had, as a matter of fact, published two years previously in a paper in our *Proceedings*, extending the Theory of Content to open sets by means of what I call the inner and outer content. These concepts constitute a finer tool than the corresponding notions used by Cantor, Peano, and Jordan,

with which alone Hartogs was acquainted. By the words in which Hartogs refers to Theorem 6 of my paper on "Open Sets and the Theory of Content," it is evident, not only that he had not seen my work previous to completing his own, but that he did not realize that, the inner content of a set G, as defined by Peano and Jordan, being nothing more than the content of the set of all the intervals contained in the set G, the generalization of Arzelà in question was contained in Theorem 3 of my paper.

The work of those early days seems to me very clumsy now. The Theory of Content is merely a special case of that of integration, and ought not to precede it. It is merely the theory of the integration of functions assuming only two values, zero and unity; the content of a set G is the Lebesgue integral of the function which is unity at points of the set and zero elsewhere. In Hartogs's work we really have no need of inner content at all. The theorem required is simply this: that the content of an *i-set*, that is an *inner limiting set*, is the limit of the contents of the defining sets of intervals, and so, if the *i-set* is the null-set, that is, if it contains no points at all, the contents of the defining sets of intervals must tend to zero.

The whole episode admirably illustrates the plea which I have advanced above. We cannot expect to prove theorems involving higher concepts by means of lower methods, but we often can prove theorems merely involving lower concepts only by higher methods. Osgood used one of the simpler theorems of the Theory of Sets and the most elementary machinery from the Theory of Functions, and he failed to prove the complete theorem, which is one involving only the most elementary concepts of the Theory of Functions. By employing the theory of Lebesgue integration and the method of sequences, I got a step farther. Hartogs uses a somewhat more refined implement than Osgood from the Theory of Sets, and applies with great skill a full technical knowledge of Harmonic Functions and Poisson's Integral to obtain the last word on the subject.

The principle which I have been illustrating, that it is essential to go on to a higher discipline in order to master a lower one, is vital in mathematical training. The common practice of proving by elementary means theorems that have been obtained by advanced methods is only salutary if the result increases our insight into the lower discipline to such a degree that the elementary proof aids still further in the work of illumination. I might refer here to Hurwitz, who published a proof of algebraic methods alone of a problem in algebra, obvious by the Calculus; while, on the other hand, his successful attack on a problem in the

Theory of Numbers, which had baffled Cayley and Roberts, was due to his taking as his guiding star a higher discipline, the theory of quadratic forms, the ignoring of which had led the English mathematicians to waste their energies by trying to effect a breach from wrong points of attack. A still more remarkable case was his work on correspondences on an algebraic curve, where Hurwitz applied Abelian integrals in the Riemann theory.

Hilbert's dictum that it was precisely a knowledge of Riemann theory which placed Hurwitz in a higher class, is one worthy of meditation. This knowledge is now expected of every serious mathematician, while it may be contended that at the moment it is by a knowledge of the whole band of concepts and theorems embraced by the Theory of Sets that a mathematician may be classed among the *élite*, with whatever branch he be concerned.

I may remark, too, without going further into the utilization of the Theory of Sets in Applied Mathematics, that it was again by the use of the Baire-Osgood theorem that the continuum is not a set of the first category, that, in 1913, Plancherel in Switzerland and Rosenthal in Bavaria, simultaneously and independently, proved the impossibility of the Boltzmann-Maxwell theory of gases.

It is a characteristic feature of modern advance that the resources attained by the most potent means have become, even to those who could have little or no insight into their mode of production, comparatively easy of manipulation, using, as they may, automatically, rules that would not be true except in view of the extended applicability of the fundamental analytical processes, due to generalized conceptions of the twentieth century, and to the work that has been done in formulating and justifying them. There has, in fact, been a great simplification going on in mathematics, a simplification obtained by the introduction of what have been called advanced ideas and methods.

One example is that afforded by the Theory of Prime Numbers; the introduction of Algebra first, then Analysis, Dedekind's Theory of Ideals, and now the most modern Theory of Sets of Points, have added to our knowledge of prime numbers, and given us results which can be explained to one who has gone no further than Arithmetic.

Examining closely the influence of the Theory of Sets of Points, it may be contended that it is not so much a knowledge of its fascinating details, as the clearer outlook due to its study, which renders it so potent a method of advance in mathematics generally. From this point of view there was what may be called a half-way house between Analysis and the Theory of Sets: Analysis in its classical presentation of the concept

of *limit*, logically and mathematically sound, and the Theory of Sets of Points, by which we recognize that *the non-existence of a limit* is a conventional mode of expression, not to be defended as it stands and leading to serious misconceptions. A limit, in fact, necessarily *exists* when the number of variable points, or values, is infinite, but the limit is not necessarily *unique*. The half-way house is the halting station of those who so far realize the truth of this statement that they proceed to single out and utilize convenient particular limits, without necessarily going further into the possibilities afforded by the Theory of Sets itself. Of the momentous nature of this step it is impossible to be too conscious. It is only under certain conditions that such a selection is feasible, and the non-recognition of this fact—a fact clearly seen by Cantor and embodied by him in his creation of the concept of a well ordered series—is at the basis of that view of the Theory of Sets of Points which identifies it axiomatically with the study of well ordered sets sometimes adopted, at least provisionally, as, for instance, by Fraenkel in the first edition of his enthusiastic little book.

The Lebesgue integral is an instance of the selection of one of the limits between the Darboux upper and lower integrals, when the Riemann integral does not exist. The Lebesgue integral is accordingly only one of a whole set of limits.

Would that we could say that the investigation of matters of this kind were closed, that the tree had granted us all its fruit! When we turn to derivation as distinct from integration, we find that no such progress has been made. We have learned, it is true, to utilize the plurality of limits; most interesting and important work has been done on upper and lower right-hand and left-hand derivates, as distinct from differential coefficients. In particular we know that a differential coefficient exists wherever all the derivates are finite, barring a set of content zero. But we have not got beyond the amoebic attempt of Riemann to devise a generalized differential coefficient whose value lies between the extreme derivates, a concept for the rest to which he seems to have attached no importance, except as enabling him to get an isolated theorem in the theory of a class of trigonometrical series. A suggestion has indeed been thrown out by my wife that we should use what she calls the mean symmetric derivate, which she has defined and shown to exist, in the case of a continuous function, except at a countable set of points. This suggestion has not been taken up, nor has there been any evidence as yet that it has any practical application.

What we want is a process, or a concept, applicable to the most general type of continuous function. Our knowledge of the classes of

continuous functions has been materially increased by work in the Theory of Sets of Points connected with the concept of Denjoy integration, although the only application of this process is to continuous functions which possess differential coefficients everywhere. Such functions are, from one point of view, more special than absolutely continuous functions, that is, Lebesgue integrals, since these latter need only to possess a differential coefficient at a set of points not belonging to a set of content zero. But whereas the process of Denjoy integration leads back unfailingly from the differential coefficient to the primitive function, when the latter is continuous, we have no certainty as to the corresponding result if we use Lebesgue integration, unless the points at which the derivate, or differential coefficient, to be integrated, is infinite, are countable.

We have no means of dealing with the problem of derivation and the associated concept of integration, which we would fain believe devisable, in the case, for instance, of Weierstrass's non-differentiable function, expressed by the uniformly convergent Fourier series

$$f(x) = \sum_{n=0}^{\infty} b^n \cos a^n x,$$

where

$$ab > 1 + 3\pi/2,$$

a being an odd integer and b lying between 0 and 1. The upper and lower derivates of this function on both sides are at no point all simultaneously finite, and, except at a set of content zero, they are respectively $+\infty$ and $-\infty$, and although at a countable everywhere dense set of points there is a tangent in a geometrical sense, it is a cuspidal vertical tangent, and does not correspond to a differential coefficient, but to a forward and backward differential coefficient, both infinite but of opposite sign. We do not get much further with Riemann's generalized differential coefficient; this exists at a countable everywhere dense set of points. The mean symmetric derivate is 1 almost everywhere, and is clearly of no particular use.

With an obvious extension of the notion of derivate, we may say that, except at the cusps, every value is assumed by one of the derivates of Weierstrass's function. What we desire is to be able to choose out a succession of h's which will not only provide finite upper and lower derivates almost everywhere, but will also suggest a corresponding integrating process, leading from such a derivate back to the primitive function. The interesting work communicated by Denjoy to the Congress at Strasbourg in 1920, on Riemann's generalized second differential coefficient, has not yet, it would appear, taken us appreciably nearer to

the solution of this problem. Closely connected with this subject is the question of defining *integration with respect to a continuous function*, which is not of bounded variation in every interval. The solution of the one problem may be expected to lead to that of the other, though the latter would appear on the surface to be the simpler.

For the solution of these two problems we cannot hope for much help from the theory of sequences alone. The success of the method of sequences is really and intrinsically due to the mechanism of monotony, though frequently hidden away, and monotony in such a connexion as this serves no useful purpose. It is by a more profound study of the Theory of Sets that we can be led to the solution of these problems, or, which comes to much the same thing, to the determination of the limits within which their solution is possible.

One thing at least we may say: no theory of integration is complete which does not enable us to define, by some kind of integrating process, the integral of a continuous function with respect to itself, or, more generally, the integration of an ordinary function, such as the square, of a continuous function with respect to that continuous function, when the latter has not bounded variation.

I should like to say, by the way, that the existence of a great number of limits is an *embarras de richesses* as long as we are unprovided with tools for dealing with them. When the tools are there, it should be possible to turn this plurality to advantage.

A case in point is the theory of new modes of convergence and the discussion of divergent series. I remember that Oliver Heaviside used to say:—"This series is fortunately divergent, so we can do something with it!" We have now many modes of summation, the best known of which are the methods of Cesàro and its variations. It still remains to have what I should call a theory of *suitability*, dealing with criteria for the choice among the different possibilities. It does not follow that, because you can make a series converge, it will converge to a suitable function. Thus, for instance, in the Theory of Trigonometrical Series, Cesàro summation is plainly indicated, when a_n and b_n are of the order of a power of n. In the case of a higher order of infinity, increasingly powerful methods of summation are, of course, required; but such series cannot arise in the theory of Fourier series and their derived series.

To take as an instance again the Weierstrassian function. The Fourier series expressing it, regarded as a trigonometrical series, presents a lot of gaps; all the coefficients are zero except those whose places are indicated by values of n which are positive integral powers of a. If we apply the classical method of Cesàro, we have to insert all these zero

terms. We can, however, try to deal with the series by taking account only of the existing terms; in this case a more powerful method of summation will at once be necessary.

The question immediately arises: if you succeed in making the derived series converge, what does it converge to in the case when the original series has no differential coefficient? This question was in the mind of Riemann, as some of his work illustrates, but only in a very special degree. If we differentiate term-by-term the series representing Weierstrass's function, we find it converges to Riemann's generalized differential coefficient at certain points, forming a countable everywhere dense set of points.

Hand in hand the theories of differential coefficients and integrals, and that of methods of summation, must be much further developed than they are at the present moment. There should be interaction of each on the other.

The correspondence of the theory of series and that of integration—obvious when the integrals are infinite integrals, in which case it is a question of applying particular modes of approach, sometimes of an obviously Cesàro nature—is less evident when the limits of integration are finite. The application of Cesàro ideas in this latter case has still to be made. The whole subject is in its infancy; it may be foreseen that all turns on the transformation of the series, or other mathematical entity under consideration, by the application to it, or to the terms, or auxiliary functions, which characterise it, of a suitable analytical process, say Q. A series which is not convergent may, for instance, be *Q-convergent*, that is, it may, by the process Q applied to its separate terms, or to its partial summations, be transformed into a convergent series.

This brings me in conclusion to speak of the concept which I have called *super-summability* and its application in Analysis. I must ask your indulgence for taking this opportunity, since it has lately come to my knowledge that the original papers in which I introduced and dealt with this concept have never been referred to in the *Jahrbuch über die Fortschritte der Mathematik*, except by title only, although subsequent papers in which the same ideas occur have been from time to time discussed.

The first germ of the concept of super-summability is to be found in the work that centres round Parseval's Theorem in the Theory of Fourier Series, and its converse, the Theorem of Riesz-Fischer. These theorems state that *the necessary and sufficient condition that a Fourier series*

$$\Sigma\{a_n \cos nx + b_n \sin nx\}$$

should have for associated function $f(x)$ one whose square is summable, is that the series of the squares of the coefficients

$$\Sigma\{a_n^2+b_n^2\}$$

should be convergent.

This idea that the summability of a function $f(x)$ may be such that another function of $f(x)$ is also summable, leads to what I call *a type of super-summability*. I have given to this concept a wide generality which I call, for brevity, *Q-summability*. Here the function $Q(u)$ becomes a summable function of x, when we write $u=f(x)$. The limitations on the function $Q(u)$ are, firstly, that it is the integral $\int U(z)dz$ of a positive monotone increasing function whose differential coefficient is nowhere zero. Such functions are in the simplest cases u^2, u^{1+p} (where p is fractional or integral), $u\log u$, $u\log\log u$, and so forth. Such functions naturally associate themselves in pairs of inverse functions, U and V, where if

$$v=U(u),$$

$$u=V(v),$$

and for these I formulated certain inequalities of the type of Schwarz's inequality for the case of two squares. These inequalities play a fundamental part in the theory.

I showed also that, *if we integrate a succession of functions $f_n(x)$, each of which possesses Q-summability, then there is a sub-succession which converges to an integral, whose integrand also possesses Q-summability, provided only the succession of integrals*

$$\int Q\{|f_n(x)|\}\,dx$$

is bounded.

This theorem, as anyone who has worked with series of functions will at once perceive, has important applications in the Theory of Series of Normal Functions. In particular, I have proved that the *necessary and sufficient condition that a Fourier series should be the Fourier series of a function $f(x)$ possessing Q-summability is that*

$$\int_{-\pi}^{\pi} Q\left(|f_n(x)|\right)dx$$

is bounded, $f_n(x)$ denoting a partial Cesàro summation.

But I find that I have been led to refer to what is, after all, a very special point in my own work. This will not, I hope, prevent your bearing in mind the main aim of my Address, which was to bring out the almost revolutionary character of recent progress in pure mathematics, and thereby to encourage the lonely worker. Exigencies of time and space have narrowed down the execution of the plan I had in view, but I have, I trust, said enough. Single-handed, however, no one can successfully stem the tide of materialism and intellectual pessimism which has invaded the world, and, inevitably, most of all the world of mathematics, and thus brought about a state of affairs that I would I could persuade myself to be but a temporary after-effect of the Great War. It is the day for organized effort, and it is for our Society to do all that may be, not only in publishing, but also by exerting influence. As for myself, what I have been able to do I have done, and, in laying down my office as President,

> "This rough magic
> I here abjure. . . .
> I'll break my staff,
> Bury it certain fathoms in the earth,
> And deeper than did ever plummet sound,
> I'll drown my book."

SOME BOUNDARY PROBLEMS OF MATHEMATICAL PHYSICS.

By D. M. WRINCH.

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1. A large class of problems in two-dimensional mathematical physics may be described in the following manner. A function V is required which satisfies Laplace's equation $\nabla^2 V = 0$, is evanescent on C the circle at infinity, has no singularities in the region between C and a certain closed curve c, and on this curve c differs by a constant from a certain function which is the real part of the function

$$f(z) = d \log z + d_0 e^{i\delta_0} + d_1 z e^{i\delta_1} + d_2 z^2 e^{i\delta_2} + \ldots + d_n z^n e^{i\delta_n}.$$

This general class includes problems of electrical distributions on infinite cylindrical conductors, both when the cylinder is freely charged and when its electrification is due to a specific external field of electric force. It further covers problems of hydrodynamics associated with an infinite cylinder, when there is circulation about the cylinder, when there is streaming past the cylinder, and when the cylinder has uniform translational or rotational motion.

2. Suppose that the equation of c is taken in the form

$$x = x(u),$$
$$y = y(u)$$

in terms of a parameter u which varies monotonically from $-\pi$ to $+\pi$ on a complete description of the curve. The curve is closed and consequently

$$x(\pi) = x(-\pi),$$
$$y(\pi) = y(-\pi).$$

Introducing the complex variable

$$w = u + iv,$$

and writing

$$z(w) = x(w) + iy(w),$$

we ensure that the curve c is the v-level, $v = 0$, in the transformation

$$z = z(w).$$

Now, provided that, for some specific values of the various constants $(a, a_n, \alpha, \alpha_n)$,

$$z(w) = ae^{iw+i\alpha} + a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1 - iw} + a_2 e^{i\alpha_2 - 2iw} + \ldots, \tag{2}$$

the circle at infinity C is the v-level, $v = -\infty$. And we will confine attention to those cases where the constants $(a, a_n, \alpha, \alpha_n)$ are such that for values of v between zero and $-\infty$, dz/dw, which is given by

$$dz/dw = i(ae^{iw+i\alpha} - a_1 e^{i\alpha_1 - iw} - 2a_2 e^{i\alpha_2 - 2iw} - \ldots), \tag{3}$$

is neither infinite nor zero.*

If now we write

$$-iW = c_1 e^{-iw+i\gamma_1} + c_2 e^{-2iw+i\gamma_2} + \ldots,$$

with

$$W = U + iV.$$

V is certainly a solution of Laplace's equation, evanescent on C the circle at infinity and without singularities between c and C. Further, on c,

$$V = R(c_1 e^{-iu+i\gamma_1} + c_2 e^{-2iu+i\gamma_2} + \ldots).$$

Now on c,

$$\begin{aligned} f(z) = \; & diu + d\log(ae^{i\alpha} + a_0 e^{i\alpha_0 - iu} + a_1 e^{i\alpha_1 - 2iu} + \ldots) + d_0 e^{i\delta_0} \\ & + d_1 e^{i\delta_1}(ae^{iu+i\alpha} + a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1 - iu} + \ldots) \\ & + d_2 e^{i\delta_2}(ae^{iu+i\alpha} + a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1 - iu} + \ldots)^2 + \ldots \\ & + d_n e^{i\delta_n}(ae^{iu+i\alpha} + a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1 - iu} + \ldots)^n, \end{aligned}$$

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* It will be noticed that this investigation is concerned *only* with those cases where

(1) $z(w)$ can be expressed in the form given in (2) *supra*, and

(2) the resulting form for dz/dw given in (3) *supra* is such that dz/dw is never zero or infinite for values of v between zero and $-\infty$, *i.e.* outside the curve c.

Thus, we deal only with those curves

$$x = x(u), \quad y = y(u),$$

in which the functions $x(u)$ and $y(u)$ are such that the related function

$$z(w) = x(w) + iy(w)$$

satisfies these two fundamental provisos.

and by selecting the coefficients (c_n, γ_n) correctly we can ensure that, on c, V differs from the real part of the function $f(z)$ by a constant only. Thus, for example, if

$$f(z) = d_1 z e^{i\delta_1}$$

simply, which on c becomes in terms of u

$$d_1 e^{i\delta_1}(a e^{iu+i\alpha} + a_0 e^{ia_0} + a_1 e^{ia_1 - iu} + \ldots),$$

we put

$$-iW = d_1(a e^{-iw-i\alpha-i\delta_1} + a_1 e^{-iw+ia_1+i\delta_1} + a_2 e^{-2iw+ia_2+i\delta_1} + \ldots).$$

Then, on c,

$$V = d_1\big(a\cos(u+\alpha+\delta_1) + a_1\cos(u-a_1-\delta_1) + a_2\cos(2u-a_2-\delta_1) + \ldots\big),$$

$$R\big(f(z)\big) = d_1\big(a\cos(u+\alpha+\delta_1) + a_0\cos(a_0+\delta_1) + a_1\cos(u-a_1-\delta_1) + \ldots\big),$$

and these differ only by a constant.

3. A very simple solution of a whole range of problems of electrostatics and hydrodynamics is therefore available when the curve of cross section of the cylinder is such that

$$x = x(u) = a\cos(u+\alpha) + a_0\cos a_0 + a_1\cos(a_1-u) + a_2\cos(a_2-2u) + \ldots,$$

$$y = y(u) = a\sin(u+\alpha) + a_0\sin a_0 + a_1\sin(a_1-u) + a_2\sin(a_2-2u) + \ldots, \quad (4)$$

provided that the constants $(a, a_n, \alpha, \alpha_n)$ are such that the expression (3) has no zeros and no infinities for values of v between zero and $-\infty$. And this proviso evidently at once cuts out any curve c which possesses nodes.

These problems then may be reduced to the problem of finding a parameter u for the specific curve c in question which is such that the Fourier series for x and for y in terms of u are such that, apart from the leading terms

$$a\cos(u+\alpha), \quad a\sin(u+\alpha),$$

in x and y respectively, any term $a_n\cos(a_n-nu)$ in x is associated with the term harmonically conjugate $a_n\sin(a_n-nu)$ in y. Alternatively we may say that we have solved this set of problems associated with c when a parameter u has been discovered which is such that c represents the motion of a particle possessing a fundamental uniform circular motion,

given in terms of the time u by

$$x_0 = a\cos(u+\alpha)+a_0\cos\alpha_0,$$

$$y_0 = a\sin(u+\alpha)+a_0\sin\alpha_0,$$

or more simply by

$$z_0 = ae^{iu+i\alpha}+a_0e^{i\alpha_0},$$

on which is superposed any number of uniform circular motions given by

$$a_1e^{i\alpha_1-iu}, \quad a_2e^{i\alpha_2-2iu}, \quad \ldots \quad a_ne^{i\alpha_n-niu}, \quad \ldots,$$

which are in the opposite sense and whose periods are aliquot parts of the period of the fundamental vibration, giving for the complete motion

$$z = ae^{iu+i\alpha}+a_0e^{i\alpha_0}+a_1e^{i\alpha_1-iu}+\ldots+a_ne^{i\alpha_n-niu}+\ldots.$$

It will be convenient to call u a suitable parameter for this set of problems if it satisfies this condition.

4. The problem of finding to order the parameter for any specific curve which is suitable for the solution of specific problems of the type discussed in this paper has so far baffled inquiry. There are solutions of specific problems associated with certain fundamental curves such as the straight line, the circle, the ellipse, and the hyperbola, the lemniscate,* the hypotrochoids,† and the inverse of an ellipse,‡ which have happened to be soluble by means of some transformation which has come under consideration. There are, further, solutions of specific problems associated with curves made up of segments of the fundamental curves which have been treated; for example, the sector of a circle,§ the arc of a circle,‖ the semi-ellipse,¶ the rectangle with two sides rectilinear and two sides concentric circular arcs,** arcs of a confocal ellipse and hyperbola,†† and the equilateral triangle; and the solutions belonging to this second group of solutions can evidently be derived from members of the first group. By means of the important extension of the

* Basset, *Quart. J. of Math.*, 20 (1885), 234; and *Hydrodynamics*, 1 (1888), 106.

† Wrinch, *Phil. Mag.*, 48 (1924), 692–703.

‡ Basset, *Quart. J. of Math.*, 19 (1883), 190; 21 (1886), 336; and *Hydrodynamics*, 1 (1888), 102.

§ Stokes, *Trans. Camb. Phil. Soc.*, 8 (1849), 578 [*Math. and Phys. Papers*, 1 (1880), 305].

‖ Love, *Proc. London Math. Soc.* (2), 22 (1924), 337–369 (358); and Richmond, *ibid.*, 483–494 (489).

¶ Richmond, *ibid.*, 486.

** Greenhill, *Messenger of Math.*, 9 (1880), 35.

†† Ferrers, *Quart. J. of Math.*, 17 (1881), 227.

Schwarz-Christoffel Method, recently published by Richmond,* a large number of solutions can now be added to this second group.

5. The subject may, however, be presented in a systematic fashion by treating at the same time all the curves c, which, with any particular choices of the coefficients $(a, a_n, \alpha, \alpha_n)$, are covered by the equations (4).

In this way it is possible to obtain extensive information as to the properties of cylinders of various types of cross section which are relevant in problems of hydrodynamics and electrostatics. It has recently been proved by Richmond† that, given the field when a cylindrical conductor is freely charged, it is possible to deduce the field when the same conductor is introduced with any orientation into a uniform field. This proposition, together with a number of others, is covered in the general treatment given in this paper, and a number of other results relevant to hydrodynamics as well as to electrostatics are obtained. Thus, attention is directed to the expressions obtained for the coefficients of hydrodynamical mass for any cylinder whose cross section has the equation (4), the constants (a_n, α_n) being suitably chosen. The exploitation of the results obtained, even in the case of the simpler curves—for example, the curves for which $a_3 = a_4 \ldots = 0 = \alpha_2 = \alpha_3 \ldots$—will evidently give a wealth of information relating to cylinders whose cross sections present interesting geometrical features.

The procedure for practical purposes would, in fact, be to investigate systematically the infinite class of curves, all of which have been dealt with by means of the transformation (2). Thus the transformation

$$z = ae^{iw-i\alpha} + a_0 e^{i\alpha_0}$$

yields for $v = 0$ the circle

$$x = a\cos(u+\alpha) + a_0 \cos\alpha_0,$$

$$y = a\sin(u+\alpha) + a_0 \sin\alpha_0.$$

Next, the transformation

$$z = ae^{iw+i\alpha} + a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1 - iw}$$

yields for $v = 0$ the curve

$$x = a\cos(u+\alpha) + a_0\cos\alpha_0 + a_1\cos(\alpha_1 - u),$$

$$y = a\sin(u+\alpha) + a_0\sin\alpha_0 + a_1\sin(\alpha_1 - u),$$

* Cf. Richmond, *loc. cit.*

† *Loc. cit.* See also Whipple, *Proc. London Math. Soc.* (2), 22 (1924), iv–vi.

which is evidently an ellipse. Next the transformation

$$z = ae^{iw+i\alpha}+a_0e^{i\alpha_0}+a_ne^{i\alpha_n-inw}$$

yields for $v = 0$ the hypotrochoid

$$x = a\cos(u+\alpha)+a_0\cos\alpha_0+a_n\cos(\alpha_n-nu),$$

$$y = a\sin(u+\alpha)+a_0\sin\alpha_0+a_n\sin(\alpha_n-nu),$$

and includes the cusped hypocycloid as a particular case. Another particular case of the general transformation which has come up for discussion is the transformation

$$z = ce^{iw}\{1+e^{-2iw}\}^{\frac{1}{2}},$$

which yields the lemniscate for $v = 0$. A considerable variety of curves can be included by making all the constants (a, a_n, α, α_n) vanish except, say, a, a_m, and α_n. It appears indeed feasible for practical purposes, and possibly the only way of systematically exploring the subject, to make an inventory of the various curves given by the equations (4) when only two constants are different from zero, then when three alone are different from zero, and so on.

And it will be evident that no generality is lost if, in the original transformation, we put $\alpha = \alpha_1 = 0 = a_0$ and so replace it by the transformation

$$z = ae^{iw}+a_1e^{-iw}+a_2e^{i\alpha_2-2iw}+\dots. \tag{5}$$

It is merely that we find it convenient to take as the centre of coordinates associated with a curve c, the point such that the average value of the x and y coordinates, as a function of u, are each zero and to choose the direction of the axes x and y and the zero of u in such a way that the ellipse, which is the curve described when $a_2 = a_3 = a_4 \dots = a_n \dots = 0$, is referred to its principal axes, with u as the eccentric angle measured from the major axis.

6. We have now therefore under control the wide class of curves c whose equations are given by

$$\left.\begin{aligned} x = x(u) &= (a+a_1)\cos u+a_2\cos(a_2-2u)+a_3\cos(a_3-3u)+\dots\\ y = y(u) &= (a-a_1)\sin u+a_2\sin(a_2-2u)+a_3\sin(a_3-3u)+\dots \end{aligned}\right\}, \tag{6}$$

when, and only when, the function $z(w)$, given by

$$z(w) = x(w)+iy(w)$$
$$= ae^{iw}+a_1e^{-iw}+a_2e^{ia_2-2iw}+a_3e^{ia_3-3iw}+\ldots,$$

is such that dz/dw, given by

$$dz/dw = i(ae^{iw}-a_1e^{-iw}-2a_2e^{ia_2-2iw}-3a_3e^{ia_3-3iw}-\ldots),$$

has no zeros or infinities for values of v between zero and $-\infty$, *i.e.* outside the curve c.

The solutions of various problems of mathematical physics associated with infinite cylinders with these cross sections are easily to be obtained and are exhibited in the sequel.

7. In the Theory of Functions there are two methods *inter alia* which are used in the solution of problems. For certain purposes, the Weierstrassian point of view is adopted and we expand a function as a power series. Such a treatment is useful when information is needed as to the behaviour of a function in the neighbourhood of a certain point. On the other hand, if the zeros of a function are under discussion, we may conveniently express it as a product. And in each case there may be a finite or an infinite number of terms.

In this paper our object has been to find a method of systematically cataloguing closed curves with reference to which the specific set of problems mentioned in § 1 can be solved. In our discussion so far we have adopted the first point of view, and the various transformations in order of simplicity give the solution of problems in connexion with circles, ellipses, hypotrochoids, and so on. If now we exploit the second point of view it is natural to consider first the transformation

$$z-be^{i\beta} = ae^{iw},$$

which gives the circle $|z-be^{i\beta}| = a$ as the curve $v = 0$. Then the transformation

$$(z-b_1e^{i\beta_1})(z-b_2e^{i\beta_2}) = a^2e^{2iw}$$

would be discussed, and so on. And it is in this scheme of classification that the lemniscate which has already been mentioned makes its most natural appearance. For, consider the transformation

$$(z-a)(z+a) = a^2e^{2iw}.$$

We have $$r_1 r_2 = a^2 e^{-2v}$$

and $$\theta_1 + \theta_2 = 2u,$$

where r_1, r_2 are the radii vectores from the points $z = a$, $z = -a$, and θ_1, θ_2 the vectorial angles made by these radii with the initial line. Then the curve $v = 0$ has the equation

$$r_1 r_2 = a^2$$

and is recognizable as the lemniscate.

In general we should consider the transformation

$$(z-b_1 e^{i\beta_1})(z-b_2 e^{i\beta_2})\dots(z-b_n e^{i\beta_n})/(z-b_{-1}e^{i\beta_{-1}})(z-b_{-2}e^{i\beta_{-2}})\dots(z-b_{-m}e^{i\beta_{-m}})$$

$$= a^{n-m} e^{(n-m)iw}. \qquad (7)$$

The points $b_s e^{i\beta_s}$ need not be distinct, and, provided that the expression for e^{iw} is convergent, there may be an infinite number of such points. The circle at infinity is still given by $v = -\infty$ and, as before, we restrict our attention to those choices of the coefficients (b_s, β_s) which are such that dz/dw has no zeros and no infinities for values of v between zero and $-\infty$. It will be evident that it is a necessary condition that the curve $v = 0$ should lie outside all these points $b_s e^{i\beta_s}$ both when s is positive and when it is negative. With this general transformation (7) we evidently have

$$(r_1 r_2 \dots r_n)/(r_{-1} r_{-2} \dots r_{-m}) = a^{n-m} e^{-(n-m)v},$$

$$\theta_1 + \theta_2 \dots \theta_n - (\theta_{-1} + \theta_{-2} \dots + \theta_{-m}) = (n-m)u,$$

where r_s and θ_s are the radius vector and vectorial angle associated with the point $b_s e^{i\beta_s}$.

9. We now therefore have under control the curves c which have the equation

$$(r_1 r_2 \dots r_n)/(r_{-1} r_{-2} \dots r_{-m}) = a^{n-m}, \qquad (8)$$

where (r_s) are the radii vectores from a set of points, provided that the corresponding transformation,

$$\prod_{s=1}^{n} (z-b_s e^{i\beta_s}) \Big/ \prod_{s=-1}^{-m} (z-b_s e^{i\beta_s}) = a^{n-m} e^{(n-m)iw},$$

allows no zeros and no infinities to dz/dw when v lies between zero and $-\infty$. And in all such cases the suitable parameter u for the curve is given by

$$(n-m)u = \sum_{s=1}^{n} \theta_s - \sum_{s=-1}^{-m} \theta_s.$$

It will be easily recognized that this classification of transformations (familiar in the Method of Images) readily yields the solution of our problems in the case of simple curves with well defined elementary geometrical properties.

Thus, consider the transformation

$$(z-a)^2(z+a) = a^3 e^{3iw}.$$

The curve $v = 0$ has equation

$$r_1^2 r_2 = a^3,$$

where r_1 and r_2 are the radii vectores from the points a and $-a$ respectively. This curve has two loops, but, unlike the lemniscate, these loops are unequal: their diameters are indeed in the ratio $|x_1/x_2|$ where x_1, x_2, and zero are the roots of the equation

$$(x-a)^2(x+a) = a^3,$$

or

$$x(x^2-ax-a^2) = 0.$$

Thus

$$|x_1/x_2| = (\sqrt{5}+1)/(\sqrt{5}-1).$$

And curves consisting of a pair of loops whose diameters are in different ratios can easily be treated on these lines.

The systematic exploitation of the transformation (7) will, in fact, evidently yield a very comprehensive catalogue of c curves.

The problems of applied mathematics which we have in mind can in some cases be solved conveniently for the curves associated with this transformation. In other cases it is more convenient to derive the solutions for the curves associated with the transformation (5): for it is a simple matter to throw the product into the form

$$z = ae^{iw} + a_0 e^{ia_0} + a_1 e^{ia_1 - iw} + \ldots,$$

and the coefficients, so far as they are required, can be calculated *ad hoc* in any particular case.

I.—Application to Problems in Electrostatics.

1.1. *A Cylindrical Conductor in a Uniform Field of Force.*

An infinite cylindrical conductor whose cross section is the curve c with equation (6) is placed in a uniform field of electric force given by

$$iW_0 = f_0 + f_1 z e^{-i\beta_1},$$

so that

$$-V_0 = f_0 + f_1(x\cos\beta_1 + y\sin\beta_1),$$

$$U_0 = f_1(y\cos\beta_1 - x\sin\beta_1).$$

Let

$$iW_1 = iU_1 - V_1$$

represent the disturbance due to the presence of the cylinder and let

$$iW = iU - V$$

be the W-function for the field when the cylinder is present, so that

$$iW = iW_0 - iW_1.$$

The conditions of the problem will be satisfied if V is a solution of Laplace's equation evanescent on C the circle at infinity, having no singularities in the region between C and c, and on this curve c differs from V_0 by a constant.

Making use of the transformation (5)

$$z = ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots,$$

we may write

$$iW_0 = f_0 + f_1 e^{-i\beta_1}(ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots)$$

and

$$-V_0 = f_0 + f_1\big(ae^{-v}\cos(u-\beta_1) + a_1 e^{v}\cos(u+\beta_1) + a_2 e^{2v}\cos(2u+\beta_1-a_2) + \ldots\big).$$

Thus on c

$$-V_0 = (-V_0)_{v=0}$$

$$= f_0 + f_1\big(a\cos(u-\beta_1) + a_1\cos(u+\beta_1) + a_2\cos(2u+\beta_1-a_2) + \ldots\big).$$

Now choosing

$$V_1 = -f\big(ae^{v}\cos(u-\beta_1) + a_1 e^{v}\cos(u+\beta_1) + a_2 e^{2v}\cos(2u+\beta_1-a_2)\big),$$

we have a satisfactory solution.

Thus $$-V = -V_0 + V_1 = f_0 - 2f_1 a \sinh v \cos(u-\beta_1), \tag{9}$$

and on the curve c, V is constant and equal, in fact, to $-f_0$, which is the potential in the undisturbed field at the centre of coordinates. It is convenient to express this fact by saying that the centre of coordinates is, in this case, the electrostatic centre for any uniform field. It will be remarked that, whatever the direction of the uniform field, the electrostatic field has the same electrostatic centre with respect to the specific curve c. The electrostatic centre of any uniform field is, in fact, thus seen to be the point which is such that the average values of the x and y coordinates as functions of u measured from that point are each zero.

The W-function associated with the field when the cylinder is present is given by

$$iW = f_0 + 2aif_1 \sin(w-\beta_1), \tag{10}$$

and consequently

$$U = 2af_1 \cosh v \sin(u-\beta_1). \tag{11}$$

The equipotentials and the lines of force in any plane perpendicular to the axis of the cylinder can be sketched from the equations (9), (10), and (11).

The surface density $\sigma(f_1, \beta_1)$ at any point u on the curve is given by

$$\begin{aligned} 4\pi\sigma(f_1, \beta_1) &= 2af_1 \cos(u-\beta_1)/|dz/dw|_{v=0} \\ &= 2af_1 \cos(u-\beta_1)/|ds/du|, \end{aligned} \tag{12}$$

where s is the arc of the curve c.

1 . 2. *General Electrostatic Fields.*

The general problem of an infinite cylindrical conductor whose cross section is the curve c, placed in any electrical field, can be solved by a slight extension of the analysis. For, let the W function for the external field be given by

$$iW_0 = f_0 + f_1 z e^{-i\beta_1} + f_2 z^2 e^{-i\beta_2} + \ldots f_n z^n e^{-i\beta_n}.$$

In terms of w the complex variable introduced in the equation (5),

$$\begin{aligned} iW_0 = f_0 &+ f_1 e^{-i\beta_1}(ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots) \\ &+ f_2 e^{-i\beta_2}(ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots)^2 + \ldots \\ &+ f_n e^{-i\beta_n}(ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots)^n. \end{aligned}$$

We introduce the W-function straight away in the form

$$iW_1 = c_1 e^{-iw+i\gamma_1} + c_2 e^{-2iw+i\gamma_2} + \ldots,$$

and choose the coefficients (c_n, γ_n) so as to make V_1 differ from V_0 by a constant when $v = 0$. Thus, suppose

$$iW_0 = f_2 z^2 e^{-i\beta_2}.$$

In terms of w, we have

$$iW_0 = f_2 e^{-i\beta_2}(a^2 e^{2iw} + 2aa_1 + 2aa_2 e^{ia_2 - iw} + a_1^2 e^{-2iw} + \ldots).$$

Thus, writing

$$iW_1 = f_2\{a^2 e^{-2iw+i\beta_2} + 2aa_2 e^{ia_2 - i\beta_2 - iw} + a_1^2 e^{-2iw-i\beta_2} + \ldots\}$$

we find that on $v = 0$, V_1 differs from V_0 by a constant only.

Thus the field, when the cylinder is present, is given by

$$iW = iW_0 - iW_1 = 2af_2\{ai \sin(2w - \beta_2) + a_1 e^{-i\beta_2}\},$$

so that

$$-V = 2af_2\{a_1 \cos\beta_2 - a \sinh 2v \cos(2u - \beta_2)\}$$

$$U = 2af_2\{-a_1 \sin\beta_2 + a \cosh 2v \sin(2u - \beta_2)\}.$$

The surface density $\sigma(f_2, \beta_2)$ is given by

$$4\pi\sigma(f_2, \beta_2) = 4a^2 f_2 \cos(2u - \beta_2)/|dz/dw|_{v=0}$$

$$= 4a^2 f_2 \cos(2u - \beta_2)/|ds/du|.$$

Thus there is an interesting relation between the surface densities on the cylindrical conductor when the external fields are given respectively by

$$iW_0 = f_1 z e^{-i\beta_1}, \quad f_2 z^2 e^{-i\beta_2}.$$

The corresponding densities are then related by the equation

$$\sigma(f_2, \beta_2)/\sigma(f_1, \beta_1) = 2af_2 \cos(2u - \beta_2)/f_1 \cos(u - \beta_1). \tag{13}$$

1.3. *A Freely Charged Cylinder.*

The problem when the cylinder of cross section c is freely charged is of the same type. For, suppose we have to consider the field given by

$$iW_0 = 2e \log z + \text{constant},$$

and also the W_1-function which is evanescent at infinity, without singu-

larities between C the circle at infinity and c the curve of cross section of the cylinder, and such that V_1 differs from V_0 by a constant only, on the curve c. Then evidently the W-function

$$iW = iW_0 - iW_1$$

is the W-function for the freely charged cylinder carrying a charge of e per unit length. For on the cylinder V is constant; on the circle at infinity

$$V = -2e \log r + \text{constant},$$

and V is without singularities between c and C and satisfies Laplace's equation.

To find the form of W_1, we remark that

$$\log z = iw + \log(ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots).$$

Then putting

$$iW_1 = 2e \log\left(1 + \frac{a_1}{a} e^{-2iw} + \frac{a_2}{a} e^{ia_2 - 3iw} + \ldots\right)$$

we satisfy the conditions of the problem and get

$$iW = iW_0 - iW_1 = 2eiw + \text{constant}.$$

The surface density, which it will be convenient to denote by σ_{free}, is given by

$$4\pi\sigma_{\text{free}} = 2e/|dz/dw|_{v=0} = 2e/|ds/du|.$$

This surface density is evidently infinite only at those points on c (if any) at which $|dz/dw|$ vanishes. It will be noticed that it follows from equations (12) and (13) that

$$\sigma(f_1, \beta_1)/\sigma_{\text{free}} = af_1 \cos(u - \beta_1)/e, \tag{14}*$$

$$\sigma(f_2, \beta_2)/\sigma_{\text{free}} = 2a^2 f_2 \cos(2u - \beta_2)/e. \tag{15}$$

1.4. *Cylindrical Condensers.*

In the case of a cylinder whose cross section c has the parametric equations

$$\left.\begin{aligned} x = x(u) &= (a + a_1)\cos u + a_2 \cos(2u - a_2) + a_3 \cos(3u - a_3) + \ldots \\ y = y(u) &= (a - a_1)\sin u - a_2 \sin(2u - a_2) - a_3 \sin(3u - a_3) - \ldots \end{aligned}\right\}, \tag{16}$$

* Cf. Richmond, *loc. cit.*, and also Whipple, *loc. cit.*

the equipotentials when it is freely charged are cylinders whose cross sections are given by

$$v = \text{constant}.$$

Thus the equipotentials associated with the curve c are the v-levels of the transformation

$$z = ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \dots,$$

when v has any value between zero and $-\infty$. And, in consequence, a typical member of this group of equipotentials has the parametric equations

$$\left.\begin{aligned} x &= ae^{-v}\cos u + a_1 e^{v}\cos u + a_2 e^{2v}\cos(a_2 - 2u) + \dots \\ y &= ae^{-v}\sin u - a_1 e^{v}\sin u - a_2 e^{2v}\sin(a_2 - 2u) - \dots \end{aligned}\right\} \quad (17)$$

where v is some constant between zero and $-\infty$.

Further, a curve with the parametric equation

$$\left.\begin{aligned} x &= (b + b_1)\cos u + b_2\cos(2u - a_2) + b_3\cos(3u - a_3) + \dots \\ y &= (b - b_1)\sin u - b_2\sin(2u - a_2) - b_3\sin(3u - a_3) - \dots \end{aligned}\right\} \quad (18)$$

will belong to the equipotential set associated with the curve c with equation (16) provided that

$$\log(a_1/b_1) = \tfrac{1}{2}\log(a_2/b_2) = \tfrac{1}{3}\log(a_3/b_3) \dots = \log(b/a) \geqslant 0.$$

Now suppose that two such cylinders whose cross sections have equations (16) and (18) have potentials V_a and V_b respectively, and that the inner cylinder (16) has a charge e per unit length, then we evidently have for the potential in the region between them

$$V = 2ev + \text{constant},$$

and further

$$V_a - V_b = 2e\log(b/a).$$

The capacity of the condenser formed by the two cylinders is therefore

$$\tfrac{1}{2}\log(b/a)$$

per unit length of the cylinders.

The particular case when $a_1 = a_2 = a_3 \dots = 0$, and consequently $b_1 = b_2 = b_3 \dots = 0$, is the case of concentric circular cylinders of radius a and b. And the capacity of the condenser formed by the two cylinders

in general is evidently identically equal to the capacity of the condenser formed by these fundamental circular cylinders associated with them.

It is worthy of notice that the set of generalized epicyclics of any specific integral order n given by the equations

$$x = (a+a_1)\cos u + a_2\cos(2u-a_2) + \ldots + a_n\cos(nu-a_n),$$

$$y = (a-a_1)\sin u - a_2\sin(2u-a_2) - \ldots - a_n\sin(nu-a_n)$$

form an equipotential set, provided that the sets of coefficients

$$(a,\ a_1,\ a_2,\ \ldots,\ a_n)$$

associated with the various members of the family are such that the expressions

$$aa_1,\quad aa_2^{\frac{1}{2}},\quad aa_3^{\frac{1}{3}}\ \ldots\ aa_n^{1/n}$$

are severally constant throughout the family. And this result is also true when n tends to infinity. These results will be seen to provide a wide generalization of the equipotential character of sets of confocal ellipses which constitute the particular case $n = 1$*.

1.5. In the problem of finding the surface density of electrification in the various problems of electrostatics which have been treated in this section, the fundamental form of transformation is evidently

$$z = ae^{iw} + a_1e^{-iw} + a_2e^{ia_2-2iw} + \ldots.$$

The alternative transformation

$$\prod_{s=1}^{n}(z-b_se^{i\beta_s})\Big/\prod_{s=-1}^{-m}(z-b_se^{i\beta_s}) = a^{n-m}e^{(n-m)iw}$$

is, however, of value when attention is focused on the form of the equipotentials and lines of force associated with a curve c in these various problems, for the simple reason that e^{iw} is given conveniently in terms of z in this transformation. For, consider the problems associated with the cylinder whose cross section has the equation

$$(r_1r_2\ \ldots\ r_n)/(r_{-1}r_{-2}\ldots r_{-m}) = a^{n-m}. \tag{19}$$

The equipotentials associated with the curve c in the problem of free

* The equipotential character of certain sets of epicyclics which fall into the category of hypotrochoids has been exposed in a paper by the present writer, *loc. cit.*

electrification are the v-levels of the function

$$\prod_{1}^{n}(z-b_s e^{i\beta_s})\Big/\prod_{-1}^{-m}(z-b_s e^{i\beta_s}) = a^{(n-m)}e^{(n-m)iw}$$

for values of v between zero and $-\infty$. Thus their equations are

$$(r_1 r_2 \dots r_n)/(r_{-1}r_{-2}\dots r_{-m}) = a^{n-m}e^{-(n-m)v},$$

where $0 > v > -\infty$. The treating of curves by means of equations of the form (19) is therefore particularly suitable in this case. Further, it is evident that we can find the capacity of a condenser formed by two infinite cylinders with curves of cross section c_a and c_b if c_a and c_b are given by

$$(r_1 r_2 \dots r_n)/(r_{-1}r_{-2}\dots r_{-m}) = a^{n-m},\ b^{n-m}$$

if the points $b_s e^{i\beta_s}$ from which the radii vectores are measured are suitably chosen. And the capacity, in fact, takes the simple form

$$\tfrac{1}{2}\log(b/a)$$

per unit length.

Thus an inventory on simple geometrical lines of soluble problems of condensers can be made; and an approximate solution of any practical problem of this type which presents itself can in general be deduced.

II.—Application to Problems in Hydrodynamics.

2.1. *Streaming Past a Cylinder.*

Fluid, of density ρ, is streaming past a cylinder with cross section c with velocity q_0 in a direction making an angle β with the axis of x. The function associated with the motion of the fluid when the cylinder is not present is given by

$$\Phi_0 = -q_0(x\cos\beta + y\sin\beta),$$

$$\psi_0 = -q_0(y\sin\beta - x\sin\beta),$$

so that

$$W_0 = -q_0 z e^{-i\beta}.$$

Let W_1 represent the disturbing effect of the cylinder, so that W given by

$$W = W_0 + W_1$$

represents the motion when the cylinder is present. Then ψ_1 must be evanescent on the circle at infinity, and have no singularities in the

region between the curve c and the circle at infinity, and, further, ψ given by

$$\psi = \psi_0 + \psi_1$$

must be constant on the curve c.

The curve c has the equations

$$x = x(u) = (a+a_1)\cos u + a_2 \cos(a_2 - 2u) + \ldots,$$

$$y = y(u) = (a-a_1)\sin u + a_2 \sin(a_2 - 2u) + \ldots,$$

and using, as before, the transformation

$$z = ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots$$

we have

$$W_0 = -q_0 e^{-i\beta}\{ae^{iw} + a_1 e^{-iw} + a_2 e^{ia_2 - 2iw} + \ldots\}$$

and

$$\psi_0 = -q_0\{ae^{-v}\sin(u-\beta) - a_1 e^{v}\sin(u+\beta) - a_2 e^{2v}\sin(2u - a_2 + \beta) - \ldots\}.$$

Then, if we put

$$\psi_1 = \quad q_0\{ae^{v}\sin(u-\beta) - a_1 e^{v}\sin(u+\beta) - a_2 e^{2v}\sin(2u - a_2 + \beta) - \ldots\},$$

ψ_1 is certainly evanescent on the circle at infinity and without singularities in the region between c and C the circle at infinity. Further,

$$\psi = \psi_0 + \psi_1 = 2q_0 a \sin(u-\beta)\sinh v,$$

so that ψ is constant on the curve c. Thus we have for the motion in the presence of the cylinder

$$W = -2q_0 a \cos(w-\beta),$$

and hence $$\Phi = -2q_0 a \cos(u-\beta)\cosh v.$$

On the cylinder

$$\partial\Phi/\partial u = 2q_0 a \sin(u-\beta)\cosh v, \quad \partial\psi/\partial u = 0,$$

and the velocity is therefore given by

$$q = -2q_0 a \sin(u-\beta)/|\,dz/dw\,|_{v=0}$$

along the curve in the direction u increasing.

In the event of the curve c being such that $|dz/dw|_{v=0}$ has a simple zero at a point on it, say at $u = \alpha$, the velocity there will be infinite, except in the special case when $\alpha = \beta$.

2 . 2. *Translational Motion of a Cylinder.*

The function W_1 of § 2 . 1 itself represents an interesting motion. We have

$$\psi_1 = q_0\{ae^v \sin(u-\beta) - a_1 e^v \sin(u+\beta) - a_2 e^{2v} \sin(2u-\alpha_2+\beta) - \ldots\},$$

and consequently

$$W_1 = -q_0\{ae^{-iw+i\beta} - a_1 e^{-iw-i\beta} - a_2 e^{-2iw-i\beta-i\alpha_2}\},$$

$$\Phi_1 = -q_0\{ae^v \cos(u-\beta) - a_1 e^v \cos(u+\beta) - a_2 e^{2v} \cos(2u-\alpha_2+\beta) - \ldots\}.$$

This function W_1 is given in terms of the function W and W_0 of § 2 . 1 by the relation

$$W_1 = W - W_0,$$

and the motion it represents is therefore that of the cylinder moving with velocity q_0 in a direction making an angle $\pi+\beta$ with the axis of x.

The kinetic energy of the fluid per unit length of cylinder is T, given by

$$T = -\tfrac{1}{2}\rho \int_c \Phi_1 d\psi_1$$

$$= -\tfrac{1}{2}\rho \int_0^{2\pi} (\Phi_0 - \Phi)(d\psi_0/du - d\psi/du)_{v=0}\, du.$$

Now

$$\int_c \Phi_0 d\psi_0 = q_0^2 A,$$

where A is the area of the curve c. Further, on the curve c, ψ is constant, and consequently $d\psi/du = 0$. Therefore

$$T = -\tfrac{1}{2}\rho \left\{q_0^2 A - \int_0^{2\pi} [\Phi\, d\psi_0/du]_{v=0}\, du\right\}$$

$$= -\tfrac{1}{2}\rho q_0^2 A + \rho a q_0^2 \int_0^{2\pi} \cos(u-\beta)\{a\cos(u-\beta) - a_1 \cos(u+\beta) - a_2 \cos(2u+\beta-\alpha_2) - \ldots\}\, du$$

$$= -\tfrac{1}{2}\rho q_0^2 A + \rho a q_0^2 \pi\{(a-a_1)\cos^2\beta + (a+a_1)\sin^2\beta\}.$$

Thus

$$T = \tfrac{1}{2}\rho q_0^2 A\{\lambda \cos^2\beta + \mu \sin^2\beta\},$$

where
$$\lambda A = 2\pi(a-a_1)\,a - A, \tag{20}$$
$$\mu A = 2\pi(a+a_1)\,a - A. \tag{21}$$

It will be remarked that $\lambda = \mu$ except when $a_1 \neq 0$. It is noteworthy that these coefficients λ and μ are determinate if the two leading coefficients a and a_1 associated with the equation of the curve c be given in addition to the area of curve. λ and μ may be called the coefficients of hydrodynamical mass associated with translational motion of the cylinder. These coefficients are evidently geometrical properties of the curve of cross section in question, in the same way as the area, the moments of inertia of the area enclosed by the curve, and the length of the curve are geometrical properties of the curve c.

It is on certain occasions convenient to have A in terms of the coefficients $(a,\ a_n,\ \alpha_n)$, which occur in the equation of the curve. We evidently have

$$\begin{aligned} A &= \tfrac{1}{2}\int_0^{2\pi} (x\,dy/du - y\,dx/du)\,du \\ &= \tfrac{1}{2}\int_0^{2\pi} [\{(a+a_1)\cos u + a_2\cos(a_2-2u)+\ldots\} \\ &\qquad \{(a-a_1)\cos u - 2a_2\cos(a_2-2u)-\ldots\} \\ &\quad + \{(a-a_1)\sin u + a_2\sin(a_2-2u)+\ldots\} \\ &\qquad \{(a+a_1)\sin u - 2a_2\sin(a_2-2u)-\ldots\}]\,du \\ &= \pi(a^2-a_1^2-2a_2^2-3a_3^2\ldots). \end{aligned}$$

It is of interest to notice the form of this result for the lemniscate. We have, in the case of the lemniscate
$$r^2 = 2a^2\cos 2\theta,$$
the transformation
$$z^2 - a^2 = a^2 e^{2iw}$$
or
$$\begin{aligned} z &= ae^{iw}\{1+e^{-2iw}\}^{\frac{1}{2}} \\ &= ae^{iw}\{1+\tfrac{1}{2}e^{-2iw}+\ldots\}. \end{aligned}$$

Thus for the lemniscate
$$\lambda A = 2\pi(a-a/2)\,a - A = \pi a^2 - A,$$
$$\mu A = 3\pi a^2 - A,$$

and $$A = 4\int_0^{\pi/4} a^2 \cos 2\theta \, d\theta = 2a^2.$$

Thus $$\lambda = \pi/2 - 1 = 0{\cdot}5708,$$

$$\mu = 3\pi/2 - 1 = 3{\cdot}7124.$$

It is evidently possible to solve certain problems of the initial motion when a cylinder with cross section c is surrounded by a cylinder of cross section c_b when c_b has the equation

$$x = (b+b_1)\cos + b_2 \cos(a_2 - 2u) + \ldots,$$

$$y = (b-b_1)\sin + b_2 \sin(a_2 - 2u) + \ldots,$$

where

$$\log(a_1/b_1) = \tfrac{1}{2}\log(a_2/b_2) = \tfrac{1}{3}\log(a_3/b_3) \ldots = \log(b/a) \geqslant 0.$$

The analysis follows very closely that of § 1 . 4 and need not be given step by step.

2 . 3. *Rotating Cylinders.*

Suppose the cylinder with cross section c is rotating with uniform angular velocity ω about its axis. The boundary condition to be satisfied by the stream function ψ_1 on the curve c is, on this occasion, that

$$\psi_1 + \tfrac{1}{2}\omega r^2$$

is constant. Now, on the curve c,

$$\begin{aligned} r^2 &= x^2(u) + y^2(u) \\ &= \{(a+a_1)\cos u + a_2 \cos(a_2 - 2u) + a_3 \cos(a_3 - 3u) + \ldots\}^2 \\ &\quad + \{(a-a_1)\sin u + a_2 \sin(a_2 - 2u) + a_3 \sin(a_3 - 3u) + \ldots\}^2 \\ &= a^2 + \Sigma a_n^2 + 2a\{a_1 \cos 2u + a_2 \cos(a_2 - 3u) + \ldots\} \\ &\quad + 2a_1 \sum_{r=2}^{\infty} a_r \cos(\overline{r-1}\, u - a_r) + 2a_2 \sum_{r=3}^{\infty} a_r \cos(\overline{r-2}\, u - a_1 + a_2) \ldots \end{aligned}$$

Now put

$$\begin{aligned} \psi_1 = -\omega\{a[a_1 e^{2v} \cos 2u + a_2 e^{3v} \cos(a_2 - 3u) + \ldots] \\ + a_1 \sum_{r=2}^{\infty} a_r e^{(r-1)v} \cos[(r-1)u - a_r] \\ + a_2 \sum_{r=2}^{\infty} a_r e^{(r-2)v} \cos[(r-2)u - a_r + a_2] + \ldots\}. \end{aligned}$$

Then on the curve c, given by $v = 0$,

$$\psi_1 + \tfrac{1}{2}\omega r^2 = \tfrac{1}{2}\omega [a^2 + a_1^2 + a_2^2 + \ldots],$$

and ψ_1 evidently satisfies all the conditions. With this determination of ψ_1 we have

$$i\Phi_1 - \psi_1 = i\omega_1 = \omega \{a (a_1 e^{-2iw} + a_2 e^{-3iw + ia_2} + \ldots) + \sum_{\substack{r>s \\ (a_1=0)}} a_r a_s e^{-(r-s)iw + ia_r - ia_s}\},$$

and on the curve c,

$$\Phi_1 = (\Phi_1)_{v=0} = \omega \{a[-a_1 \sin 2u + a_2 \sin (a_2 - 3u) + \ldots] - \sum_{\substack{r>s \\ (a_1=0)}} a_r a_s \sin [(r-s)u - a_r + a_s]\}.$$

Then the kinetic energy of the liquid, due to the rotation of the liquid (of density ρ) is T per unit length, given by

$$T = -\tfrac{1}{2}\rho \int_c \Phi \, d\psi$$

$$= -\tfrac{1}{2}\rho\omega^2 \int_0^{2\pi} \{a [-a_1 \sin 2u + a_2 \sin (a_2 - 3u) + \ldots] - \sum_{\substack{r>s \\ (a_1=0)}} a_r a_s \sin [(r-s)u - a_r + a_s]$$

$$\{a [2a_1 \sin 2u - 3a_2 \sin (a_2 - 3u) - \ldots] + \sum_{\substack{r>s \\ (a_1=0)}} (r-s) a_r a_s \sin [(r-s)u - a_r + a_s]\} \, du$$

$$= +\tfrac{1}{2}\rho\omega^2 \{\sum_1^\infty a_r^2 a_{r+1}^2 + 2 [\Sigma a_r a_{r+2} \cos (a_{r+2} - a_r) + a a_1]^2 + 2\Sigma a_r^2 a_{r+2}^2 \sin^2 (a_{r+2} - a_r) + \ldots\}$$

$$= \tfrac{1}{2}\rho\omega^2 \{\sum_1^\infty a_r^2 a_{r+1}^2 + 2 \sum_1^\infty a_r^2 a_{r+2}^2 + \ldots + 2a^2 a_1^2 + 3a^2 a_2^2 + \ldots + 2 [2a a_1 \Sigma a_r a_{r+2} \cos (a_{r+2} - a_r) + 3a a_2 \Sigma a_r a_{r+3} \cos (a_{r+3} - a_r) + \ldots]\}.$$

We may also write this result in the form

$$T = \tfrac{1}{2} r \rho A k^2 \omega^2,$$

where (as before) A is the area of the cross section of the cylinder, and k is the radius of gyration of the area of cross section of the cylinder about the axis of the cylinder. Then ν may conveniently be called the coefficient of hydrodynamical mass associated with rotational motion of the cylinder. This geometrical constant associated with the general curve c can be worked out very simply when the constants (a_n, α_n) have suitable values.

2 . 4. *Translation and Circulation.*

Suppose that there is circulation k round the cylinder, and streaming past the cylinder with velocity q_0 in a direction making an angle β with the axis of x. Superposing these two motions, we have

$$W = -2aq_0 \cos(w-\beta) - \kappa w/2\pi,$$

giving

$$\Phi = -2aq_0 \cos(u-\beta) \cosh v - \kappa u/2\pi.$$

The velocity q at any point on the curve c is given by

$$q = (-2aq_0 \sin(u-\beta) + \kappa/2\pi)/|dz/dw|_{v=0}$$

along the curve in the direction u increasing.

An interesting point arises when the curve c is such that, at some specific point, say $u = \alpha$, $|dz/dw|_{v=0}$ has a simple zero. In general, such a zero will involve an infinite velocity at that point, whether there is merely streaming (as we have already seen), or merely circulation, or a combination of streaming and circulation. If, however, the ratio of the velocity of streaming to the circulation is suitably adjusted, this infinity of q can be avoided. For if it is the case that

$$-2aq_0 \sin(\alpha-\beta) + \kappa/2\pi = 0,$$

q will be finite at the cusp $u = \alpha$ on the curve. With this particular combination of streaming and circulation the velocity is never infinite, and we therefore have under consideration a motion which is physically possible with a finite pressure at infinity, provided that this pressure is sufficiently large. This fact is of great importance in Aeronautics in connexion with the Joukowsky hypothesis in the Circulation Theory of Lift.*

* See A. R. Low, "The Circulation Theory of Lift", International Air Congress, London, 1923, and *Dictionary of Applied Physics*, 5 (1923), 187.

2.5. In the case of the hydrodynamical problems which have been discussed, the alternative classification of curves by means of the equation

$$(r_1 r_2 \dots r_n)/(r_{-1} r_{-2} \dots r_{-m}) = a^{n-m}$$

is also of value. As in the electrostatic problems, this classification is particularly convenient when attention is directed to the equipotentials associated with the curve in various fields. But it is also noteworthy that the coefficients of hydrodynamical mass λ and μ are easily to be obtained from this type of classification. As we saw in the case of the lemniscate, we throw the transformation

$$\prod_1^n (z - b_s e^{i\beta_s}) \Big/ \prod_1^{-m} (z - b_s e^{i\beta_s}) = a^{(n-m)iw}$$

into the form $$z = ae^{iw} + a_0 e^{ia_0} + a_1 e^{ia_1 - iw} + \dots.$$

But beyond the expression for the area under the curve c, which can be found by a variety of methods, and is not necessarily only to be found in terms of these coefficients, we require only the coefficients a and a_1. Further, as in the electrostatic problems, this form of equation for curves is peculiarly suited to the making of an inventory of pairs of curves, with reference to which we can solve certain hydrodynamical problems associated with pairs of cylinders.

SUMMARY.

A large class of problems in two-dimensional mathematical physics may be described in the following manner. A function V is required, which satisfies Laplace's equation $\nabla^2 V = 0$, is evanescent on C, the circle at infinity, has no singularities in the region between C and a certain closed curve c, and on this curve c differs by a constant from a certain function which is the real part of the function

$$f(z) = d \log z + d_0 e^{i\delta_0} + d_1 z e^{i\delta_1} + d_2 z^2 e^{i\delta_2} + \dots.$$

This general class includes problems of electrical distributions on infinite cylindrical conductors, both when the cylinder is freely charged and when its electrification is due to a specific external field of electric force. It further covers problems of hydrodynamics associated with an infinite cylinder, when there is circulation about the cylinder,

when there is streaming past the cylinder, and when the cylinder has uniform translational or rotational motion.

In this paper this problem is solved in the case of certain curves whose equations are of the form

$$x = (a+a_1)\cos u + a_2\cos(a_2-2u) + a_3\cos(a_3-3u) + \ldots,$$

$$y = (a-a_1)\sin u + a_2\sin(a_2-2u) + a_3\sin(a_3-3u) + \ldots,$$

and simple results are obtained in various electrostatic and hydrodynamic problems. This method of attack allows the systematic exploration of the class of curves with reference to which these problems may be solved, and a methodological inventory of some of the various constants associated with curves which have interpretations in mathematical physics.

THE NUMERICAL SOLUTION OF A CERTAIN CLASS OF DIFFERENTIAL EQUATION

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In the practical application of differential equations of the type

$$\frac{d^2y}{dx^2}+\phi(x, y) = 0, \tag{1}$$

it is very frequently the case that a numerical solution is required, subject to boundary conditions of the type : $x = x_0$, $y = y_0$; $x = x_1$, $y = y_1$; error in y everywhere less than some given small number. A direct solution by expansion in a power series may fail because the range (x_0-x_1) may be greater than the radius of convergence of the series, while an application of the process of continuation, where the number representing the slope at (x_0, y_0) is carried forward from series to series, is burdensome and inconvenient.

The simplest method of approach to a graphical solution would be by means of the sequence

$$y_{r+1} = -\int_{x_0}^{x} dt \int_{x_0}^{t} dt\, \phi(t, y_r)+Ax+B, \tag{2}$$

where A and B are determined at each stage by making y_{r+1} conform to the required boundary conditions. This can best be effected graphically by sketching in any first approximation y_r satisfying the required conditions. Using the sequence (2) in the form

$$\frac{d^2y_{r+1}}{dx^2} = -\chi(x, y_r), \tag{3}$$

d^2y_{r+1}/dx^2 is then sketched in on an x base. Integrating this curve by means of a planimeter or otherwise, the shape of the curve dy_{r+1}/dx in the range (x_0, x_1) is determined, but its position is not. To make dy_{r+1}/dx conform to the required boundary conditions, any one possible curve for dy_{r+1}/dx is integrated between x_0 and x_1, and the displacement of this

curve parallel to the y axis required to make this integral equal $y_1 - y_0$ is easily calculated. The next approximation y_{r+1} is thus definitely determined, and the process may be repeated until the difference between two successive approximations is less than the error initially postulated as allowable. If an error of $\frac{1}{2}$ per cent. is permissible in the final result, this may be attained graphically with comparative rapidity; but if greater accuracy is desirable it is usually necessary to compute the values of the integrals arithmetically.

An examination of the conditions under which this sequence of steps is convergent leads, however, to the conclusion that it is subject to restrictions in application similar to those referred to in the case of an attempted solution by expansion in a power series, viz. that the sequence is convergent only over a limited range. If (x_0, x_1) be greater than this range, there does not appear to be any simple modification of the process that will enable the solution to be constructed. It is eminently desirable for practical purposes that a process be derived which will lead to a solution of (1), subject to the given boundary conditions, and whose convergence is independent of the range (x_0, x_1).

Consider the equation

$$\frac{d^2y}{dx^2} + n^2\phi(x, y) = 0, \tag{4}$$

and let the solution be required to satisfy the conditions

$$x = 0, \quad y = a; \qquad x = l, \quad y = b. \tag{5}$$

Writing (4) in the form

$$(D^2 + n^2)\, y = n^2 [y - \phi(x, y)]$$

the differential equation is directly transformable into the integral equation

$$y = A \sin nx + B \cos nx + n \int_0^x dt\, [y - \phi(t, y)] \sin n(t - x), \tag{6}$$

where A and B are arbitrary constants.
To satisfy conditions (5) we require

$$\left.\begin{aligned} A \sin nl &= b - a \cos nl - n \int_0^l dt\, [y - \phi] \sin n(t - l) \\ B &= a \end{aligned}\right\} \tag{7}$$

and (6) takes the form

$$y = \frac{b \sin nx + a \sin (l-x)}{\sin nl} + \frac{n \sin n(x-l)}{\sin nl} \int_0^x dt \sin nt(y-\phi)$$

$$+ \frac{n \sin nx}{\sin nl} \int_x^l dt \sin n(t-l)(y-\phi). \qquad (8)$$

Consider the sequence

$$y_{r+1} = \frac{b \sin nx + a \sin (l-x)}{\sin nl} + \frac{n \sin n(x-l)}{\sin nl} \int_0^x dt \sin nt[y_r - \phi(t, y_r)]$$

$$+ \frac{n \sin nx}{\sin nl} \int_x^l dt \sin n(t-l)[y_r - \phi(t, y_r)]. \qquad (9)$$

It is clear that at each stage y_{r+1} satisfies the conditions

$$x = 0, \quad y_{r+1} = a; \qquad x = l, \quad y_{r+1} = b.$$

Let $\epsilon_{r+1} = y_{r+1} - y$, $\epsilon_r = y_r - y$, ..., where y is defined by the integral equation (8), then, from (8) and (9),

$$\epsilon_{r+1} = -n \sin nx \int_x^l dt\, \epsilon_r \left[1 - \left(\frac{\partial \phi}{\partial y}\right)_\mu\right] \cos nt$$

$$- n \cos nx \int_0^x dt\, \epsilon_r \left[1 - \left(\frac{\partial \phi}{\partial y}\right)_\mu\right] \sin nt$$

$$+ n \cot nl \sin nx \int_0^l dt\, \epsilon_r \left[1 - \left(\frac{\partial \phi}{\partial y}\right)_\mu\right] \sin nt, \qquad (10)$$

provided ϕ is continuous over the range (y_r, y) and $(\partial\phi/\partial y)_\mu$ is a value of $\partial\phi/\partial y$ at some value of y in that range for each value of x.

Let $$J_r(x) \equiv \epsilon_r \left[1 - \left(\frac{\partial \phi}{\partial y}\right)_\mu\right]; \qquad (11)$$

then since $y_{r+1} = y$ at $x = 0$ and $x = l$, $\epsilon_r = 0$ at these points, and, therefore, if $\partial\phi/\partial y$ is finite,

$$J_r(0) = 0 \quad \text{and} \quad J_r(l) = 0.$$

Let there be a finite number p of maxima and minima in $J_r(x)$ between $x = 0$ and $x = l$ occurring at

$$x = x_t \quad (t = 1, 2, \ldots, p),$$

then in the successive ranges

$$(0, x_1),\ (x_1, x_2) \ldots (x_t, x),\ (x, x_{t+1}),\ \ldots,\ (x_p, l),$$

$J_r(x)$ is monotonic.

Consider the first integral in the expression for ϵ_{r+1}, viz.

$$K = \int_x^l J_r(t) \cos nt\, dt$$

$$= \int_x^{x_{t+1}} + \int_{x_{t+1}}^{x_{t+2}} + \ldots + \int_{x_p}^{l}.$$

Since $J_r(t)$ is monotonic within each one of the successive ranges,

$$\begin{aligned} nK = \quad & J_r(x)[\sin n\xi_1 - \sin nx] + J_r(x_{t+1})[\sin nx_{t+1} - \sin n\xi_1] \\ & + J_r(x_{t+1})[\sin n\xi_2 - \sin nx_{t+1}] + J_r(x_{t+2})[\sin nx_{t+2} - \sin n\xi_2] \\ & + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + J_r(x_p)[\sin n\xi_{p-t+1} - \sin nx_p] + J_r(l)[\sin nl - \sin n\xi_{p-t+1}], \end{aligned}$$

where $$x_{t+s-1} < \xi_s < x_{t+s}.$$

Similarly, for the second integral in the expression for ϵ_{r+1}, viz.

$$L = \int_0^x J_r(t) \sin nt\, dt$$

$$= \int_0^{x_1} + \int_{x_1}^{x_2} + \ldots + \int_{x_t}^{x}.$$

Since $J_r(t)$ is monotonic within the range of each sub-integral,

$$\begin{aligned} nL = \quad & J_r(0)[1 - \cos n\eta_1] + J_r(x_1)[\cos n\eta_1 - \cos nx_1] \\ & + J_r(x_1)[\cos nx_1 - \cos n\eta_2] + J_r(x_2)[\cos n\eta_2 - \cos nx_2] \\ & + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + J_r(x_t)[\cos nx_t - \cos n\eta_{t+1}] + J_r(x)[\cos n\eta_{t+1} - \cos nx], \end{aligned}$$

where $$x_s > \eta_s > x_{s-1}.$$

If M be the greatest absolute value of $J_r(x)$ which can occur in the

range $(0, l)$, it follows that

$$|nK \sin nx + nL \cos nx| < M(2p+1).$$

In the same manner it will readily be seen that the third integral in the expression for ϵ_{r+1} is less than $2pM/n$.
Hence

$$|\epsilon_{r+1}| < M(2p+1+2p\lambda), \tag{12}$$

where

$$\lambda = |\cot nl|. \tag{13}$$

Since $J_r(x) = \epsilon_r\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\mu\right]$, from the definition of M, it follows that

$$M < \epsilon_r'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max},$$

where ϵ_r' is the maximum value of ϵ_r that occurs in the range $(0, l)$.

Thus from (12),

$$|\epsilon_{r+1}| < \epsilon_r'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}\{1+2p(1+\lambda)\},$$

where $|\epsilon_{r+1}|$ is, of course, a function of x.
In particular therefore,

$$\epsilon_{r+1}' < \epsilon_r'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}\{1+2p(1+\lambda)\},$$

$$\epsilon_r' < \epsilon_{r-1}'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}\{1+2p(1+\lambda)\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\epsilon_2' < \epsilon_1'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}\{1+2p(1+\lambda\}.$$

Hence

$$\epsilon_{r+1}' < \epsilon_1'\{1+2p(1+\lambda)\}^r\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}^r, \tag{14}$$

all the numbers involved being positive.

Provided

$$\{1+2p(1+\lambda)\}\left|1-\frac{\partial\phi}{\partial y}\right|_{\max} < 1,$$

there exists a number ρ, such that for $r > \rho$, ϵ_{r+1}', and *a fortiori* $|\epsilon_{r+1}|$ can be made less than any given positive number, however small, no matter what (finite) quantity ϵ_1' (*i.e.* what finite function of x, ϵ_1) may

be. Hence the sequence (9) converges, provided

$$1-\frac{1}{1+2p(1+\lambda)}<\frac{\partial\phi}{\partial y}<1+\frac{1}{1+2p(1+\lambda)}, \tag{15}$$

within the range $(0, l)$. In this condition the only number explicitly involving l is $\lambda=|\cot nl|$, and with this we shall deal presently.

Differentiate the sequence (9) twice with respect to x, then

$$\frac{d^2y_{r+1}}{dx^2}=-n^2y_{r+1}+n^2[y_r-\phi(x, y_r)]$$

i.e.
$$\frac{d^2y_{r+1}}{dx^2}+n^2\phi(x, y_{r+1})=n^2[y_r-y_{r+1}-\phi(x, y_r)+\phi(x, y_{r+1})]$$

$$=n^2(\epsilon_r-\epsilon_{r+1})\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\nu\right], \tag{16}$$

where $(\partial\phi/\partial y)_\nu$ is a value of $\partial\phi/\partial y$ at a value of y intermediate between y_r and y_{r+1}, and is therefore finite. Hence from (14) and (15) a number σ can be found, such that for $r>\sigma$ the right-hand side of (16) can be made less than any given positive number τ, however small. Hence the value of y derived from the sequence (9) can be made to approach the solution of the differential equation

$$\frac{d^2y}{dx^2}+n^2\phi(x, y)=0, \tag{4}$$

to any desired degree of closeness, and satisfy the boundary conditions

$$x=0,\ y=a,\ x=l,\ y=b.$$

It remains to consider how far the convergence depends on the range, as implied by (15).

Consider the differential equation

$$\frac{d^2y}{dx^2}+\chi(x, y)=0. \tag{17}$$

This may be written in the form

$$\frac{d^2y}{dx^2}+n^2\frac{\chi(x, y)}{n^2}=0,$$

where n is arbitrary. Condition (15) then states that the present method of solution will certainly be valid, if M and m being the maximum and minimum values (positive) of $\partial\chi/\partial y$ which can occur in the range, it is

possible to find a real number n such that

$$M < n^2 + \frac{n^2}{1+2p(1+|\cot nl|)}, \tag{18}$$

$$m > n^2 - \frac{n^2}{1+2p(1+|\cot nl|)}, \tag{19}$$

p being some unknown positive interger.

In practice, none of the numbers in (18) and (19), except l, are known exactly; but M and m can usually be estimated roughly, and a suitable value of n selected intermediate between $\sqrt{M}$ and $\sqrt{m}$. Moreover, the range of variation of $\partial\chi/\partial y$ permissible may be in point of fact much greater than that specified by (18) and (19), which are merely conditions sufficient to ensure convergence. For reasons apparent however from these conditions, care must be taken to avoid choosing n in the immediate neighbourhood of $s\pi/l$, where s is an integer. If it is convenient to choose $n = (2s+1)\pi/2l$, the sequence solution for (4), convergent over the whole range takes the simpler form

$$\begin{aligned} y_{r+1} = (-1)^s b \sin(2s+1)\frac{\pi x}{2l} + a\cos(2s+1)\frac{\pi x}{2l} \\ -(2s+1)\frac{\pi}{2l}\Big[\sin\frac{(2s+1)\pi x}{2l}\int_x^l dt\,[y_r - \phi(t, y_r)]\cos\frac{(2s+1)\pi t}{2l} \\ +\cos(2s+1)\frac{\pi x}{2l}\int_0^x dt\,[y_r - \phi(t, y_r)]\sin\frac{(2s+1)\pi t}{2l}\Big], \end{aligned} \tag{20}$$

and (4) becomes

$$\frac{d^2y}{dx^2} + \frac{(2s+1)^2\pi^2}{4l^2}\phi(x, y) = 0. \tag{21}$$

For practical convenience in calculation it is better to use the sequence (9) in the form

$$\begin{aligned} y_{r+1} = A_r \sin nx + a\cos nx + n\sin nx\int_0^x dt\,[y_r - \phi(t, y_r)]\cos nt \\ -n\cos nx\int_0^x dt\,[y_r - \phi(t, y_r)]\sin nt, \end{aligned}$$

where

$$\begin{aligned} A_r = \frac{b - a\cos nl}{\sin nl} - n\int_0^l dt\,[y_r - \phi(t, y_r)]\cos nt \\ +n\cot nl\int_0^l dt\,[y_r - \phi(t, y_r)]\sin nt. \end{aligned}$$

The two integrals in the expression for y_{r+1} can be evaluated graphically or arithmetically for any approximate solution y_r, and in doing so, the

evaluation of the two integrals in the expression for A_r is incidentally accomplished.

It may be remarked that the same method may be applied, subject to corresponding restrictions, to equations of the type

$$\frac{d^2y}{dx^2}+\phi\left(x, y, \frac{dy}{dx}\right)=0,$$

and even to equations of higher order.

In the simpler case, where the boundary conditions refer to one point only, the process here explained may be applied with equal success.

Let the differential equation be

$$\frac{d^2y}{dx^2}+n^2\phi(x, y)=0, \tag{4}$$

subject to the conditions

$$x=0, \quad y=A, \quad \frac{dy}{dx}=B.$$

Consider the sequence

$$y_{r+1}=A\cos nx+B\sin nx+n\sin nx\int_0^x\cos nt[y_r-\phi(t, y_r)]\,dt$$

$$-n\cos nx\int_0^x\sin nt[y_r-\phi(t, y_r)\,dt, \tag{22}$$

A and B being now given constants.

We note that when

$$x=0, \quad y_{r+1}=A, \quad \text{and} \quad \frac{dy_{r+1}}{dx}=B.$$

As before, defining y by equation (22), when y has been substituted for y_{r+1} and y_r, we have

$$\epsilon_{r+1}=n\sin nx\int_0^x\epsilon_r\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\mu\right]\cos nt\,dt$$

$$-n\cos nx\int_0^x\epsilon_r\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\mu\right]\sin nt\,dt$$

where $(\partial\phi/\partial y)_\mu$ is the value of $\partial\phi/\partial y$ at some value of y intermediate between y and y_r for each value of t.

Thus $$\epsilon_{r+1}=n\int_0^x J_r(t)\sin n(x-t)\,dt,$$

where $$J_r(x)=\epsilon_r\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\mu\right],$$

since $\epsilon_{r+1}=0$ when $x=0$, $J_r(0)=0$.

Let the successive maxima and minima of $J_r(x)$ occur at $x_1, x_2, \ldots, x_p$; $J_r(x)$ is monotonic in the ranges $(0, x_1), (x_1, x_2), \ldots, (x_p, x)$.

Therefore $$\epsilon_{r+1} = n[I_1+I_2+\ldots+I_p+I_x],$$

where

$$nI_1 = n\int_0^{x_1} J_r(t)\sin n(x-t)\,dt$$

$$= nJ_r(0)\int_0^{\xi_1}\sin n(x-t)\,dt+nJ_r(x_1)\int_{\xi_1}^{x_1}\sin n(x-t)\,dt$$

$$= J_r(x_1)[\cos n(x-x_1)-\cos n(x-\xi_1)]$$

$$nI_2 = J_r(x_1)[\cos n(x-\xi_2)-\cos n(x-x_1)]$$
$$+J_r(x_2)[\cos n(x-x_2)-\cos n(x-\xi_2)]$$

$$nI_3 = J_r(x_2)[\cos n(x-\xi_3)-\cos n(x-x_2)]$$
$$+J_r(x_3)[\cos n(x-x_3)-\cos n(x-\xi_3)]$$

... ...

$$nI_p = J_r(x_{p-1})[\cos n(x-\xi_p)-\cos n(x-x_{p-1})]$$
$$+J_r(x_p)[\cos n(x-x_p)-\cos n(x-\xi_p)]$$

$$nI_x = J_r(x_p)[\cos n(x-\xi_{p+1})-\cos n(x-x_p)]+J_r(x)[1-\cos n(x-\xi_{p+1})].$$

Hence if M be the maximum absolute value of $J_r(x)$ which can occur in the range $(0, x)$

$$|\epsilon_{r+1}| < 2M(p+1).$$

Now since $$J_r(x) = \epsilon_r\left[1-\left(\frac{\partial\phi}{\partial y}\right)_\mu\right],$$

it follows as before that

$$\epsilon_{r+1} < 2(p+1)\epsilon_r'\left|1-\left(\frac{\partial\phi}{\partial y}\right)\right|_{\max}.$$

Hence

$$\epsilon_r' < 2(p+1)\epsilon_{r-1}'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}$$

... ...

$$\epsilon_2' < 2(p+1)\epsilon_1'\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}.$$

Thus $$\epsilon_{r+1}' < 2^r(p+1)^r\left|1-\frac{\partial\phi}{\partial y}\right|_{\max}^r\epsilon_1', \qquad (23)$$

where ϵ_r' is the maximum absolute value of ϵ_r that occurs in the range $(0, x)$.

Hence provided $$2(p+1)\left|1-\frac{\partial\phi}{\partial y}\right|_{\max} < 1, \tag{24}$$

there exists a quantity ρ such that for $r > \rho$, ϵ_{r+1}' can be made less than any assigned positive number, however small, no matter what (finite) value ϵ_1' may have.

It follows that, subject to the restriction (24), the sequence (22) converges. It converges, moreover, to the solution of equation (4), for from (22)

$$\frac{d^2y_{r+1}}{dx^2} = -n^2y_{r+1}+n^2y_r-n^2\phi(x, y_r).$$

Thus

$$\begin{aligned}\frac{d^2y_{r+1}}{dx^2}+n^2\phi(x, y_{r+1}) &= n^2[\phi(x, y_{r+1})-\phi(x, y_r)-y_{r+1}+y_r] \\ &= n^2(\epsilon_{r+1}-\epsilon_r)\left[\left(\frac{\partial\phi}{\partial y}\right)_\nu-1\right],\end{aligned} \tag{25}$$

where $(\partial\phi/\partial y)_\nu$ is the value of $\partial\phi/\partial y$ at some value of y intermediate between y_{r+1} and y_r for each value of x.

From (24), $\partial\phi/\partial y-1$ always remains finite, and from (23) $\epsilon_{r+1}-\epsilon_r$ can be diminished without limit by suitably increasing r. Hence y_{r+1}, as derived from the sequence (22), can be made to approach as closely as we require to the solution of

$$\frac{d^2y}{dx^2}+n^2\phi(x, y) = 0, \tag{4}$$

$$x = 0, \quad y = A, \quad \frac{dy}{dx} = B.$$

There is no restriction on the range of validity of this sequence other than that implied by the inequality (24). This is equivalent to the requirement that $\partial\phi/\partial y$ shall be positive, and

$$1-\frac{1}{2(p+1)} < \frac{\partial\phi}{\partial y} < 1+\frac{1}{2(p+1)},$$

$$p \geqslant 0.$$

This condition, it should be noted, is independent of the number n^2.

If the equation for solution is

$$\frac{d^2y}{dx^2}+\chi(x, y) = 0,$$

the condition merely requires that a quantity n^2 shall be chosen, such that

$$\left.\begin{aligned} M &< n^2+\frac{n^2}{2(p+1)} \\ m &> n^2-\frac{n^2}{2(p+1)} \end{aligned}\right\} \tag{26}$$

where M and m are the maximum and minimum values of $\partial\chi/\partial y$ in the range over which the solution is required. M and m can usually be estimated roughly at the commencement of the operation and a value of n^2 chosen midway between them. Should it appear, after the integration has been effected over a portion of the range required, that the values of y are such as to cause a violation of the conditions (26), an adjusted value of n may be adopted from that point onwards, and by this process the integration may be effected over as wide a range as desired, provided only that $\partial\chi/\partial y$ remains positive.

A similar process may, of course, be adopted for the solution of the equation

$$\frac{d^2y}{dx^2}+\phi\left(x,\, y,\, \frac{dy}{dx}\right)=0,$$

and, indeed, the whole method is capable of simple extension to differential equations of higher order.

The following numerical example will provide a fair indication of the rapidity of convergence of the sequence and the amount of labour involved in arriving at a solution.

$$\frac{d^2y}{dx^2}+(0{\cdot}98+xe^{-10x})y=0,$$

$$x=0,\quad y=0\,;\qquad x=\pi/2,\quad y=1.$$

Taking $n=1$ the sequence solution is

$$y_{r+1}=A_r\sin x+\sin x\int_0^x y_r(0{\cdot}02-te^{-10t})\cos t\,dt$$

$$-\cos x\int_0^x y_r(0{\cdot}02-te^{-10t})\sin t\,dt,$$

$$A_r=1-\int_0^{\pi/2} y_r(0{\cdot}02-te^{-10t})\cos t\,dt.$$

The first approximation is taken as $y_1=\sin x$, tabulated below to two places of decimals only. The two integrals in y_{r+1} were evaluated graphically by means of a planimeter, A_r being determined incidentally

in the course of evaluating the first integral. It is to be noted that the second and third approximations y_2 and y_3 agree to four places, and therefore represent the solution to the degree of accuracy obtainable by means of a planimeter.

x	y_1	$(\cdot02-xe^{-10x})\times y_1\cos x$	$(\cdot02-xe^{-10x})\times y_1\sin x$	$\int_1$	$\int_2$	$\sin x\times\int_1$	$-\cos x\times\int_2$	$A_1\sin x$	y_2	Col. 3 corrected	Col. 4 corrected	y_3
0·0	0·00	·0000	0	0	0	0	0	0	0	0	0	0
0·1	0·10	−·0017	−·0002	-6×10^{-5}	-1×10^{-5}	-1×10^{-5}	$+1\times10^{-5}$	·0991	·0991	−·0017	−·0002	·0991
0·2	0·20	−·0015	−·0003	−25	−4	−5	+4	·1971	·1971	−·0015	−·0003	·1971
0·3	0·30	+·0015	+·0004	−26	−4	−8	+4	·2931	·2931	+·0014	−·0004	·2931
0·4	0·39	+·0046	+·0019	+4	+7	+2	−6	·3862	·3862	+·0045	+·0019	·3862
0·5	0·48	+·0070	+·0038	+62	+36	+30	−32	·4755	·4755	+·0070	+·0038	·4755
0·6	0·56	+·0086	+·0058	+130	+85	+73	−70	·5601	·5601	+·0086	+·0059	·5601
0·7	0·64	+·0095	+·0080	+223	+155	+144	−119	·6390	·6392	+·0036	+·0081	·6392
0·8	0·72	+·0100	+·0102	+322	+247	+231	−172	·7116	·7122	+·0099	+·0102	·7122
0·9	0·78	+·0096	+·0122	+421	+361	+330	−224	·7770	·7781	+·0097	+·0122	·7781
1·0	0·84	+·0091	+·0141	+515	+491	+433	−265	·8347	·8364	+·0091	+·0141	·8364
1·1	0·89	+·0081	+·0158	+607	+641	+541	−291	·8840	·8865	+·0081	+·0159	·8865
1·2	0·93	+·0067	+·0173	+682	+807	+636	−292	·9245	·9279	+·0067	+·0173	·9279
1·3	0·96	+·0052	+·0185	+742	+987	+715	−264	·9558	·9633	+·0052	+·0186	·9603
1·4	0·99	+·0034	+·0195	+785	+1177	+774	−200	·9775	·9832	+·0033	+·0194	·9832
1·5	1·00	+·0014	+·0200	+809	+1375	+807	−97	·9894	·9965	+·0014	+·0200	·9965
π/2	1·00	+·0000	+·0200	+814	+1575	+814	0	·9919	1·00	+·0000	+·0200	1·00

THE SUMMATION OF CERTAIN KAPTEYN SERIES OF BESSEL FUNCTIONS, WITH AN APPLICATION TO ELECTROMAGNETIC RADIATION

By R. F. BUDDEN.

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1. The problem of the radiation emitted by a number of electrons moving with uniform speed in the same circular orbit is of considerable importance in current theories of the atom, and this importance is not diminished by the fact that, in some theories, the radiation is neglected. Sir J. J. Thomson has given an approximate solution which is valid when the speed of the electrons is small compared with that of light. Schott has given an exact formula in terms of Kapteyn Series of Bessel Functions, but it does not readily lend itself to numerical computation.

In this paper we have summed the Kapteyn series occurring in Schott's result in terms of a finite number of subsidiary variables. We have also discussed another type of Kapteyn series which, although it does not occur in this particular result, yields to similar methods.

2. *The series* $\sum_{r=1}^{\infty} J_{nr}(nr\beta)$,

where n is an integer which may be taken as positive.

By a known theorem*

$$\frac{1}{1-\beta\cos\phi} = 1+2\sum_{s=1}^{\infty} J_s(s\beta)\cos s(\phi-\beta\sin\phi), \tag{1}$$

where $-1<\beta<1$.

* See *e.g.* Watson, *Theory of Bessel Functions*, 555, equation 6.

Let us give ϕ the value ϕ_p, given by the equation

$$\phi_p - \beta \sin \phi_p = 2p\pi/n, \tag{2}$$

where p is a positive integer not greater than n. It can easily be shown that, if $|\beta| < 1$, equation (2) defines one and only one real value of ϕ_p. From (1) and (2) we have

$$\frac{1}{(1-\beta \cos \phi_p)} = 1 + 2 \sum_{s=1}^{\infty} J_s(s\beta) \cos 2sp\pi/n.$$

If we sum this equation for p from $p = 0$ to $p = n-1$, we find that the coefficient of $J_s(s\beta)$ vanishes unless s is a multiple of n; if s is a multiple of n, the coefficient is equal to n.

Writing $s = rn$ we have

$$n + 2n \sum_{r=1}^{\infty} J_{rn}(rn\beta) = \sum_{p=0}^{n-1} (1-\beta \cos \phi_p)^{-1}. \tag{3}$$

The range of summation for p can be halved. For we have

$$\phi_p + \phi_{n-p} - \beta(\sin \phi_p + \sin \phi_{n-p}) = 2\pi.$$

This equation is satisfied if $\phi_p + \phi_{n-p} = 2\pi$; and since ϕ_p and ϕ_{n-p} are uniquely determined, this relation must hold between them, and therefore $\cos \phi_{n-p} = \cos \phi_p$.

Taking out of the summation the terms $p = 0$ and $p = \frac{1}{2}n$ (when n is even), for which the values of ϕ_p are 0 and π, we have

$$n + 2n \sum_{r=1}^{\infty} J_{rn}(rn\beta) = 2 \sum_{p=1}^{<\frac{1}{2}n} \frac{1}{1-\beta \cos \phi_p} + \frac{1}{1-\beta} \left[+ \frac{1}{1+\beta} \right], \tag{4}$$

the term in square brackets appearing only when n is even.

3. *The series* $\sum_{r=1}^{\infty} r^{\lambda} J_{nr}(nr\beta)$,

where n and λ are positive integers.

If we differentiate (1) with respect to ϕ, treating β as constant, we get

$$2 \sum_{s=1}^{\infty} s J_s(s\beta) . \cos \{ s(\phi - \beta \sin \phi) + \tfrac{1}{2}\pi \} = \frac{1}{1-\beta \cos \phi} \cdot \frac{\partial}{\partial \phi} \left(\frac{1}{1-\beta \cos \phi} \right).$$

Whence, by repeated differentiation, we see that

$$2 \sum_{s=1}^{\infty} s^{\lambda} J_s(s\beta) . \cos \{ s(\phi - \beta \sin \phi) + \tfrac{1}{2}\lambda\pi \} = \left[\frac{1}{1-\beta \cos \phi} \cdot \frac{\partial}{\partial \phi} \right]^{\lambda} \frac{1}{1-\beta \cos \phi}. \tag{5}$$

The series on the left is uniformly convergent, and the procedure is justified.

If λ is even, we use (2) and sum for p as before. Writing $r = sn$ we get

$$2n^{\lambda+1} \sum_{r=1}^{\infty} r^{\lambda} J_{rn}(rn\beta) \cdot \cos \tfrac{1}{2}\lambda\pi = \sum_{p=0}^{n-1} \left[\left\{ \frac{1}{1-\beta\cos\phi} \cdot \frac{\partial}{\partial\phi} \right\}^{\lambda} \frac{1}{1-\beta\cos\phi} \right]_{\phi=\phi_p}, \quad (6)$$

and the summation for p can be halved, as before.

If λ is odd, we must define ϕ_p by the equation

$$\phi_p - \beta \sin \phi_p + \tfrac{1}{2}\pi = 2p\pi/n \quad (7)$$

instead of by (2), and proceed in the same way. The summation for p cannot be halved in this case.

4. *The series* $\sum_{r=1}^{\infty} r^{2\lambda} \{J_r(r\beta)\}^2$,

where λ *is a positive integer (or zero).*

Integrate (1) with respect to ϕ from 0 to π. We get

$$\pi + 2\pi \sum_{s=1}^{\infty} \{J_s(s\beta)\}^2 = \pi/(1-\beta^2)^{\frac{1}{2}},$$

or

$$\sum_{s=1}^{\infty} \{J_s(s\beta)\}^2 = \tfrac{1}{2}(1-\beta^2)^{-\frac{1}{2}} - \tfrac{1}{2}.* \quad (8)$$

In (5), let λ be even, and write 2λ in place of λ. Then by the same process

$$2(-)^{\lambda}\pi \sum_{s=1}^{\infty} s^{2\lambda} \{J_s(s\beta)\}^2 = \int_0^{\pi} \left[\frac{1}{1-\beta\cos\phi} \cdot \frac{\partial}{\partial\phi} \right]^{2\lambda} \frac{1}{1-\beta\cos\phi} \cdot d\phi. \quad (9)$$

Alternatively, let λ be odd, and write $2\lambda+1$ in place of λ. (5) is then

$$2(-)^{\lambda} \sum_{s=1}^{\infty} s^{2\lambda+1} J_s(s\beta) \cdot \sin s(\phi - \beta\sin\phi) = \left[\frac{1}{1-\beta\cos\phi} \cdot \frac{\partial}{\partial\phi} \right]^{2\lambda+1} \frac{1}{1-\beta\cos\phi}.$$

Multiply by $\sin\phi$, and integrate with respect to ϕ from 0 to π. The left-hand side becomes

$$2(-)^{\lambda}\pi \sum_{s=1}^{\infty} s^{2\lambda+1} J_s(s\beta) \cdot J_s'(s\beta).$$

* Obtained by Schott, *Electromagnetic Radiation* (1912), 122.

Integrating with respect to β we then get

$$2(-)^{\lambda}\pi \sum_{s=1}^{\infty} s^{2\lambda} \{J_s(s\beta)\}^2$$
$$= 2\int_0^{\beta} d\beta \int_0^{\pi} \left[\left\{ \frac{1}{1-\beta\cos\phi} \cdot \frac{\partial}{\partial\phi} \right\}^{2\lambda+1} \frac{1}{1-\beta\cos\phi} \right] . \sin\phi \, . \, d\phi. \quad (10)$$

The right-hand sides of (9) and (10) must therefore be identical.

The evaluation of these integrals for any given positive value of λ will be considerably simplified by the transformation

$$1-\beta\cos\phi = (1-\beta^2)/(1+\beta\cos\psi).$$

The right-hand side of (9) becomes

$$\frac{1}{(1-\beta^2)^{3\lambda+\frac{1}{2}}} \int_0^{\pi} \left[\left\{ (1+\beta\cos\psi)^2 \frac{\partial}{\partial\psi} \right\}^{2\lambda} (1+\beta\cos\psi) \right] . \frac{d\psi}{1+\beta\cos\psi}. \quad (11)$$

5. *The series*

$$\sum_{r=1}^{\infty} r^{\lambda} J_{nr}(nr\beta) \quad \text{and} \quad \sum_{r=1}^{\infty} r^{2\lambda} \{J_r(r\beta)\}^2, \quad (12)$$

where λ *is a negative integer.*

We proceed successively as follows :—

(*a*) Multiplying (1) by $1-\beta\cos\phi$ and integrating with respect to ϕ, we have

$$2\sum_{s=1}^{\infty} s^{-1} J_s(s\beta) . \sin s(\phi-\beta\sin\phi) = \beta\sin\phi, \quad (13)$$

the constant vanishing on putting $\phi = 0$; whence we can evaluate $\Sigma s^{-1} J_{ns}(ns\beta)$ by using (7) and summing for p.

(*b*) Multiply (13) by $\sin\phi$, and integrate with respect to ϕ from 0 to π.

Then $$2\pi \sum_{s=1}^{\infty} s^{-1} J_s(s\beta) . J_s'(s\beta) = \tfrac{1}{2}\pi\beta.$$

Integrating with respect to β,

$$\sum_{s=1}^{\infty} s^{-2} \{J_s(s\beta)\}^2 = \tfrac{1}{4}\beta^2,* \quad (14)$$

the constant vanishing on putting $\beta = 0$.

* Obtained by Nielsen, *Cylinderfunktionen* (1904), 307, equation 7, with $\nu = \rho = 1$.

(*c*) Multiply (13) by $1-\beta\cos\phi$, and integrate with respect to ϕ. Then

$$-2\sum_{s=1}^{\infty} s^{-2}J_s(s\beta).\cos s(\phi-\beta\sin\phi) = -\beta\cos\phi+\frac{\beta\cos 2\phi}{4}+A. \quad (15)$$

To determine A, integrate with respect to ϕ from 0 to π and use (14). This gives $A = -\frac{1}{2}\beta^2$. From (15) we can now obtain $\Sigma s^{-2}J_{ns}(ns\beta)$ by using (2) and summing for p. This threefold process may be repeated indefinitely.

6. *Radiation from a Ring of Electrons.*

If we have n electrons, equally spaced round a circle of radius ρ, moving round the circle with constant speed, the radiation of energy per unit time is given by*

$$R = 2ce^2n^2\beta\rho^{-2}\sum_{r=1}^{\infty}\left[nr\beta^2J'_{2nr}(2nr\beta)-n^2r^2(1-\beta^2)\int_0^{\beta}J_{2nr}(2nr\beta)\,d\beta\right], \quad (16)$$

where c is the speed of light, e the charge on an electron, and β the ratio of the speed of the electrons to that of light.

There are two sums involved. For the first we use (4). On putting $n = 2n$ in that equation, we get

$$2n+4n\sum_{r=1}^{\infty}J_{2nr}(2nr\beta) = 2\sum_{p=1}^{n-1}(1-\beta\cos\phi_p)^{-1}+2(1-\beta^2)^{-1},$$

the term in brackets being included.

Differentiate with respect to β, remembering that ϕ_p is a function of β in virtue of (2). Then we get

$$4n^2\sum_{r=1}^{\infty}rJ'_{2nr}(2nr\beta) = \sum_{p=1}^{n-1}\left[\frac{\cos\phi-\beta}{(1-\beta\cos\phi)^3}\right]_{\phi=\phi_p}+\frac{2\beta}{(1-\beta^2)^2}. \quad (17)$$

For the second sum we use (6), with $\lambda = 2$. Writing $2n$ for n, and carrying out the differentiation on the right-hand side (in which β is to be taken as constant), we get

$$-16n^3\sum_{r=1}^{\infty}r^2J_{2nr}(2nr\beta) = \sum_{p=0}^{n-1}\left[-\frac{\beta\cos\phi}{(1-\beta\cos\phi)^4}+\frac{3\beta^2\sin^2\phi}{(1-\beta\cos\phi)^5}\right]_{\phi=\phi_p}.$$

We take out the terms $p = 0$ and $p = n$, and halve the summation

* Schott, *Electromagnetic Radiation*, 110.

for p. To obtain the sum required, we have to integrate with respect to β. We now have

$$16n^3 \sum_{r=1}^{\infty} r^2 \int_0^{\beta} J_{2nr}(2nr\beta)\, d\beta$$

$$= \frac{3\beta-1}{6(1-\beta)^3} + \frac{3\beta+1}{6(1+\beta)^3} + 2\sum_{p=1}^{n-1} \int_0^{\beta} \left[\frac{\beta\cos\phi}{(1-\beta\cos\phi)^4} - \frac{3\beta^2\sin^2\phi}{(1-\beta\cos\phi)^5}\right]_{\phi=\phi_p} d\beta.$$

To evaluate the right-hand side, change the variable from β to ϕ_p by means of (2) (with $2n$ in place of n). The integral becomes

$$\int_{p\pi/n}^{\phi} \left[\frac{\beta\cos\phi}{(1-\beta\cos\phi)^3\sin\phi} - \frac{3\beta^2\sin\phi}{(1-\beta\cos\phi)^4}\right] d\phi. \tag{18}$$

Now we have

$$\frac{d}{d\phi}\left[\sin\phi(1-\beta\cos\phi)\right] = \beta\sin^2\phi.$$

We use this to integrate by parts, starting with the second term, and always making the denominator a power of $\sin\phi\,(1-\beta\cos\phi)$ by multiplying numerator and denominator by a suitable power of $\sin\phi$. We note also that $\phi = p\pi/n$ means that $\beta = 0$. The final result is

$$8n^3 \sum_{r=1}^{\infty} r^2 \int_0^{\beta} J_{2rn}(2rn\beta)\, d\beta$$

$$= \sum_{p=1}^{n-1} \left[\frac{\beta}{(1-\beta\cos\phi)^3} + \frac{\cos\phi}{(1-\beta\cos\phi)^2\sin^2\phi}\right]_{\phi=\phi_p} + \frac{4\beta^3}{3(1-\beta^2)^3}, \tag{19}$$

excluding a term from the integral (18), which vanishes on summation.

Substituting from (17) and (19) into (16), and simplifying, we get

$$R = \frac{2nce^2\beta^4}{3\rho^2(1-\beta^2)^2}$$

$$+ \frac{ce^2n\beta}{4\rho^2} \sum_{p=1}^{n-1} \left[\frac{-(\beta+\beta^3)+(3\beta^2-1)\cos\phi+2\beta\cos^2\phi-2\beta^3\cos^3\phi}{(1-\beta\cos\phi)^3\sin^2\phi}\right]_{\phi=\phi_p}, \tag{20}$$

where $$\phi_p - \beta\sin\phi_p = p\pi/n.$$

The first term is n times the radiation from a single charge.* It gives the sum of the radiations from each charge taken separately. The remainder of the expression represents the effect due to interference.

This result may be compared with an approximate formula for small values of β obtained by J. J. Thomson.† Thomson was working in

* Schott, *loc. cit.*, 110.

† *Phil. Mag.* (6), 6 (1903), 681.

electromagnetic units, so that a factor c^2 must be omitted. Also, owing to a misprint, his result should be doubled. With these alterations, his result may be reduced to

$$R = \frac{2ce^2}{\rho^2} \cdot n^{2n+3} \cdot \beta^{2n+2} \cdot \frac{n+1}{(2n+1)!}. \tag{21}$$

The result in (20) can be shown to agree with this, for the case $n = 2$, on expanding in powers of β and picking out the lowest power; for higher values of n the algebra becomes too heavy.

If we expand the expression (16) for R and take the lowest power of β, we find without difficulty that it agrees with (21).

7. In computing R from (20), the first thing to do is to solve the equation for ϕ_p for various values of p, n, and β. The best method of procedure then is to put $\beta \cos \phi_p = z$. The expression under the sign of summation becomes

$$\frac{-\beta[\beta^2+\beta^4+z(1-3\beta^2-2z+2z^3)]}{(1-z)^3(\beta^2-z^2)}.$$

The following is a table of the results for $n = 1, 2, 3, 4$, at intervals of 0·1 for β. The value of e has been taken as $4{\cdot}77 \times 10^{-10}$ E.S.U., and c as $2{\cdot}9986 \times 10^{10}$ cm. per sec. The values for $n = 1$ are taken from the formula already referred to, obtained by Liénard, Schott, and others.

TABLE of $N = R\rho^2$ = Radiation in ergs per sec × (radius of orbit in cms.)^2.

β	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0·1	$4{\cdot}64 \times 10^{-14}$	*$4{\cdot}32 \times 10^{-15}$*	*$1{\cdot}68 \times 10^{-16}$*	
0·2	$7{\cdot}90 \times 10^{-13}$	$2{\cdot}71 \times 10^{-13}$	*$4{\cdot}65 \times 10^{-14}$*	*$7{\cdot}77 \times 10^{-15}$*
0·3	$4{\cdot}45 \times 10^{-12}$	$3{\cdot}02 \times 10^{-12}$	$1{\cdot}23 \times 10^{-12}$	$3{\cdot}84 \times 10^{-13}$
0·4	$1{\cdot}65 \times 10^{-11}$	$1{\cdot}70 \times 10^{-11}$	$1{\cdot}13 \times 10^{-11}$	$5{\cdot}98 \times 10^{-12}$
0·5	$5{\cdot}05 \times 10^{-11}$	$6{\cdot}78 \times 10^{-11}$	$6{\cdot}26 \times 10^{-11}$	$5{\cdot}00 \times 10^{-11}$
0·6	$1{\cdot}44 \times 10^{-10}$	$2{\cdot}29 \times 10^{-10}$	$2{\cdot}61 \times 10^{-10}$	$2{\cdot}53 \times 10^{-10}$
0·7	$4{\cdot}20 \times 10^{-10}$	$7{\cdot}48 \times 10^{-10}$	$9{\cdot}74 \times 10^{-10}$	$1{\cdot}11 \times 10^{-9}$
0·8	$1{\cdot}44 \times 10^{-9}$	$2{\cdot}72 \times 10^{-9}$	$3{\cdot}88 \times 10^{-9}$	$4{\cdot}85 \times 10^{-9}$
0·9	$8{\cdot}27 \times 10^{-9}$	$1{\cdot}63 \times 10^{-8}$	$2{\cdot}42 \times 10^{-8}$	$3{\cdot}17 \times 10^{-8}$

The figures in italics are given for the sake of comparison with Thomson's formula, but are not trustworthy in view of the large possible error. This applies also to the table on p. 478.

The error in the last three columns is not greater than

$$4\beta n(n-1) \times 10^{-15}.$$

This must not be taken as giving the exact way in which the maximum error depends on β and n, but only as an upper limit. It will be apparent that our result is not suitable for computation when β is small and n is large; in such cases it is better to fall back on Thomson's formula (21).

The radiation for small values of β falls as n increases, but for large values of β it increases with n. For $\beta = 0{\cdot}4$ and $0{\cdot}5$, the radiation is largest for $n = 2$; for $\beta = 0{\cdot}6$, it is largest when $n = 3$. As a matter of fact, the radiation *per electron* falls in all cases as n increases; but the proportional decrease is small for large values of β, and the presence of the factor n in the final result causes the total to increase. The maximum observed for $\beta = 0{\cdot}4$, $0{\cdot}5$, and $0{\cdot}6$ occurs in all probability for the higher values of β, and would show itself if our table extended beyond $n = 4$. It is natural to expect this, since if n were infinite we should have the case of a circular electric current, in which there is no radiation at all.

J. J. Thomson gives also another table, giving the ratio of the total radiation from a ring to that from a single charge.* The following table gives the ratio of the various values of N to the value in the case $n = 1$, $\beta = 0{\cdot}1$; *i.e.* the radiation from a single charge moving with velocity one-tenth that of light is taken as unity.

TABLE showing comparative values of N.

β	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0·1	1	*0·093*	*0·004*	
0·2	17	5·84	*1·00*	*0·17*
0·3	96	65·1	26·5	8·3
0·4	356	366	243·5	129
0·5	1090	1460	1350	1080
0·6	3100	4935	5625	5450
0·7	9050	16100	21000	24000
0·8	31000	58600	83600	105000
0·9	178000	351000	521000	683000

For $n = 2$, $\beta = 0{\cdot}1$, Thomson gives 0·096; for $n = 3$, $\beta = 0{\cdot}1$, he gives 0·0046.

* In the article referred to, the table is stated to give the radiation *per electron* from the ring. This appears to be in error, as the table is calculated direct from the formula giving the *total* radiation of the ring.

A CLASS OF NULL SERIES

By C. Fox.

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1. Various Schlömilch series, representing null functions, have been discovered by Nielsen,* who gave proofs of them by arguments similar to those by which the Riemann-Lebesgue lemma is proved. The methods of contour integration, however, give extremely simple proofs of these results, and also give rise to many interesting results which, I believe, are new.

Consider the integral

$$\int_c \frac{\phi(x, t)}{J_\nu(t)} t^\nu dt$$

taken round a contour C, consisting of the imaginary axis and a large semicircle of radius R on the right of it, the origin being the centre. Denote the n-th zero of $J_\nu(t)$ by j_n. The radius R of the contour is to be made to tend to infinity through such values that the whole of the contour remains at a distance exceeding some positive number δ from any pole of the integrand. If the origin is a pole an indentation is made in the imaginary axis to exclude it.

Suppose, now, that for a certain range of values of x, $\phi(x, t)$ satisfies the following conditions :—

(i) $\phi(x, t)$ is regular in that part of the t plane for which $R(t) \geqslant 0$, except possibly at the origin, where it may have a simple pole.

(ii) $\int \frac{\phi(x, t)}{J_\nu(t)} t^\nu dt$ taken round the large semi-circle of C vanishes in the limit as $R \to \infty$ in the manner above mentioned.

(iii) $\phi(x, -t) = -\phi(x, t)$.

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* *Math. Annalen*, 52 (1899), 582–587. *Annali di Mat.* (3), 6 (1901), 301–329. For full references see Watson, *Theory of Bessel Functions*, 636.

It follows then, from the last condition, that the values of the integral along the positive and negative portions of the imaginary axis will cancel each other. From condition (ii) the only contribution to the value of the integral will be made by the portion of the integral over the indentation which excludes the origin. Calling the radius of this indentation ρ, it follows from (i) that, as $\rho \to 0$, this portion of the integral is of value $-\pi i a$, where a is the coefficient of t^{-1} in $2^\nu \Gamma(\nu+1)\ \phi(x, t)$.

The theory of residues then asserts that

$$\tfrac{1}{2}a + \sum_{n=1}^{\infty} \frac{\phi(x, j_n)}{J'_\nu(j_n)} j_n^\nu \qquad (1.1)$$

will converge for this range of values of x, and that its value in this range is zero.

2. Before considering special types of functional forms for $\phi(x, t)$ it will be useful to see what must be expected of it in order that the conditions laid down should be satisfied.

We have

$$J_\nu(t) = \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \left[\cos(t - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\{1 + O(t^{-2})\}\right.$$
$$\left. - \frac{4\nu^2 - 1}{8t} \sin(t - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\{1 + O(t^{-2})\}\right], \qquad (2.1)$$

from which, on putting $t = Re^{i\theta}$, it follows immediately that when R is sufficiently large

$$\left|\frac{t^\nu}{J_\nu(t)}\right| \leqslant A \,.\, R^{\nu+\frac{1}{2}} \exp(-R|\sin\theta|)$$

on the circular part of C, A being independent of R and θ. With the help of Jordan's lemma* it is now easily seen that *sufficient* conditions in order that the integral

$$\int \frac{\phi(x, t) \,.\, t^\nu}{J_\nu(t)} dt$$

should satisfy (ii) of § 1 are as follows :—

$$|\phi(x, t)| < C \,.\, R^p \exp(\lambda R|\sin\theta|), \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi) \quad (1)$$

* Whittaker and Watson, *Modern Analysis* (1914), 115.

where C is a constant independent of θ and λ is a function of x, together with either

$$\lambda < 1, \quad p+\nu+\tfrac{1}{2} < 0, \tag{2}$$

or

$$\lambda \leqslant 1, \quad p+\nu+\tfrac{1}{2} < -1. \tag{3}$$

3. Suppose now that x lies in the range which makes $\lambda < 1$ but that $p+\nu+\frac{1}{2}$ $(= q)$ is not negative. The series (1 . 1) will then diverge; it will, however, be summable, as I now proceed to show.

Consider the integral

$$\int_{C_1} \frac{\phi(x, t)}{J_\nu(t)}\, t^\nu \left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} \qquad (\epsilon > 0). \quad (3 . 1)$$

The contour C_1 is the same as C, except that the radius of its large semicircle $R_1 = R-\frac{1}{2}\delta$ (where δ is as defined as § 1). C_1 encloses the same number of poles as C does, and if $R \to \infty$ in the manner described in § 1, the whole of the contour C_1 remains at a distance exceeding the positive value $\frac{1}{2}\delta$ from any pole of the integrand. The object of using the new contour is to avoid the branch point which would have been introduced by the factor $\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon}$ had C been used.

Let $\phi(x, t)$ satisfy the conditions of § 1. Then $\phi(x, t)\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon}$ will satisfy condition (iii) and will be regular inside and on the contour C_1, except possibly at the origin [where it will have a simple pole if $\phi(x, t)$ does] for all values of R.

Also the value of the new integral over the indentation, excluding the origin, will tend to $-\pi i a$ as $\rho \to 0$, where a is defined as in § 1.

It remains now to examine the behaviour of the new integral over the large semicircle of C_1.

Putting $t = Re^{i\theta}$ divide the semicircle into three arcs, the points on the inner arc being defined by the inequality

$$R_1 |\sin\theta| < R_1^{\epsilon'} \qquad (0 < \epsilon' < 1),$$

and those on the outer arcs by

$$R_1 |\sin\theta| \geqslant R_1^{\epsilon'}.$$

On the outer arcs

$$\left| \frac{\phi(x, t).t^\nu}{J_\nu(t)} \left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} \right| \leqslant K.R_1^{p+\nu+\frac{1}{2}} \exp\left\{-(1-\lambda) R_1^{\epsilon}\right\},$$

from which it follows that, as $R \to \infty$, the value of the integral over the two outer arcs tends to zero, since $\lambda < 1$.

Over the middle arc (whose magnitude is of order $R_1^{\epsilon'}$) we have

(i) $$\exp\{-(1-\lambda) R|\sin\theta|\} \leqslant 1.$$

(ii) $$\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} = O(\theta^{q+\epsilon}) = O\{R^{-(q+\epsilon)(1-\epsilon')}\}.$$

Hence the value of the portion of the integral (3 . 1) taken over the middle arc is of order

$$R_1^{\epsilon'} . R^{p+\nu+\frac{1}{2}-(q+\epsilon)(1-\epsilon')},$$

which tends to zero as $R \to \infty$, provided ϵ' is chosen, so that

$$(q+\epsilon)(1-\epsilon')-(p+\nu+\tfrac{1}{2}) > \epsilon',$$

i.e., so that $$\epsilon(1-\epsilon')-q\epsilon' > \epsilon',$$

which is always possible.

From the theory of residues then

$$\frac{a}{2} + \lim_{R\to\infty} \sum_{j_n<R} \frac{\phi(x, j_n)}{J'_\nu(j_n)} j_n^\nu \left(1-\frac{j_n^2}{R^2}\right)^{q+\epsilon}$$

exists and is zero. Denoting j_n^2/R^2n^2 by $1/w^2$, we have

$$\frac{a}{2} + \lim_{w\to\infty} \sum_{n<w} \frac{\phi(x, j_n)}{J'_\nu(j_n)} . j_n^\nu \left(1-\frac{n^2}{w^2}\right)^{q+\epsilon}$$

exists and is zero. In other words, if as $R \to \infty$ through real values

$$\phi(x, R) = O(R^p),$$

then the series $$\frac{a}{2} + \sum_{n=0}^{\infty} \frac{\phi(x, j_n)}{J'_\nu(j_n)} j_n^\nu$$

is summable $(C,\ p+\nu+\frac{1}{2}+\epsilon)$, its sum in this sense being zero for the range of values of x for which λ [defined in (1) § 2] is less than unity.

4. To get simple applications of these results put $\nu = \frac{1}{2}$, when the integral of § 1 becomes

$$\sqrt{\left(\frac{\pi}{2}\right)} \int_c \frac{\phi(x, t)}{\sin t} . t\, dt.$$

In this case it is better to deal with the integral

$$\int \frac{f(x, t)}{\sin \pi t}\, dt.$$

The conditions of § 1 are easily expressed in terms of $f(x, t)$; thus, in condition (i), $f(x, t)$ may have no pole at the origin, whilst condition (iii) becomes

$$f(x, t) = f(x, -t).$$

The limit of the integral round the indentation, excluding the origin, will, in this case, be $-f(x, 0)$, and so we have a set of conditions which, if satisfied, lead to the fact that

$$\tfrac{1}{2}f(x, 0) + \sum_{n=1}^{\infty} (-1)^n f(x, n) \qquad (4.1)$$

converges to zero for those values of x which make λ lie between 0 and π, sometimes including and sometimes excluding the end points.

Since $f(x, t) = t \,.\, \phi(x, t)$ and $\nu = \frac{1}{2}$, sufficient conditions for convergence corresponding to (1), (2), and (3) of § 2 will be

$$|f(x, t)| < C.R^p \exp(\lambda R |\sin\theta|) \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi) \qquad (1)$$

at the point $t = Re^{i\theta}$, where λ is a function of x, together with either

$$\lambda < \pi, \quad p < 0, \qquad (2)$$

or

$$\lambda \leqslant \pi, \quad p < -1. \qquad (3)$$

If $\lambda < \pi$ but p is not negative, then the series (4 . 1) will be summable $(C, p+\epsilon)$, its sum $(C, p+\epsilon)$ being zero.

When $f(x, t) = J_\nu(xt)/(\frac{1}{2}xt)^{\nu-2m}$, where m is a positive integer, we get the well known formulae, due to Nielsen,

$$\frac{\frac{1}{2}}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n J_\nu(nx)}{(\frac{1}{2}nx)^\nu} = 0, \qquad (4.2)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n J_\nu(nx)}{(\frac{1}{2}nx)^{\nu-2m}} = 0. \qquad (4.3)$$

These hold when x is real and such that $-\pi < x < \pi$, except in the case (4 . 2) when the value $x = 0$ must be excluded.

As $R \to \infty$,

$$f(x, R) = O(R^{-\nu+2m-\frac{1}{2}}),$$

and so the series converge, provided $\nu > -\frac{1}{2}$ and $\nu - 2m > -\frac{1}{2}$ respectively, whilst otherwise they are summable $(C, 2m-\nu-\frac{1}{2}+\epsilon)$ to zero. Summability includes, of course, the possibility that $\nu \leqslant -\frac{1}{2}$.

Similar results can be established for Struve's function $\mathbf{H}_\nu(x)$, not only for the equations above, but throughout the paper.

By putting $$f(x, t) = \frac{t^{2m} J_\nu\{x\sqrt{(t^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(t^2+a^2)}\}^\nu},$$

where a is real, we have

$$\frac{\frac{1}{2}}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n J_\nu\{x\sqrt{(n^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(n^2+a^2)}\}^\nu} = 0, \tag{4.4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n . n^{2m} J_\nu\{x\sqrt{(n^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(n^2+a^2)}\}^\nu} = 0. \tag{4.5}$$

These equations are subjected to the same conditions as regards convergence, summability, and range of validity as equations (4 . 2) and (4 . 3), except that the value $x = 0$ is also to be excluded from (4 . 5).

These two null series [(4 . 4) and (4 . 5)] correspond to the discontinuous integrals of Sonine and Gegenbauer* in the same way as Nielsen's formulae (4 . 2) and (4 . 3) correspond to the Weber-Schafheitlin† integral.

The substitution of $x\sqrt{(n^2+a^2)}$ for xn in the way it has been done from equations (4 . 2) and (4 . 3) to (4 . 4) and (4 . 5) leads to correct results for all the null series, constructed to Bessel functions, which will be obtained in the next paragraph.

5. Suppose that $\phi(x, t)$ satisfies all the conditions of § 1, and that in addition when $|t|$ is large it satisfies the inequality (1) of § 2. Then, provided λ is real and less than unity, it has been shown that

$$\frac{a}{2} + \sum_{n=1}^{\infty} \frac{\phi(x, j_n)\, j_n^\nu}{J'_\nu(j_n)}$$

is summable $(C, p+\nu+\frac{1}{2}+\epsilon)$, its sum in this sense being zero. Since, if m is a positive integer other than zero, $t^{2m}\phi(x, t)$ satisfies similar conditions to $\phi(x, t)$, it is easily seen that

$$\sum_{n=1}^{\infty} \frac{\phi(x, j_n)\, j_n^{\nu+2m}}{J'_\nu(j_n)} \tag{5.1}$$

* Watson, *Theory of Bessel Functions*, 415 (§ 13.47).
† *L.c.*, 391 (§ 13.24).

is summable $(C, p+2m+\nu+\frac{1}{2}+\epsilon)$ to zero. If $p+m+\nu+\frac{1}{2}$ is negative the series will, of course, be convergent and its sum will be zero. The range of values of x, for which this is true, is still determined, as before, by the fact that λ must be real and less than unity.

Suppose, again, that $\phi_r(x_r, t)$ satisfies all the conditions of § 1 and on putting $t = Re^{i\theta}$, we have, when R is sufficiently large,

$$| \phi_r(x_r, t) | < C \,.\, R^p \exp \{\lambda_r R | \sin\theta |\},$$

where λ_r is a function of x_r.

From condition (iii), $\phi_r(x_r, t)$ is an odd function of t, and so there is little difficulty in seeing that we can get results of the type

$$\frac{a}{2} + \sum_{n=1}^{\infty} \frac{\Pi_r \phi_r(x_r, j_n)\, j_n^{\nu+m}}{J'_\nu(j_n)} = 0, \qquad (5\,.\,2)$$

where m is an integer odd or even, according as the number of factors of the product $\Pi_r \phi_r(x_r, j_n)$ is even or odd. The range of values of the x_r's for which these results hold, is determined by the facts that λ_r must be real and such that

$$\Sigma | \lambda_r | < 1.$$

The range of m is limited below by the fact that $\Pi_r \phi_r(x_r, t)\, t^m$ may not have more than a simple pole at the origin. If m becomes so large that the series of (5 . 1) diverges, it will still be summable (C, p) for some value of p found by investigating the order of $\Pi_r \phi_r(x_r, t)\, t^m$ as $t \to \infty$ through real values, as in § 3.

As particular instances of these results we have, on putting $\nu = \frac{1}{2}$ and $\phi_r(x_r, t) = J_{\nu_r}(x_r t)/(\frac{1}{2}x_r t)^{\nu_r}$,

$$\tfrac{1}{2} \sum_{r=1}^{q} \frac{1}{\Gamma(\nu_r+1)} + \sum_{n=1}^{\infty} \left\{(-1)^n \prod_{r=1}^{q} \frac{J_{\nu_r}(nx_r)}{(\frac{1}{2}nx_r)^{\nu_r}}\right\} = 0, \qquad (5\,.\,3)$$

$$\sum_{n=1}^{\infty} \left\{(-1)^n . n^{2m} \prod_{r=1}^{q} \frac{J_{\nu_r}(nx_r)}{(\frac{1}{2}nx_r)^{\nu_r}}\right\} = 0. \qquad (5\,.\,4)$$

These equations* hold provided that the x_r's are real and such that the sum of their absolute values does not exceed π. From (5 . 3), in addition, we must exclude the cases when all the x_r's are zero and when the sum of their absolute values equals π. In (5 . 4), m, of course, must

* These results correspond to Nicholson's Integrals. Cf. Watson, *Theory of Bessel Functions*, 413 (equation 8).

be a positive integer. The series converge if $\sum_{r=1}^{q} \nu_r + \frac{1}{2}q - 2m > 0$, otherwise they are summable $(C, 2m - \sum_{r=1}^{q} \nu_r - \frac{1}{2}q + \epsilon)$, their sum in this sense being zero.

Watson* has raised the question under what circumstances does

$$\frac{\frac{1}{2}a_0}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty} \frac{a_n J_\nu(nx)}{(\frac{1}{2}nx)^\nu} \qquad (5.5)$$

represent a null function? He shows that, if this represents a null function in the interval $-\pi < x < \pi$ (except possibly the origin), and if $-\frac{1}{2} < \nu \leqslant \frac{1}{2}$, we must have $a_n = (-1)^n a_0$. In other words, with these restrictions (4 . 2) gives the only null series of the type (5 . 5). By considering x_1 as a variable and $x_2, x_3 \ldots, x_n$ as constants, we see from equation (5 . 3) and (5 . 4) there are an infinite number of null series of the type (5 . 5) in any interval $-\pi + \epsilon < x < \pi - \epsilon$, where ϵ is any positive number other than zero and less than π. The values of $\nu_2, \nu_3, \ldots, \nu_q$ can be so chosen that convergent series can be obtained for all values of ν_1. In general, as $\epsilon \to 0$, these results lead to Nielsen's equations (4 . 2) and (4 . 3).

6. In this and the next paragraph we investigate series closely related to (1 . 1) and (5 . 1), and will obtain results which throw some light on why (1 . 1) and (5 . 1) have zero as their sum for a whole range of values of x.

Consider the integral $\int_C \frac{\phi(x, t)}{J_\nu(t)} t^{\nu-2m}\, dt$,

where $\phi(x, t)$ satisfies all the conditions of § 1 and m is a positive integer. The value of the portion of this integral taken round the semicircular part of C obviously vanishes in the limit as R (the radius of the semicircle) tends to infinity, whilst the portions along the imaginary axis cancel one another.

Since $\phi(x, t)$ is an odd function of t (condition iii), the integrand, on being expanded in the neighbourhood of the origin, is of the form

$$\frac{a_m}{t^{2m+1}} + \frac{a_{m-1}}{t^{2m-1}} + \ldots + \frac{a_0}{t} + \text{positive powers of } t.$$

* Watson, *l. c.*, 636, &c. The question which Watson raises and answers is more general than this.

Hence the contribution of the integral round the semicircular indentation, excluding the origin, will be, in the limit, as the radius of this indentation tends to zero, $-\pi i a_0$. It follows that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\phi(x, j_n)}{J'_\nu(j_n)} \cdot j_n^{\nu-2m} = \tfrac{1}{2} \times \text{coefficient of } \frac{1}{t} \text{ in } \frac{\phi(x, t)\, t^{\nu-2m}}{J_\nu(t)}, \quad (6.1)$$

the series converging and the result being true for that interval of x in which $\phi(x, t)$ satisfies the assumed conditions.

If, for example, at the point $t = Re^{i\theta}$, when R is large $\phi(x, t)$ is of the form (1) § 2, then sufficient conditions for convergence are

$$\lambda < 1, \quad p-2m+\nu+\tfrac{1}{2} < 0, \quad (1)$$

or

$$\lambda \leqslant 1, \quad p-2m+\nu+\tfrac{1}{2} < -1. \quad (2)$$

If, now, $\lambda < 1$, but p is such that $p-2m+\nu+\frac{1}{2}(=q)$ is not negative, then, by attaching the factor $\{1-(t^2/R^2)\}^{q+\epsilon}$ to the integrand and using the methods of § 2 with the contour C' we have

$$\sum_{n<w} (-1)^{n-1} \frac{\phi(x, j_n)}{J'_\nu(j_n)} j_n^{\nu-2m} \left(1-\frac{n^2}{w^2}\right)^{q+\epsilon}$$

$$= \tfrac{1}{2} \times \text{coefficient of } \frac{1}{t} \text{ in } \frac{\phi(x, t) \cdot t^{\nu-2m}}{J_\nu(t)} \left(1-\frac{t^2}{R^2}\right)^{q+\epsilon},$$

where $w = nR/j_n$. In making $R \to \infty$, there is little difficulty in seeing that the coefficient $1/t$ in the right-hand side becomes the same as that of the coefficient of $1/t$ in $\phi(x, t)\, t^{\nu-2m}/J_\nu(t)$. It follows, then, that in this case the series of (6.1) will diverge but will be summable $(C, p-2m+\nu+\frac{1}{2}+\epsilon)$, the sum in this sense being the right-hand side of (6.1).

The illustration to this, when $\nu = \frac{1}{2}$ and $t\,\phi(x, t)$ is a product of functions of the type $J_{\nu_r}(x_r t)/(\frac{1}{2}x_r t)^{\nu_r}$ is, when $m = 1$,

$$\sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n^2} \prod_{r=1}^{q} \frac{J_{\nu_r}(nx_r)}{(\tfrac{1}{2}nx_r)^{\nu_r}} \right\} = \left(\frac{\pi^2}{12} - \tfrac{1}{8} \sum_{r=1}^{q} x_r^2\right) \Big/ \prod_{r=1}^{q} \Gamma(\nu_r+1). \quad (6.2)$$

This equation holds provided all the x_r's are real and $0 < \sum_r |x| < \pi$. When $2+\sum_{r=1}^{q} \nu_r + \frac{1}{2}q$ is positive the series converges, otherwise it is summable $(C, -2-\sum_{r=1}^{q} \nu_r - \frac{1}{2}q+\epsilon)$, its sum being the right-hand side of (6.2).

7. Consider $$\int_C \frac{\phi(x,t)}{J_\nu(t)}\, t^{\nu+2s}\, dt,$$

where s is not necessarily an integer but must be positive and other than zero, whilst $\phi(x,t)$ satisfies the conditions of § 1 for some range of values of x. It must also be assumed that in addition $\phi(x,t)\,.\,t^{2s}$ satisfies condition ii. It follows then that the portion of the integral round the large semicircle vanishes in the limit as $R\to\infty$, but now the portions of the integral along the imaginary axis do not cancel, since the origin is a branch point. Now $\phi(x,t)$ is an odd function of t, and so we may put $\phi(x,it) = i\Phi(x,t)$, where $\Phi(x,t)$ is a real function of x and t. Also the value of the integral round the indentation, which excludes the origin, vanishes. Hence, substituting $te^{\frac{1}{2}\pi i}$ for t along the upper part of the imaginary axis and $te^{-\frac{1}{2}\pi i}$ for t along the lower part, we have

$$2\pi i \sum_{n=1}^{\infty} \frac{\phi(x,j_n).j_n^{\nu+2s}}{J'_\nu(j_n)} = i\int_\infty^0 \frac{\Phi(x,t).t^{\nu+2s}}{I_\nu(t)}\, e^{(s+\frac{1}{2})\pi i}\, dt - i\int_0^\infty \frac{\Phi(x,t).t^{\nu+2s}}{I_\nu(t)}.e^{-(s+\frac{1}{2}\pi i)}\, dt,$$

and so $$\sum_{n=1}^{\infty} \frac{\phi(x,j_n).j_n^{\nu+2s}}{J'_\nu(j_n)} = \frac{\sin s\pi}{\pi}\int_0^\infty \frac{\Phi(x,t).t^{\nu+2s}}{I_\nu(t)}\, dt. \qquad (7\,.\,1)$$

The accompanying conditions are, if, when $|t|$ is large, $\phi(x,t)$ is of the form (1) of § 2, then either

$$\lambda<1, \quad p+2s+\nu+\tfrac{1}{2}<0,$$

or $$\lambda\leqslant 1, \quad p+2s+\nu+\tfrac{1}{2}<-1.$$

When s is a positive integer (7 . 1) reduces to (5 . 1).

Since from (1) § 2 and the asymptotic expression of $I_\nu(t)$ we have

$$\left|\frac{\Phi(x,t).t^{\nu+2s}}{I_\nu(t)}\right| < K.t^{p+\nu+\frac{1}{2}}e^{-(1-\lambda)t},$$

where K is independent of t, the integral converges absolutely for all values of p when $\lambda<1$ but only for $p+2s+\nu+\frac{1}{2}<-1$ when $\lambda=1$.

Suppose now that $\lambda<1$, but $p+2s+\nu+\frac{1}{2}(=q)$ is positive. Then, multiplying the integrand used at the beginning of this paragraph by $\{1-(t^2/R^2)\}^{q+\epsilon}$, and using the contour and methods of § 2, we have

$$\sum_{n<w} \frac{\phi(x,j_n)}{J'_\nu(j_n)}\, j_n^{\nu+2s}\left(1-\frac{n^2}{w^2}\right)^{q+\epsilon} = \frac{\sin s\pi}{\pi}\int_0^{R'} \frac{\Phi(x,t)\,t^{\nu+2s}}{I_\nu(t)}\left(1+\frac{t^2}{R^2}\right)^{q+\epsilon} dt. \qquad (7\,.\,2)$$

It follows, then, that in this case the series on the left of (7 . 1) will be summable $(C, q+\epsilon)$ provided the limit on the right of (7 . 2) exists as $R \to \infty$. This we now proceed to show.

For a given δ, a number N can be chosen, so that

$$\int_N^\infty \frac{\Phi(x, t).t^{\nu+2s}}{I_\nu(t)}\,dt \leqslant \int_N^\infty \frac{|\Phi(x, t)|}{I_\nu(t)}\,t^{\nu+2s}\,dt < \frac{\delta}{2^p}.$$

Denote the integral on the right of (6 . 2) by U. Then

$$U = \left(\int_0^N + \int_N^R\right) \frac{\Phi(x, t).t^{\nu+2s}}{I_\nu(t)} \left(1+\frac{R^2}{t^2}\right)^{q+\epsilon} dt.$$

The second integral is in absolute value less than

$$2^{q+\epsilon}\int_N^R \frac{|\Phi(x, t)|\,t^{\nu+2s}}{I_\nu(t)}\,dt,$$

and so less than δ. The first integral, since N is independent of R, obviously tends to $\int_0^N \frac{\Phi(x, t).t^{\nu+2s}}{I_\nu(t)}\,dt$ as $R \to \infty$.

It is now evident that

$$\lim_{R\to\infty} U = \int_0^\infty \frac{\Phi(x, t).t^{\nu+2s}}{I_\nu(t)}\,dt.$$

In other words, the series on the left of (7 . 1) is summable

$$(C, p+2s+\nu+\tfrac{1}{2}+\epsilon)$$

its sum, in this sense, being the integral on the right of (7 . 1).

When $\nu = \frac{1}{2}$ and $\phi(x, t)$ is a product of Bessel functions, the theorem, with a small change of notation, becomes

$$\sum_{n=1}^\infty (-1)^{n-1}.n^{2s} \prod_{r=1}^q \frac{J_{\nu_r}(nx_r)}{(\frac{1}{2}nx_r)^{\nu_r}} = \sin \pi s \int_0^\infty \frac{t^{2s}}{\sinh \pi t} \prod_{r=1}^q \frac{I_{\nu_r}(x_r t)}{(\frac{1}{2}x_r t)^{\nu_r}}\,dt. \qquad (7 . 3)$$

When, in (7 . 3), there is only one Bessel function, the integral can be expressed in terms of Legendre functions, for, on expanding $1/\sinh \pi t$ in powers of $e^{-\pi t}$, we have

$$\int_0^\infty \frac{I_\nu(xt)}{(\frac{1}{2}xt)^\nu} \frac{t^{2s}\,dt}{\sinh \pi t} = \frac{2}{(\frac{1}{2}x)^\nu} \sum_{n=0}^m \int_0^\infty e^{-(2n+1)\pi t} I_\nu(xt).t^{2s-n}\,dt$$

$$+ \frac{2}{(\frac{1}{2}x)^\nu}\int_0^\infty e^{-(2m+2)\pi t} I_\nu(xt)\,\frac{t^{2s-\nu}}{2\sinh \pi t}\,dt.$$

Let m' be any positive integer, such that $m'\pi$ exceeds $x-\pi$ in magnitude, then from the asymptotic expansion of $I_\nu(xt)$ it can be seen that $e^{-m'\pi t}\{I_\nu(xt)/\sinh \pi t\}\,.\,t^{1-\nu}$ is bounded, as t varies from zero to infinity, its upper bound being M, say. Therefore

$$\int_0^\infty e^{-(2m+2)\pi t}\,\frac{I_\nu(xt)}{\sinh \pi t}\,t^{2s-\nu}\,dt \leqslant M\int_0^\infty e^{-(2m+2-m')\pi t}\,t^{2s-1}\,dt \to 0$$

as $m\to\infty$ if s is positive.

Now it is known that*

$$\int_0^\infty e^{-\lambda t}I_\nu(t)\,t^{\mu-1}\,dt = \frac{\cos \nu\pi}{\sin(\mu+\nu)\,\pi}\cdot\frac{1}{\sqrt{(\tfrac{1}{2}\pi)}}\,\frac{Q_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(\lambda)}{(\lambda^2-1)^{\frac{1}{2}\mu-\frac{1}{4}}}.$$

Combining all these results with (7 . 3) we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n^{2s}.J_\nu(nx)}{(\tfrac{1}{2}nx)^\nu} = -\frac{2^{\nu+\frac{1}{2}}}{\sqrt{(\pi)}\,x^{\nu+\frac{1}{2}}}\cdot\frac{\cos \nu\pi}{\cos s\pi}\sum_{n=0}^{\infty}\frac{Q_{\nu-\frac{1}{2}}^{2s-\nu+\frac{1}{2}}\{(2n+1)\,\pi/x\}}{\{(2n+1)^2\,\pi^2-x^2\}^{s-\frac{1}{2}\nu+\frac{1}{4}}}, \quad (7\,.\,4)$$

an equation which holds under the same circumstances as (7 . 3).

All the Legendre functions in this equation can be expressed as convergent hypergeometric series provided that their arguments are greater than unity (which, in this case, amounts to saying that $-\pi < x < \pi$), and each has $\sin 2s\pi/\cos \nu\pi$, as a factor.

8. In the preceding paragraphs the theory of series derived from the function $\phi(x,t)$ satisfying given conditions has been developed by considering certain integrals involving $J_\nu(t)$ in the denominator. The theory has been illustrated by the case $\nu=\frac{1}{2}$ at each stage. The case $\nu=-\frac{1}{2}$ gives interesting results, which I examine below.

With a slight change of notation we have to deal with integrals of the type

$$\int_c \frac{\phi(x,\,t)}{\cos \pi t}\,dt$$

and allied integrals. When $\phi(x,t)$ is a product of Bessel functions and a power of t, we get theorems analogous to all those that have been

* Watson, *Theory of Bessel Functions*, 388 (equation 6). The actual definition of $Q_\mu^m(\lambda)$ used is that given by Barnes, *Quarterly Journal*, 39 (1908), 109.

obtained for the case $\nu = \frac{1}{2}$. Those corresponding to (4 . 2) and (4 . 3) have already been given by Nielsen, who has also given special cases of (5 . 3) and (5 . 4), namely, when the product of only two Bessel functions is considered, both of which have the same argument, although of different order.

By putting $$\phi(x, t) = \frac{t^{2m-1} J_\nu(2x . \sqrt{t^2+\frac{1}{4}a^2})}{\{x . \sqrt{(t^2+\frac{1}{4}a^2)}\}^\nu},$$

where a is real and m is a positive integer or zero, we get in the interval $-\pi < 2x < \pi$.

$$\frac{\pi}{4\Gamma(\nu+1)} + \sum_{n=0}^{\infty} (-1)^{n-1} \frac{J_\nu[x . \sqrt{\{(2n+1)^2+a^2\}}]}{(2n+1)[\frac{1}{2}x . \sqrt{\{(2n+1)^2+a^2\}}]^\nu} = 0, \quad (8 . 1)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(n+\frac{1}{2})^{2m-1} J_\nu[x . \sqrt{\{(2n+1)^2+a^2\}}]}{[\frac{1}{2}x . \sqrt{\{(2n+1)^2+a^2\}}]} = 0. \quad (8 . 2)$$

When the series do not converge they are summable $(C, 2n-\frac{1}{2}-\nu+\epsilon)$ to zero.

Corresponding to (5 . 3) and (5 . 4) we have

$$\frac{\pi}{4} \prod_{r=1}^{q} \frac{1}{\Gamma(\nu_r+1)} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} . \prod_{r=1}^{q} \frac{J_\nu\{(2n+1)x_r\}}{\{(n+\frac{1}{2})x_r\}^{\nu_r}} = 0. \quad (8 . 3)$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} (n+\tfrac{1}{2})^{2n-1} \prod_{r=1}^{q} \frac{J_{\nu_r}\{(2n+1)x_r\}}{\{(n+\frac{1}{2})x_r\}^{\nu_r}} = 0. \quad (8 . 4)$$

These series are either convergent or are summable to zero when all the x_r's are real, provided $0 < 2\sum_r |x_r| < \pi$.

Also $(2n+1)x_r$ can be replaced in the r-th factor of the product in each term by the expression $x_r . \sqrt{\{(2n+1)^2+a_r^2\}}$, similar to (8 . 1) and (8 . 2).

The simplest illustration to (6 . 1), when $\nu = -\frac{1}{2}$ [corresponding to (6 . 2)] is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \prod_{r=1}^{q} \frac{J_{\nu_r}\{(2n+1)x_r\}}{\{(n+\frac{1}{2})x_r\}^{\nu_r}} = \left(\frac{\pi^3}{32} - \frac{\pi}{16} \sum_{r=1}^{q} x_r^2\right) \Big/ \prod_{r=1}^{q} \Gamma(\nu_r+1) \quad (8 . 6)$$

holding in the same interval for the x_r's as (8 . 3). The series on the left, when not convergent, is summable $(C, -3-\sum_{r=1}^{q} \nu_r - \frac{1}{2}q+\epsilon)$ to the right-hand side.

Corresponding to (7 . 3) we have, if s is positive,

$$\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})^{2s-1}\prod_{r=1}^{q}\frac{J_{\nu_r}\{(2n+1)x_r\}}{\{(n+\frac{1}{2})x_r\}^{\nu_r}}$$

$$=\sin s\pi\int_0^{\infty}\prod_{r=1}^{q}\frac{I_{\nu_r}(2x_rt)}{(x_rt)^{\nu_r}}\frac{t^{2s-1}}{\cosh\pi t}\,dt. \quad (8\,.\,7)$$

When $q=1$ we have, corresponding to (7 . 4),

$$\sum_{n=0}^{\infty}(-1)^n\frac{(n+\frac{1}{2})^{2s-1}J_\nu\{(2n+1)x\}}{\{(n+\frac{1}{2})x\}^{\nu}}$$

$$=\frac{1}{\sqrt{(\pi)}\,x^{\nu+\frac{1}{2}}}\cdot\frac{\cos\nu\pi}{\cos s\pi}\sum_{n=0}^{\infty}(-1)^n\frac{Q^{2s-\nu-\frac{1}{2}}_{\nu-\frac{1}{2}}\{(n+\frac{1}{2})\pi/x\}}{\{(2n+1)^2\pi^2-4x^2\}^{s-\frac{1}{2}\nu-\frac{1}{4}}}, \quad (8\,.\,8)$$

which holds when $-\pi<2x<\pi$.

9. Interesting cases also occur when $\phi(x,t)$ is of the form $\sin\frac{1}{3}\pi t\,.\,f(x,t)$ and $\nu=\frac{1}{2}$. There is little difficulty in framing a set of conditions for $f(x,t)$ such that when satisfied we have from the study of $\int_c\frac{\sin\frac{1}{3}\pi t}{\sin\pi t}f(x,t)\,dt$, the result

$$f(x,1)-f(x,2)+f(x,4)-f(x,5)+f(x,7)-f(x,8)+\ldots$$

$$=\frac{1}{\sqrt{3}}\ \pi\times\text{coefficient of }\frac{1}{t}\text{ in the expression }\frac{\sin\frac{1}{3}\pi t.f(x,t)}{\sin\pi t}$$

in the interval $-2\pi<3x<2\pi$.

When $f(x,t)$ has only a simple pole or no pole at the origin the right-hand side will generally be a constant or zero, and then we have null functions. When $f(x,t)$ is a product of Bessel functions, theorems analogous to all those that precede can be obtained.

10. Nielsen* has studied the functions defined by the equations

$$F(x)=\int_0^{a}\cos(x\xi)f(\xi)\,d\xi$$

and

$$G(x)=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\cos(x\cos\omega\sin\phi)\,f(\omega,\phi)\,d\phi\,d\omega,$$

* Nielsen, *Annali di Mat.* (3), 6 (1901), 301–329.

with the two analogous functions obtained by replacing the cosines in the integrands by sines. The two functions $f(\xi)$ and $f(\omega, \phi)$ are supposed bounded in the range of integration. He shows that $F(x)$ and $G(x)$ in a certain range of values of x give rise to null series of the type

$$\sum_{n=1}^{x} (-1)^{n-1} F(nx) - \tfrac{1}{2} F(0),$$

whilst the other two functions give rise to similar results when n increases through odd integers.

There is little difficulty in seeing that, on the large semicircle of the contour C of § 1 at the point $t = Re^{i\theta}$, if $f(\xi)$ is of limited total fluctuation, $F(xt)$ is of order $\exp(\lambda R|\sin\theta|)$, where λ is at most ax. On the imaginary axis $F(xt)$ is of order $\exp(\lambda R)$, whilst along the real axis it is of order $1/R$.* The same applies to the other three expressions studied by Nielsen. Hence these are all special cases of the functions of the form $\phi(x, t)$.

In addition to this, however, Nielsen has summed these series for values of the variable outside the null range.

* This follows from a theorem given by Bromwich, *Theory of Infinite Series* (1907), 447.

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CORRIGENDUM.

In formula (18) and the following on p. 207, and in the first formula on p. 210, the denominator

$$2(c^2-\xi'^2-\eta'^2-\zeta'^2)$$

has been omitted.

M. K.

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The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1924–25 are on November 13 and December 11, 1924, and January 15, February 12, March 12, April 23, May 14, and June 11, 1925. Cards giving the Programmes of the meetings can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained. Abstracts of papers submitted for publication in full should be extremely brief; as a rule, such an Abstract should not exceed half a page of the *Proceedings*. When an Abstract refers to a communication which is not to be published in full, more space will be allowed for the Abstract, with a maximum of about two pages.

Communications for the Secretaries may be forwarded to them at the following addresses:—

New College, Oxford—G. H. Hardy, M.A., F.R.S.
The University, Birmingham—G. N. Watson, Sc.D., F.R.S.
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